Exercise 11

Submission: 26-01-2016

Problem 1 (4 Points). Let \( g : [0, \infty) \to [0, \infty) \) be strictly increasing with \( g(0) = 0 \), and let \( N = (N_t)_{t \geq 0} \) be a Poisson process with intensity 1, and \( M = (M_n)_{n=0,1,...} \) is a discrete time Markov chain with values in \( \mathbb{Z} \) and transition matrix \( \Pi = (\pi_{ij})_{i,j \in \mathbb{Z}} \). Furthermore, \( N \) and \( M \) are independent.

Show that \( X = (X_t)_{t \geq 0} \) with

\[
X_t := M_{N_g(t)}
\]

is a Markov process with respect to the natural filtration and determine the transition kernel and the transition operator.

Hint: Use the Chapman-Kolmogorov equation (Corollary 16.16, see the Skript) and Theorem 16.17 “Existence of Markov processes”.

Problem 2 (4 Points). (a) Under what conditions (with respect to \( g \) and \( \Pi \)) is the process \( (M_{N_g(t)})_t \) homogeneous.

(b) Determine the generator of \( (M_{N_g(t)})_t \).

Problem 3 (4 Points). Recall from Exercise 10: Let \( (X_t)_{t \geq 0} \) be a Brownian motion and \( Y_t = e^{-t/2}X(e^t - 1) \).

Show that \( (Y_t)_{t \geq 0} \) is homogeneous, i.e., \( (P_{s,t}f(x) = P_{0,t-s}f(x)) \) with generator

\[
G^Y f(x) = -\frac{x}{2} f'(x) + \frac{1}{2} f''(x)
\]

for \( f \in C^2(\mathbb{R}) \).

Problem 4 (4 Points). Show that every Feller process with right-continuous paths satisfies the strong Markov property.