Problem 1 (4 Points). Let $X = (X_t)_{t \geq 0}$ be a Poisson process with parameter $\lambda > 0$.

(a) Show that $Y_t = X_t - \lambda t$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ generated by the family of random variables $\{X_s : s \in [0, t]\}$.

(Hint: $Y_t$ is a martingale implies (i) $Y_t \in L^1$, for all $t \geq 0$. (ii) $E[Y_t|\mathcal{F}_s] = Y_s$, for $0 \leq s \leq t$.

(b) Show that
\[
\lim_{t \to \infty} \frac{X_t}{t} = \lambda \quad a.s.
\]

Problem 2 (4 Points). (a) Let $X = (X_t)_{t \geq 0}$ and $Y = (Y_t)_{t \geq 0}$ be two independent Poisson processes with parameters $\lambda > 0$ and $\mu > 0$. Show that $(X_t + Y_t)_{t \geq 0}$ is a Poisson process with parameter $\lambda + \mu$.

(b) Let $X = (X_t)_{t \geq 0}$ and $Y = (Y_t)_{t \geq 0}$ be two independent standard Brownian motions. Show that
\[
\frac{X_t + Y_t}{\sqrt{2}}
\]

is also a standard Brownian motion.

Problem 3 (4 Points). Let $(\xi_n)_{n \geq 0}$ be a martingale with a $\tau$ stopping time with respect to the filtration $(\mathcal{F}_n)_{n \geq 0}$ such that the following conditions holds:

(i) $\tau < \infty \quad a.s.$

(ii) $\xi_\tau$ is integrable

(iii) $E[\xi_n 1_{\tau > n}] \to 0$ as $n \to \infty$.

Then
\[
E[\xi_\tau] = E[\xi_1]
\]

Problem 4 (4 Points). A real-valued stochastic process $X = (X_t)_{t \geq 0}$ is measurable if the mapping $[0, \infty) \times \Omega \to \mathbb{R}^d; (t, \omega) \mapsto X_t(\omega)$ is $B([0, \infty]) \otimes \mathcal{F} - B(\mathbb{R}^d)$ measurable.

(a) Give an example of a predictable process.

(b) Give an example of an optional process which is not predictable

(c) Give an example of an adapted stochastic process $X$ that is not measurable.

(d) Show that every progressively measurable stochastic process $X = (X_t)_{t \geq 0}$ with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ is also measurable.