Stochastic Filtering (SS2016) Exercise Sheet 9

Lecture and Exercises: JProf. Dr. Philipp Harms
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9.1. Innovations approach for jumps observations

Let $X$ solve the martingale problem associated to $A : \mathcal{D}(A) \subseteq B(\mathbb{R}^d) \to B(\mathbb{R}^d)$ and let $Y$ be a Poisson process with rate $\lambda(X_\cdot)$, i.e., $Y_t = N_t \int_0^t \lambda(X_s) \, ds$, where $N$ is a standard Poisson process independent of $X$ and $\lambda : \mathbb{X} \to (0, \infty)$ is a measurable function. Assume that $\lambda$ and $\lambda^{-1}$ are bounded. Let $I$ denote the innovations process $I_t = Y_t - \int_0^t \pi_t(\lambda) \, dt$.

a) Show that the $(\mathbb{F}(Y), \mathbb{P})$-compensator of $[I, I]$ is $\int_0^t \pi_s(\lambda) \, ds$, i.e., the difference between the two is an $(\mathbb{F}(Y), \mathbb{P})$-local martingale.

b) Assume that $(I, \mathbb{F}(Y), \mathbb{P})$ has the strong property of predictable representation and show that the Kushner-Stratonovich equation holds, i.e.,

$$d\pi_t(f) = \pi_t(\lambda f) \, dt + \frac{\pi_t(\lambda f) - \pi_t(\lambda) \pi_t(f)}{\pi_t(\lambda)} \, dI_t, \quad \forall f \in \mathcal{D}(A).$$

9.2. Correlated noise

The innovations approach allows one to treat filtering problems with correlated noise. Let $X$ solve the martingale problem associated to $A : \mathcal{D}(A) \subseteq B(\mathbb{R}^d) \to B(\mathbb{R}^d)$, $M^f_t = f(X_t) - f(X_0) - \int_0^t Af(X_s) \, ds$, and $dY_t = h(X_t) \, dt + dW_t$, where $W$ is $d$-dimensional Brownian motion. To model the dependence between $X$ and $W$, assume that there are operators $C_i : \mathcal{D}(C_i) \subseteq B(\mathbb{X}) \to B(\mathbb{X})$, $i \in \{1, \ldots, d\}$ such that

$$\langle M^f, W^i \rangle_t = \int_0^t (C_i f)(X_s) \, ds, \quad \forall i \in \{1, \ldots, d\}, \forall f \in \mathcal{D}(A) \cap \mathcal{D}(C_1) \cap \cdots \cap \mathcal{D}(C_d).$$

The vector $(C_1 f, \ldots, C_d f)$ is denoted by $C f$. 
a) Assume that \((I, \mathcal{F}(Y), \mathbb{P})\) has the strong property of predictable representation and show that the Kushner-Stratonovich equation takes the form

\[
d\pi_t f = \pi_t (Af) dt + \left( \pi_t (fh) - \pi_t (f) \pi_t (h) + \pi_t (Cf) \right) (dY_t - \pi_t (h) dt),
\]

for all \(f \in \mathcal{D}(A) \cap \mathcal{D}(C_1) \cap \cdots \cap \mathcal{D}(C_d)\).

Hint: Adapt the proof of the Kushner-Stratonovich equation with uncorrelated noise.

b) What are the operators \(C\) in the case where \(X\) is a diffusion, say, \(dX_t = \mu(X_t) dt + \sigma(X_t) dB_t\)?

9.3. Asymptotics of the Kalman-Bucy filter

Let \(X\) and \(Y\) be real-valued processes solving the SDE

\[
dX_t = (a_0 + a_1 X_t + a_2 Y_t) dt + \sum_{i=1}^{2} b_i dW^i_t,
\]

\[
dY_t = (\Lambda_0 + \Lambda_1 X_t + \Lambda_2 Y_t) dt + \sum_{i=1}^{2} B_i dW^i_t,
\]

with normally distributed initial condition \((X_0, Y_0)\), where \(W^1\) and \(W^2\) are two independent Wiener processes on \(\mathbb{R}\) and where \(B_1^2 + B_2^2 > 0\).

a) Explain on a conceptual level how the Kalman-Bucy filter with correlated noise can be derived, i.e.,

\[
d\hat{X}_t = (a_0 + a_1 \hat{X}_t + a_2 Y_t) dt + \frac{b_1 B_1 + b_2 B_2 + \hat{\Sigma}_t \Lambda_1}{B_1^2 + B_2^2} (dY_t - (\Lambda_0 + \Lambda_1 \hat{X}_t + \Lambda_2 Y_t) dt),
\]

\[
\frac{d\hat{\Sigma}_t}{dt} = 2a_1 \hat{\Sigma}_t + b_1^2 + b_2^2 - \frac{(b_1 B_1 + b_2 B_2 + \hat{\Sigma}_t \Lambda_1)^2}{B_1^2 + B_2^2},
\]

with initial conditions

\[
\hat{X}_0 = \mathbb{E}[X_0|Y_0], \quad \hat{\Sigma}_0 = \text{Var}(X_0|Y_0).
\]
b) Calculate $\lim_{t \to \infty} \hat{\Sigma}_t$ in the following special cases and compare the results:

(i) $\displaystyle dX_t = 0, \quad dY_t = X_t \, dt + dW_t^2$

(ii) $\displaystyle dX_t = dW_t^1, \quad dY_t = X_t \, dt + dW_t^2$

(iii) $\displaystyle dX_t = dW_t^2, \quad dY_t = X_t \, dt + dW_t^2$

(iv) $\displaystyle dX_t = -dW_t^2, \quad dY_t = X_t \, dt + dW_t^2$

9.4. Kalman filter for a model of population growth

We consider the following model for population growth with noisy observations:

$$dX_t = rX_t \, dt, \quad dY_t = X_t \, dt + m dW_t,$$

with $X_0 \sim \mathcal{N}(b, a^2)$ and $Y_0 = 0$ for some constants $r, m, b, a > 0$.

a) Calculate $\lim_{t \to \infty} \hat{\Sigma}_t$. How is the asymptotic precision of the filter affected by the growth rate $r$?

b) Implement the Kálmán-Bucy filter for the model. In order to test your implementation approximate a path of $(X_t, Y_t)_{t \in [0,1]}$ using the Euler-Maruyama scheme. Use your implementation of the Kálmán-Bucy filter in order to recover the signal from the observation. Use the following parameters: $r = 0.5, \quad m = 1, \quad b = 1, \quad a = 0.5$. What do you observe?