2.1. Semimartingales are good integrators

(i) Show that finite variation processes are good integrators.

**Hint.** Prove the following inequality for $S \in \mathcal{V}$ and simple $H$:

$$\text{Var}(H \cdot S) \leq |H| \cdot \text{Var}(S) \leq \|H\|_{\infty} \text{Var}(S)$$

(ii) Show that quadratically integrable martingales are good integrators.

**Hint.** Prove the following inequality for $S \in \mathcal{M}^2$ and simple $H$:

$$\forall T \geq 0 : \quad \mathbb{E}[(H \cdot S)_T^2] \leq \|H\|_2^2 \mathbb{E}[S_T^2]$$

(iii) Show that uniformly integrable martingales are good integrators.

**Hint:** Use Burkholder’s inequality of Problem 1.3.

2.2. Integration of Brownian motion with respect to itself

Prove the following statements by expressing the stochastic integral as a limit of elementary integrals:
(i) If $A$ is a continuous finite variation process with $A_0 = 0$, then $\int_0^t A dA = \frac{1}{2} A_t^2$.

(ii) If $B$ is Brownian motion, then $\int_0^t B dB = \frac{1}{2} (B_t^2 - t)$.

2.3. Enlargement of the set of simple integrands

Suppose that the set of simple integrands consists of the larger class of all processes of the form

$$H = H_0 1_{\{0\}} + \sum_{i=1}^n H_i 1_{(T_i, T_{i+1}]}$$

with $H_i$ being $\mathcal{F}_{T_{i+1}}$-measurable (rather than only being $\mathcal{F}_{T_i}$-measurable). Characterize the class of good integrators.