A compound Poisson process is a process of the form:

\[ X_t = \sum_{k=1}^{N_t} Y_k, \quad X_0 = 0 \]

where \( N \) is a Poisson process with intensity \( \lambda \), \((Y_k)_{k \geq 1}\) are i.i.d random variables with cumulative distribution function \( F(y) = \mathbb{P}(Y_1 \leq y) \), independent of \( N \). Let \( F(dy) \) be the measure associated with \( F \) and \( F^n \) its n-th convolution, i.e.,

\[ F^n(y) = \mathbb{P}\left( \sum_{k=1}^{n} Y_k \leq y \right). \]

**Problem 1** (4 Points). (a) Show that \( X \) has independent and stationary increments.

(b) Show that the cumulative distribution function of the random variable \( X_t \) is

\[ \mathbb{P}(X_t \leq x) = \exp(-\lambda t) \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} F^n(x). \]

**Problem 2** (4 Points). If \( \mathbb{E}|Y_1| < \infty \);

(a) Show that

\[ \mathbb{E}(X_t) = \lambda t \mathbb{E}(Y_1). \]

(b) Show that the process

\[ M_t = X_t - \lambda t \mathbb{E}(Y_1) \]

for \( t \geq 0 \) is a martingale.

**Problem 3** (4 Points). Let \((N_t)_{t \geq 0}\) be a Poisson process with a constant intensity \( \lambda \) and \( M \) the associated compensated martingale, i.e., \( M_t = N_t - \lambda t \).

(a) Let \((X_i, Y_i)_{i \geq 1}\) be i.i.d random variables, independent of \( N \), and let

\[ U_t = \sum_{i=1}^{N_t} X_i \quad \text{and} \quad V_t = \sum_{i=1}^{N_t} Y_i \]

be compound Poisson processes.

Compute \([U, V]_t\) and \(\langle U, V \rangle_t\).

(b) Let \( p \) and \( q \) be two square integrable functions and

\[ X_t = \int_0^t p(s) dM_s, \quad Y_t = \int_0^t q(s) dM_s. \]

Compute \([X, Y]_t\) and \(\langle X, Y \rangle_t\).
Problem 4 (4 Points). An inhomogeneous Poisson process $N$ with intensity $\lambda$ is a counting process with independent increments which satisfies

$$
P(N_t - N_s = n) = \exp(-\Lambda(s,t)) \frac{(\Lambda(s,t))^n}{n!}
$$

where

$$
\Lambda(s,t) = \Lambda(t) - \Lambda(s) = \int_s^t \lambda(u)du
$$

and

$$
\Lambda(t) = \int_0^t \lambda(u)du \quad \text{for } t \geq s.
$$

(a) Show that

$$
M_t = N_t - \int_0^t \lambda(u)du
$$

for $t \geq 0$ is a martingale.

(b) Give two examples of non-semimartingales.