2.1. Injectivity of the Fourier transform

Show that $E$-valued random variables $X_1$ and $X_2$ are identically distributed if

\[ \mathbb{E} \left[ \exp \left( -i \langle X_1, x^* \rangle \right) \right] = \mathbb{E} \left[ \exp \left( -i \langle X_2, x^* \rangle \right) \right], \quad x^* \in E^*. \]

Note: you may use without proof that the Fourier transform is bijective on tempered distributions on $\mathbb{R}^n$, for each $n \in \mathbb{N}$.

2.2. Rotations of independent Gaussians

For iid centered Gaussian random variables $X_1$ and $X_2$, set $Y_1 := (X_1 + X_2)/\sqrt{2}$ and $Y_2 := (X_1 - X_2)/\sqrt{2}$. Show that $Y_1$ and $Y_2$ are iid and have the same distribution as $X_1$ and $X_2$.

Hint: use the previous exercise.

2.3. Convergence of Gaussians

Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of centered Gaussian $E$-valued random variables, let $X$ be an $E$-valued random variable, and assume that $\langle X_n, x^* \rangle \to \langle X, x^* \rangle$ in probability for each $x^* \in E^*$. Show that $X$ is centered Gaussian.
2.4. Sazanov’s theorem

Let $H$ be a separable Hilbert space. Show that $Q \in L(H)$ is the covariance operator of a centered Gaussian $H$-valued random variable $X$ if and only if $Q$ is symmetric, non-negative definite, and $\text{Tr}(Q) < \infty$. 