Exercise 3

Submission: 03-11-2015

Problem 1 (4 Points).  (a) We say that a random variable \( \eta \) has the exponential distribution of rate \( \lambda > 0 \) if:
\[
P\{ \eta > t \} = e^{-\lambda t} \quad \text{for all } t \geq 0.
\]
Show that a random variable \( \eta \) with exponential distribution satisfies
\[
P\{ \eta > t + s \} = P\{ \eta > t \} P\{ \eta > s \}.
\]
Hint: When the probabilities are replaced by exponents, the equality should become obvious.

(b) Let \( X = (X_t)_{t \geq 0} \) be a Poisson process with parameter \( \lambda > 0 \).
Show that
\[
Y_t = 2^{X_t} \exp(-\lambda t)
\]
is a martingale with respect to the filtration \( (\mathcal{F}_t)_{t \geq 0} \) generated by the family of random variables \( \{X_s : s \in [0,t]\} \).

Problem 2 (4 Points). Let \( X = (X_t)_{t \geq 0} \) be a Brownian motion.

(a) Show that
\[
|X_t|^2 - t
\]
is a martingale.

Hint: Use the fact that \( X_t - X_s \) is independent of \( \mathcal{F}_s \) if \( s < t \).

(b) Show that
\[
\exp(X_t) \exp(-t/2)
\]
is a martingale.

Problem 3 (4 Points). Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space with random variables \( X \) and \( Y \) such that \( X \) and \( Y \) are independents. Prove that
\[
E[g(X,Y)|Y = y] = E[g(X,y)]
\]
where \( g \) is a bounded measurable function.

Hint: Use the Monotone Class Theorem.