1.1. Komlos’ lemma

Read the proof of Komlos’ lemma in [BSV12, Section 2.1].

**Note.** Some details of the proof were left out during the lecture.

1.2. Continuity of $L^0$-valued linear mappings

(i) Give the definition of bounded subsets of a topological vector space.

(ii) Give the definition of bounded linear maps between topological vector spaces.

(iii) Prove that linear maps between metrizable topological spaces are continuous if and only if they are bounded.

(iv) Characterize bounded subsets in $L^0$.

(v) Characterize bounded linear maps from a metrizable topological vector space into $L^0$.
1.3. Burkholder’s inequality for uniformly integrable martingales

Show the following inequality, due to Burkholder: for all $c, T \in \mathbb{R}_+$, martingales $S$, and simple predictable processes $H$ with $|H| \leq 1$,

$$ c\mathbb{P}\left[ \sup_{t \in [0,T]} |(H \cdot S)_t| \geq c \right] \leq 18\|S_T\|_{L^1(\Omega)}. $$

**Hint.** Complete the following sketch of proof.

(i) Argue that it suffices to prove the inequality for discrete time martingales $S$ on a finite set of time points.

(ii) Suppose in the following steps that the martingale $S$ is nonnegative. Show that $Z := S \wedge c$ is a supermartingale.

(iii) Prove the estimate

$$ c\mathbb{P}\left[ \sup_{t \in [0,T]} |(H \cdot S)_t| \geq c \right] \leq c\mathbb{P}\left[ \sup_{t \in [0,T]} |S_t| \geq c \right] + c\mathbb{P}\left[ \sup_{t \in [0,T]} |(H \cdot Z)_t| \geq c \right]. $$

(iv) Use the Doob-Meyer decomposition in discrete time to obtain the existence of a martingale $\tilde{S}$ and a predictable increasing process $A$ such that

$$ H \cdot Z \leq H \cdot \tilde{S} + A. $$

(v) Show that

$$ c\mathbb{P}\left[ \sup_{t \in [0,T]} |(H \cdot S)_t| \geq c \right] \leq \mathbb{E}[S_T] + \frac{2}{c} \mathbb{E}[(H \cdot \tilde{S})^2_T + A^2_T]. $$

For this step, you may use that for every nonnegative discrete time submartingale $X$ we have

$$ c\mathbb{P}\left( \sup_{n=0,...,N} X_n \geq c \right) \leq \mathbb{E}[X_N]. $$
(vi) Derive the upper bound
\[ c \mathbb{P} \left[ \sup_{t \in [0,T]} |(H \bullet S)_t| \geq c \right] \leq 9 \mathbb{E}[S_0]. \]

Here you may use that for every nonnegative bounded supermartingale \( 0 \leq X \leq c \) we have
\[ \mathbb{E}[M^p_N] \leq p!c^{p-1}\mathbb{E}[X_0] \quad \text{and} \quad \mathbb{E}[A^p_N] \leq p!c^{p-1}\mathbb{E}[X_0], \]
where \( X = M - A \) denotes the Doob-Meyer decomposition.

(vii) Now derive the result in the general situation where the martingale \( S \) does no longer assumed to be nonnegative.

### 1.4. Stability properties of good integrators

Familiarize yourself with the first couple of pages of [Pro05, Chapter II]. More specifically, read the proofs of the following statements and think about how you would prove the corresponding statements for semimartingales rather than good integrators.

(i) Good integrators are stable under stopping.

(ii) Local good integrators are good integrators.

(iii) Good integrators under \( \mathbb{P} \) are also good integrators under any absolutely continuous probability measure \( \mathbb{Q} \ll \mathbb{P} \).

(iv) Good integrators under \( \mathbb{F} \) are also good integrators under any sub-filtration \( \mathbb{G} \subseteq \mathbb{F} \).
References

