

# A MULTIPLE CURVE LÉVY SWAP MARKET MODEL

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ABSTRACT. In this paper we develop an arbitrage free multiple curve model through the specification of forward swap rates. Two sets of assets are chosen as fundamentals: OIS zero coupon bonds and forward rate agreements. This is a very natural approach since on the one side OIS bonds represent the new class of risk free discount bonds, and on the other side the mid and long maturity part of the interest rate term structure is bootstrapped from quotes of swap rates that can be represented by FRA rates and OIS bond prices in the new market environment. We construct the rates via a backward induction along the tenor structure on the basis of the forward swap measures. Time-inhomogeneous Lévy processes are used as drivers of the dynamics. As an application we derive an approximative Fourier based valuation formula for swaptions. The model is implemented and calibrated by using generalized hyperbolic Lévy processes as drivers.

## 1. INTRODUCTION

In classical interest rate modeling forward rates can be expressed in terms of risk free zero coupon bond prices. No arbitrage considerations then imply that forward rates for different tenors are strongly related to each other. Underlying these restrictions is the assumption that these rates are not subject to credit or liquidity risk. Since the global financial crisis starting in summer 2007, this assumption can no longer be justified. In the presence of these risks, the tenor of a fixed income investment becomes relevant. The longer the tenor, the higher is the risk for a deterioration of the credit quality or the market liquidity.

Since swaps represent a sequence of forward rate agreements (FRAs), as a consequence of this tenor dependence, the classical formula for forward swap rates

$$S_t(T_i) = \frac{B_t(T_i) - B_t(T_n)}{\sum_{j=i+1}^n \delta B_t(T_j)}$$

(see Section 3.1) does no longer hold in the new multiple curve interest rate environment. For swaps with variable rates linked to risky Interbank Offered Rates (IBORs) such as LIBOR, EURIBOR - generically we will use just LIBOR in the following - this formula has to be modified such that it accounts for the various spreads that can be observed in the market. As swap rates can be represented on the basis of FRAs and OIS discount bond prices (see equation (3.3)), FRAs are the additional ingredients which are needed. This is the reason why we will consider a financial market consisting of FRAs and discount bonds as basic building blocks.

The aim of this paper is to develop an arbitrage free model for the interest rate market through the specification of forward swap rates. Let us emphasize

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*Date:* April 1, 2020.

that this is very natural since swaps are liquidly traded instruments for many maturities and therefore represent a very reliable source of financial data. The sheer size of the swap market – according to the BIS Global OTC Derivatives statistic the notional amount outstanding in H1/2019 was approximately USD 389 trillions – is impressive.

The construction of the forward swap rates will be done via a backward induction similar to the single curve swap market model developed in [Eberlein and Liinev \(2007\)](#). While in the latter approach only a single swap rate has to be modeled, our approach here requires to consider risk free as well as risky swap rates. Basic tools in this construction are the forward swap measures as well as the change of numeraire technique. As driving processes we will use time-inhomogeneous Lévy processes.

Another important reason to consider swap rates as primary objects in this model is their direct use in the pricing formula for swaptions. We develop a numerically efficient swaption pricing formula which we use to fit the model to market data. In the implementation of the model generalized hyperbolic processes will be used as drivers. This guarantees enough flexibility in order to fit the quoted volatility surfaces.

There is a substantial literature on multiple curve modeling since tenor dependent spreads were observed in the market quotes. Basically three streams of research can be identified: short rate models, HJM approaches and LIBOR market models. The necessity to account for different risk levels in interbank rates has first been acknowledged already in [Henrard \(2007\)](#). Thereafter, multiple curve short rate models have been introduced in [Kijima, Tanaka, and Wong \(2009\)](#), [Kenyon \(2010\)](#), and [Filipović and Trolle \(2013\)](#) to mention a few. Most of the multiple curve papers which have appeared so far are HJM approaches. Among those are [Pallavicini and Tarengi \(2010\)](#), [Crépey, Grbac, and Nguyen \(2012\)](#), [Moreni and Pallavicini \(2014\)](#), [Crépey, Grbac, Ngor, and Skovmand \(2015\)](#), [Cuchiero, Fontana, and Gnoatto \(2016\)](#), [Eberlein and Gerhart \(2018\)](#), and [Fontana, Grbac, Gümbel, and Schmidt \(2020\)](#). LIBOR market models with multiple curves have been developed in [Mercurio \(2010a\)](#), [Mercurio \(2010b\)](#), [Grbac, Papapantoleon, Schoenmakers, and Skovmand \(2015\)](#), and [Grbac and Papapantoleon \(2013\)](#). Closely related to LIBOR market models are approaches where the dynamics of the forward process is specified. We mention here [Bianchetti \(2010\)](#) and [Eberlein, Gerhart, and Grbac \(2019\)](#). For a gentle introduction into multiple curve markets and their modeling we refer to [Henrard \(2014\)](#) and [Grbac and Runggaldier \(2015\)](#).

In [Section 2](#) we sketch the key properties of the driving processes. The swap market model is developed in [Section 3](#). Valuation of swaptions is the topic of [Section 4](#). In [Section 5](#) we implement the model by using generalized hyperbolic processes as drivers and calibrate the model to market data. Finally, [Section 6](#) concludes.

## 2. THE DRIVING PROCESS

Let  $T^* \in \mathbb{R}_+ := [0, \infty)$  be a finite time horizon and  $\mathcal{B} := (\Omega, \mathcal{F}_{T^*}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0, T^*]}, P)$  a stochastic basis that satisfies the usual conditions in the sense

of [Jacod and Shiryaev \(2003\)](#), Definition I.1.2 and Definition I.1.3). As driving processes we will use time-inhomogeneous Lévy processes  $L = (L_t)_{t \in [0, T^*]}$  on  $\mathcal{B}$ . This means  $L$  is an  $\mathbb{F}$ -adapted process with independent increments and absolutely continuous characteristics (see [Jacod and Shiryaev \(2003\)](#) or [Eberlein and Kallsen \(2019\)](#)). This type of stochastic processes is also known as additive processes (see [Sato \(1999\)](#)).

For convenience we sketch the key properties in the  $d$ -dimensional case  $L = (L^1, \dots, L^d)$ .  $L$  is a semimartingale. Without loss of generality we can assume that the path of each component  $L^i$  is càdlàg and starts in zero. The law of  $L_t$  is determined by its characteristic function

$$\begin{aligned} \mathbb{E}[e^{i\langle u, L_t \rangle}] &= \exp \left( \int_0^t \left[ i\langle u, b_s(h) \rangle - \frac{1}{2} \langle u, c_s u \rangle \right. \right. \\ &\quad \left. \left. + \int_{\mathbb{R}^d} \left( e^{i\langle u, x \rangle} - 1 - i\langle u, h(x) \rangle \right) F_s(dx) \right] ds \right) \quad (u \in \mathbb{R}^d). \end{aligned} \quad (2.1)$$

Here,  $h$  is a truncation function, where usually one takes  $h(x) = x \cdot \mathbb{1}_{\{|x| \leq 1\}}$ ,  $b_s(h) = (b_s^1(h), \dots, b_s^d(h)) : [0, T^*] \rightarrow \mathbb{R}^d$ ,  $c_s = (c_s^{ij})_{i,j \leq d} : [0, T^*] \rightarrow \mathbb{R}^{d \times d}$ , a symmetric nonnegative-definite  $d \times d$ -matrix and  $F_s$  is a Lévy measure for every  $s \in [0, T^*]$ , i.e. a nonnegative measure on  $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$  that integrates  $(|x|^2 \wedge 1)$  and satisfies  $F_s(\{0\}) = 0$ . We denote by  $\langle \cdot, \cdot \rangle$  the Euclidean scalar product on  $\mathbb{R}^d$  and  $|\cdot|$  is the corresponding norm. The scalar product on  $\mathbb{R}^d$  is extended to complex numbers by setting  $\langle w, z \rangle := \sum_{j=1}^d w_j z_j$  for every  $w, z \in \mathbb{C}^d$ . Thus,  $\langle \cdot, \cdot \rangle$  is not the Hermitian scalar product here. We further assume that

$$\int_0^{T^*} \left[ \|b_s(h)\| + \|c_s\| + \int_{\mathbb{R}^d} (|x|^2 \wedge 1) F_s(dx) \right] ds < \infty,$$

where  $\|\cdot\|$  denotes any norm on the set of  $d \times d$ -matrices. The triplet  $(b, c, F) = (b_s, c_s, F_s)_{s \in [0, T^*]}$  represents the local characteristics of  $L$ . We also make the following standing assumption on the existence of exponential moments.

**Assumption (EM).** There exist constants  $M, \varepsilon > 0$  such that

$$\int_0^{T^*} \int_{|x| > 1} \exp\langle u, x \rangle F_t(dx) dt < \infty,$$

for every  $u \in [-(1 + \varepsilon)M, (1 + \varepsilon)M]^d$ . In particular, we assume without loss of generality that  $\int_{|x| > 1} \exp\langle u, x \rangle F_t(dx) < \infty$ , for all  $t \in [0, T^*]$ .

Assumption (EM) is equivalent to  $\mathbb{E}[\exp\langle u, L_t \rangle] < \infty$  for all  $t \in [0, T^*]$  and  $u \in [-(1 + \varepsilon)M, (1 + \varepsilon)M]^d$ . We will consider models with underlying processes that are exponentials of stochastic integrals with respect to  $L$ . These underlying processes have to be martingales under the risk neutral measure. Therefore, a priori they have to have finite expectations which is exactly guaranteed by assumption (EM). An immediate consequence of (EM) is that the random

variable  $L_t$  has finite expectation. Therefore, the representation (2.1) simplifies and can be written as

$$\mathbb{E}[e^{i\langle u, L_t \rangle}] = \exp \left( \int_0^t \left[ i\langle u, b_s \rangle - \frac{1}{2}\langle u, c_s u \rangle + \int_{\mathbb{R}^d} \left( e^{i\langle u, x \rangle} - 1 - i\langle u, x \rangle \right) F_s(dx) \right] ds \right). \quad (2.2)$$

We emphasise that the characteristic  $b$  is now different from the one in (2.1). We will always work with the local characteristics  $(b, c, F)$  that appear in (2.2). Another implication of assumption (EM) is that the process  $L$  is a special semimartingale. Thus, its canonical representation is given by the simple form

$$L_t = \int_0^t b_s ds + \int_0^t \sqrt{c_s} dW_s + \int_0^t \int_{\mathbb{R}^d} x(\mu^L - \nu)(ds, dx) \quad (2.3)$$

(see Jacod and Shiryaev (2003, Corollary II.2.38)), where  $W = (W_t)_{t \in [0, T^*]}$  is a standard  $d$ -dimensional Brownian motion,  $\sqrt{c_s}$  is a measurable version of the square root of  $c_s$ , and  $\mu^L$  is the random measure of jumps of  $L$  with compensator  $\nu(ds, dx) = F_s(dx)ds$ . Obviously, the integrals in (2.3) should be understood componentwise. We stress that assumption (EM) is valid for a very general class of processes and holds, in particular, for all processes that are generated by generalised hyperbolic distributions. The (extended) cumulant process associated with the process  $L$  under the probability measure  $P$  is denoted by  $\theta_s$  and given by

$$\theta_s(z) = \langle z, b_s \rangle + \frac{1}{2}\langle z, c_s z \rangle + \int_{\mathbb{R}^d} \left( e^{\langle z, x \rangle} - 1 - \langle z, x \rangle \right) F_s(dx)$$

for every  $z \in \mathbb{C}^d$  where this function is defined. The latter requires that  $\operatorname{Re}(z) \in [-(1 + \epsilon)M, (1 + \epsilon)M]^d$ . A detailed analysis of the cumulant process for semimartingales is given by Kallsen and Shiryaev (2002). Note that if  $L$  is a (homogeneous) Lévy process, i.e. if the increments of  $L$  are stationary, the triplet  $(b_s, c_s, F_s)$  and thus also  $\theta_s$  do not depend on  $s$ .

### 3. THE SWAP MARKET MODEL

In this section, we follow the framework of Eberlein and Liinev (2007) and Fontana et al. (2020) to set up a multiple curve swap market model.

**3.1. Financial Market.** We consider a financial market in which LIBOR rates are quoted for a finite set of tenors  $\mathcal{D} = \{\delta_1, \dots, \delta_m\}$  with  $0 < \delta_1 \dots < \delta_m$ . Typically, these range from one week to one year. Denote by  $L(T, T, T + \delta)$  the LIBOR rate for tenor  $\delta \in \mathcal{D}$  which is fixed at time  $T$  and applies to the time interval  $[T, T + \delta]$ .

We assume that forward rate agreements with LIBOR rates of different tenor lengths as reference rates are traded in the market. They are formally defined as follows.

**Definition 3.1.** A *forward rate agreement* (FRA) with tenor  $\delta$ , settlement date  $T$  and strike  $K$ , is a contract stating that the fixed interest rate  $K$  will apply to a certain nominal amount  $N$  for the future period  $[T, T + \delta]$ . At maturity  $T + \delta$  the payoff is given by

$$\delta(L(T, T, T + \delta) - K)N.$$

In the following we will always normalize the notional amount  $N$  to one. Denote the price of such a FRA contract at date  $t \leq T$  by  $F_t(T, K)$ . The interest rate  $K$ , which ensures that the FRA for period  $[T, T + \delta]$  has value zero at time  $t$ , is denoted by  $L(t, T, T + \delta)$ .

Furthermore we assume that swaps are traded in the market. Let a tenor structure  $\mathcal{T}^\delta = \{T_0, \dots, T_n\}$  with constant tenor length  $\delta = T_i - T_{i-1}$  be given.

**Definition 3.2.** An *interest rate (payer) swap* (IRS) is a contract where an investor agrees to pay to the other party a predetermined fixed rate  $S$  on a notional amount  $N$  at dates  $T_1, \dots, T_n$ . In return, the investor receives interest rate payments at a floating rate on the same notional principal. We assume that the payment dates for the floating and the fixed leg are the same. Again we normalize the notional  $N$  to be one. The fixed rate  $S$  such that the swap contract for the future time period  $[T_i, T_n]$  has zero value at time  $t$  is called the time  $t$  *forward swap rate* and will be denoted by  $S_t(T_i, \delta)$ .

Overnight indexed swaps (OIS) constitute a special case of an interest rate swap where the floating rate is given by simply compounding the consecutive overnight rates between the dates  $T_i$  and  $T_i + \delta$ . We denote the time  $t$  swap rate corresponding to an overnight indexed swap for the time period  $[T_i, T_n]$  by  $S_t^d(T_i)$ . It is given by

$$S_t^d(T_i) = \frac{B_t^d(T_i) - B_t^d(T_n)}{\sum_{j=i+1}^n \delta B_t^d(T_j)} \quad (3.1)$$

where  $B_t^d(T)$  denotes the time  $t$  price of a risk free (discount) bond with maturity  $T$ . Those prices are derived by bootstrapping from quoted OIS-rates (cf. [Gerhart and Lütkebohmert \(2018\)](#)).

When the floating rate in an IRS is given by the LIBOR rate for tenor  $\delta$ , the cash flow to the investor at each time  $T_i$  for  $i = 1, \dots, n$  equals

$$\delta(L(T_{i-1}, T_{i-1}, T_i) - S)N.$$

The time  $t$  forward swap rate corresponding to the period  $[T_i, T_n]$  with  $t \leq T_i$  and  $i \in \{0, \dots, n-1\}$  is then given by

$$S_t(T_i, \delta) = \frac{\sum_{j=i+1}^n \delta B_t^d(T_j) L(t, T_{j-1}, T_j)}{\sum_{j=i+1}^n \delta B_t^d(T_j)} \quad (3.2)$$

(cf. [Grbac and Runggaldier \(2015\)](#) or [Gerhart and Lütkebohmert \(2018\)](#)). Note that  $L(t, T_{j-1}, T_j)$  can be defined using the  $T_j$ -forward measure  $P_{T_j}$  as

$$L(t, T_{j-1}, T_j) = \mathbb{E}_{P_{T_j}}[L(T_{j-1}, T_{j-1}, T_j) | \mathcal{F}_t]$$

(see [Eberlein et al. \(2019\)](#) for details). The following pricing formula for a forward rate agreement with tenor  $\delta$  is well known

$$F_t(T, K) = \delta B_t^d(T + \delta)(L(t, T, T + \delta) - K)$$

(see Grbac and Runggaldier (2015, section 1.4.1)). By using it we can reformulate the swap rate formula (3.2) as

$$S_t(T_i, \delta) = \frac{\sum_{j=i+1}^n \delta B_t^d(T_j)[L(t, T_{j-1}, T_j) - K'] + \sum_{j=i+1}^n \delta B_t^d(T_j)K'}{\sum_{j=i+1}^n \delta B_t^d(T_j)}$$

which means that we can write

$$S_t(T_i, \delta) = \frac{\sum_{j=i+1}^n F_t(T_{j-1}, K')}{\sum_{j=i+1}^n \delta B_t^d(T_j)} + K', \quad (3.3)$$

where  $K' \in \mathbb{R}$  is some fixed rate. Thus, it suffices to assume that only forward rate agreements and risk free zero-coupon bonds are traded. Consequently the following definition of a multiple curve financial market turns out to be useful.

**Definition 3.3** (Compare Fontana et al. (2020), Def. 2.2). A *multiple curve financial market* as considered in this paper is a financial market containing two sets of assets: OIS zero-coupon bonds for all maturities  $T \geq 0$  as well as FRAs for all tenors  $\delta \in \mathcal{D}$ , all settlement dates  $T \geq 0$  and all strikes  $K \in \mathbb{R}$ .

The assumption that OIS zero-coupon bonds are traded for all maturities  $T \geq 0$  implicitly requires that OIS swaps are available for all maturities. Due to the formula

$$F_t(T_{j-1}, K) = F_t(T_{j-1}, K') - \delta(K - K')B_t^d(T_j)$$

(cf. Fontana et al. (2020, Remark 2.3)) the price of a forward rate agreement with an arbitrary  $K$  can be represented as a combination of the price of a forward rate agreement with a specific strike  $K'$  and a correction term depending on the discount bond. Therefore, it is sufficient to assume that only FRAs with some fixed strike  $K'$  for all tenors and settlement dates are traded.

**3.2. Backward Construction.** Recall that we consider a complete stochastic basis  $(\Omega, \mathcal{F}_{T^*}, (\mathcal{F}_t)_{0 \leq t \leq T^*}, P)$  where we identify  $P$  with the  $T^*$ -forward measure  $P_{T^*}$ . We consider again the tenor structure  $\mathcal{T}^\delta = \{T_0, \dots, T_n\}$  with  $T_n = T^*$ . Following Musiela and Rutkowski (2006) we define the forward swap measure as follows.

**Definition 3.4.** The *forward swap measure*  $\tilde{P}_{T_i}$  associated with date  $T_i \in \mathcal{T}^\delta$  is a probability measure equivalent to the underlying probability measure  $P$  under which the relative asset prices

$$\frac{B_t^d(T_j)}{\delta B_t^d(T_i) + \dots + \delta B_t^d(T_n)}$$

and

$$\frac{F_t(T_l, K)}{\delta B_t^d(T_i) + \dots + \delta B_t^d(T_n)}$$

are local martingales for all  $K \in \mathbb{R}$ ,  $j \in \{0, \dots, n\}$  and  $l \in \{0, \dots, n-1\}$ .

**Assumption (VOL).** We assume deterministic volatility structures  $\gamma^d : [0, T^*] \times [0, T^*] \rightarrow \mathbb{R}$  and  $\gamma : [0, T^*] \times [0, T^*] \rightarrow \mathbb{R}$  where

$$\begin{aligned} |\gamma^d(s, T)| &\leq M_1 < M \\ |\gamma(s, T)| &\leq M_2 < M \end{aligned}$$

for all  $(s, T) \in [0, T^*] \times [0, T^*]$  with  $s \leq T$  and  $M$  is the constant from assumption (EM).

We assume that the forward swap measure  $\tilde{P}_{T_n}$  related to the terminal date  $T_n$  to be the forward measure  $P_{T^*}$ . Under  $\tilde{P}_{T_n}$ , the forward swap rate for date  $T_{n-1}$ ,

$$S_t(T_{n-1}, \delta) = \frac{F_t(T_{n-1}, K')}{\delta B_t^d(T_n)} + K',$$

is a local martingale according to Definition 3.4.

We postulate that the forward swap rate  $S(T_{n-1}, \delta)$  is given by

$$S_t(T_{n-1}, \delta) = S_0(T_{n-1}, \delta) \exp \left( \int_0^t \alpha(s, T_{n-1}, T_n) ds + \int_0^t \gamma(s, T_{n-1}) d\tilde{L}_s^{T_n} \right)$$

with initial values

$$S_0(T_{n-1}, \delta) = L(0, T_{n-1}, T_n)$$

where

$$\tilde{L}_t^{T_n} = \int_0^t \sqrt{c_s} d\tilde{W}_s^{T_n} + \int_0^t \int_{\mathbb{R}} x(\mu^L - \tilde{\nu}^{T_n})(ds, dx)$$

is a time-inhomogeneous Lévy process under  $\tilde{P}_{T_n}$ . For simplicity we will use hereafter a one dimensional driving process. The extension to multivariate time-inhomogeneous Lévy processes would be straightforward.

Now we specify the drift function  $\alpha$  in such a way that this forward swap rate becomes a martingale under  $\tilde{P}_{T_n}$ , namely

$$\begin{aligned} \int_0^t \alpha(s, T_{n-1}, T_n) ds &= -\frac{1}{2} \int_0^t \gamma(s, T_{n-1}) c_s \gamma(s, T_{n-1}) ds \\ &\quad - \int_0^t \int_{\mathbb{R}} \left( e^{\gamma(s, T_{n-1})x} - 1 - \gamma(s, T_{n-1})x \right) \tilde{\nu}^{T_n}(ds, dx). \end{aligned}$$

Similarly, the forward swap rate

$$S_t^d(T_{n-1}) = \frac{B_t^d(T_{n-1}) - B_t^d(T_n)}{\delta B_t^d(T_n)}$$

of the OIS contract is a local martingale under  $\tilde{P}_{T_n}$ . We model the forward OIS swap rate as

$$S_t^d(T_{n-1}) = S_0^d(T_{n-1}) \exp \left( \int_0^t \alpha^d(s, T_{n-1}, T_n) ds + \int_0^t \gamma^d(s, T_{n-1}) d\tilde{L}_s^{T_n} \right)$$

with initial values

$$S_0^d(T_{n-1}) = \frac{B_0^d(T_{n-1}) - B_0^d(T_n)}{\delta B_0^d(T_n)},$$

and drift condition

$$\begin{aligned} \int_0^t \alpha^{\mathbf{d}}(s, T_{n-1}, T_n) ds &= -\frac{1}{2} \int_0^t \gamma^{\mathbf{d}}(s, T_{n-1}) c_s \gamma^{\mathbf{d}}(s, T_{n-1}) ds \\ &\quad - \int_0^t \int_{\mathbb{R}} \left( e^{\gamma^{\mathbf{d}}(s, T_{n-1})x} - 1 - \gamma^{\mathbf{d}}(s, T_{n-1})x \right) \tilde{\nu}^{T_n}(ds, dx) \end{aligned}$$

in order to ensure the martingale property. We can express  $S^{\mathbf{d}}(T_{n-1})$  as a stochastic exponential

$$S_t^{\mathbf{d}}(T_{n-1}) = S_0^{\mathbf{d}}(T_{n-1}) \mathcal{E}_t(V^{\mathbf{d}}(T_{n-1})),$$

where

$$\begin{aligned} V_t^{\mathbf{d}}(T_{n-1}) &= \int_0^t \gamma^{\mathbf{d}}(s, T_{n-1}) \sqrt{c_s} d\tilde{W}_s^{T_n} \\ &\quad + \int_0^t \int_{\mathbb{R}} \left( e^{\gamma^{\mathbf{d}}(s, T_{n-1})x} - 1 \right) (\mu^L - \tilde{\nu}^{T_n})(ds, dx). \end{aligned}$$

Hence, the dynamics of the forward swap rate under  $\tilde{P}_{T_n}$  is given by

$$\begin{aligned} dS_t^{\mathbf{d}}(T_{n-1}) &= S_t^{\mathbf{d}}(T_{n-1}) \left( \gamma^{\mathbf{d}}(t, T_{n-1}) \sqrt{c_t} d\tilde{W}_t^{T_n} \right. \\ &\quad \left. + \int_{\mathbb{R}} \left( e^{\gamma^{\mathbf{d}}(t, T_{n-1})x} - 1 \right) (\mu^L - \tilde{\nu}^{T_n})(dt, dx) \right). \end{aligned} \quad (3.4)$$

As second step of the backward induction we specify now the processes  $S(T_{n-2}, \delta)$  and  $S^{\mathbf{d}}(T_{n-2})$  as well as the forward swap measure for the date  $T_{n-1}$ . Again by definition 3.4 the forward swap rates

$$S_t(T_{n-2}, \delta) = \frac{F_t(T_{n-2}, K') + F_t(T_{n-1}, K')}{\delta B_t^{\mathbf{d}}(T_{n-1}) + \delta B_t^{\mathbf{d}}(T_n)} + K' \quad (3.5)$$

and

$$S_t^{\mathbf{d}}(T_{n-2}) = \frac{B_t^{\mathbf{d}}(T_{n-2}) - B_t^{\mathbf{d}}(T_n)}{\delta B_t^{\mathbf{d}}(T_{n-1}) + \delta B_t^{\mathbf{d}}(T_n)} \quad (3.6)$$

are local martingales under  $\tilde{P}_{T_{n-1}}$ . We write (3.5) as

$$S_t(T_{n-2}, \delta) = \frac{\frac{F_t(T_{n-1}, K')}{\delta B_t^{\mathbf{d}}(T_n)}}{\delta \frac{B_t^{\mathbf{d}}(T_{n-1})}{\delta B_t^{\mathbf{d}}(T_n)} + 1} + \frac{\frac{F_t(T_{n-2}, K')}{\delta B_t^{\mathbf{d}}(T_n)}}{\delta \frac{B_t^{\mathbf{d}}(T_{n-1})}{\delta B_t^{\mathbf{d}}(T_n)} + 1} + K'. \quad (3.7)$$

The processes  $\frac{B_t^{\mathbf{d}}(T_{n-1})}{\delta B_t^{\mathbf{d}}(T_n)}$  and  $\frac{F_t(T_l, K')}{\delta B_t^{\mathbf{d}}(T_n)}$  that appear in this expression are  $\tilde{P}_{T_n}$ -local martingales for all  $l \in \{0, \dots, n-1\}$ . Therefore, for every date  $T_j \in \{T_0, \dots, T_{n-1}\}$ , the dynamics of  $Z_1(\cdot, T_j)$  defined by

$$Z_1(t, T_j) = \frac{F_t(T_j, K')}{\delta B_t^{\mathbf{d}}(T_n)}$$



can be expressed under  $\tilde{P}_{T_n}$  in a general form

$$dZ_1(t, T_j) = Z_1(t-, T_j) \left( \varphi_1(t, T_j) d\tilde{W}_t^{T_n} + \int_{\mathbb{R}} \psi_1(t, x, T_j) (\mu^L - \tilde{\nu}^{T_n})(dt, dx) \right)$$

for some functions  $\varphi_1, \psi_1$ . In the same sense, for every date  $T_j \in \{T_0, \dots, T_n\}$ , the dynamics of  $Z_1^d(\cdot, T_j)$  defined by

$$Z_1^d(t, T_j) = \frac{B_t^d(T_j)}{\delta B_t^d(T_n)}$$

can be written under  $\tilde{P}_{T_n}$  in the form

$$dZ_1^d(t, T_j) = Z_1^d(t-, T_j) \left( \varphi_1^d(t, T_j) d\tilde{W}_t^{T_n} + \int_{\mathbb{R}} \psi_1^d(t, x, T_j) (\mu^L - \tilde{\nu}^{T_n})(dt, dx) \right) \quad (3.8)$$

for some functions  $\varphi_1^d, \psi_1^d$ .

We define now the forward swap measure  $\tilde{P}_{T_{n-1}}$  associated with date  $T_{n-1}$  by setting its Radon-Nikodym density as

$$\frac{d\tilde{P}_{T_{n-1}}}{d\tilde{P}_{T_n}} = \mathcal{E}_{T_{n-1}}(M_1^d)$$

where

$$\begin{aligned} M_1^d(t) &= \int_0^t \frac{\delta Z_1^d(s-, T_{n-1}) \varphi_1^d(s, T_{n-1})}{1 + \delta Z_1^d(s-, T_{n-1})} d\tilde{W}_s^{T_n} \\ &\quad + \int_0^t \int_{\mathbb{R}} \frac{\delta Z_1^d(s-, T_{n-1}) \psi_1^d(s, x, T_{n-1})}{1 + \delta Z_1^d(s-, T_{n-1})} (\mu^L - \tilde{\nu}^{T_n})(ds, dx). \end{aligned} \quad (3.9)$$

The specific form of  $M_1^d(t)$  can be derived as follows. Recall that  $\delta B^d(t, T_n)$  is the numeraire of  $\tilde{P}_{T_n}$  and  $\delta B^d(t, T_{n-1}) + \delta B^d(t, T_n)$  is the one of  $\tilde{P}_{T_{n-1}}$ .

Consequently, the density process of  $\frac{d\tilde{P}_{T_{n-1}}}{d\tilde{P}_{T_n}}$  is

$$\frac{d\tilde{P}_{T_{n-1}}}{d\tilde{P}_{T_n}} \Big|_{\mathcal{F}_t} = \frac{\delta B^d(t, T_{n-1}) + \delta B^d(t, T_n)}{\delta B^d(t, T_n)} \cdot C(n)$$

where

$$C(n) = \frac{\delta B^d(0, T_n)}{\delta B^d(0, T_{n-1}) + \delta B^d(0, T_n)}$$

is chosen such that this process has initial value 1. From the definition of  $Z_1^d(t, T_{n-1})$  we see that

$$U_t(n) = \frac{d\tilde{P}_{T_{n-1}}}{d\tilde{P}_{T_n}} \Big|_{\mathcal{F}_t} = \left( \delta Z_1^d(t, T_{n-1}) + 1 \right) \cdot C(n).$$

As this process is strictly positive it can be represented as a stochastic exponential  $\mathcal{E}_t(Y)$  where  $Y$  is explicitly given by the stochastic logarithm (see [Kallsen and Shiryaev \(2002, Lemma 2.2\)](#) or [Eberlein and Kallsen \(2019, Section 3.6\)](#))

$$Y_t = \int_0^t \frac{dU_s(n)}{U_{s-}(n)}.$$

This means that

$$Y_t = \int_0^t \frac{\delta dZ_1^d(s, T_{n-1})}{\delta Z_1^d(s-, T_{n-1}) + 1}$$

which coincides with [\(3.9\)](#).

As a consequence of Girsanov's Theorem for semimartingales ([Eberlein and Kallsen \(2019, Propositions 3.70 and 3.73\)](#)) the Brownian motion for the date  $T_{n-1}$  is

$$\tilde{W}_t^{T_{n-1}} = \tilde{W}_t^{T_n} - \int_0^t \frac{\delta Z_1^d(s-, T_{n-1}) \varphi_1^d(s, T_{n-1})}{1 + \delta Z_1^d(s-, T_{n-1})} ds$$

and the  $\tilde{P}_{T_{n-1}}$ -compensator of  $\mu^L$  is

$$\tilde{\nu}^{T_{n-1}}(dt, dx) = \left( 1 + \frac{\delta Z_1^d(t-, T_{n-1}) \psi_1^d(t, x, T_{n-1})}{1 + \delta Z_1^d(t-, T_{n-1})} \right) \tilde{\nu}^{T_n}(dt, dx).$$

Next we will design the dynamics of  $S_t(T_{n-2}, \delta)$  and  $S_t^d(T_{n-2})$  under the swap measure  $\tilde{P}_{T_{n-1}}$  as local martingales. We postulate

$$S_t(T_{n-2}, \delta) = S_0(T_{n-2}, \delta) \exp \left( \int_0^t \alpha(s, T_{n-2}, T_{n-1}) ds + \int_0^t \gamma(s, T_{n-2}) d\tilde{L}_s^{T_{n-1}} \right)$$

and

$$S_t^d(T_{n-2}) = S_0^d(T_{n-2}) \exp \left( \int_0^t \alpha^d(s, T_{n-2}, T_{n-1}) ds + \int_0^t \gamma^d(s, T_{n-2}) d\tilde{L}_s^{T_{n-1}} \right)$$

where

$$\tilde{L}_t^{T_{n-1}} = \int_0^t \sqrt{c_s} d\tilde{W}_s^{T_{n-1}} + \int_0^t \int_{\mathbb{R}} x (\mu^L - \tilde{\nu}^{T_{n-1}})(ds, dx)$$

and the initial values are given by

$$S_0(T_{n-2}, \delta) = \frac{\delta B_0^d(T_n) L(0, T_{n-1}, T_n) + \delta B_0^d(T_{n-1}) L(0, T_{n-2}, T_{n-1})}{\delta B_0^d(T_n) + \delta B_0^d(T_{n-1})}$$

and

$$S_0^d(T_{n-2}) = \frac{B_0^d(T_{n-2}) - B_0^d(T_n)}{\delta B_0^d(T_{n-1}) + \delta B_0^d(T_n)}.$$

We choose the drift terms as

$$\begin{aligned} \int_0^t \alpha(s, T_{n-2}, T_{n-1}) ds &= -\frac{1}{2} \int_0^t \gamma(s, T_{n-2}) c_s \gamma(s, T_{n-2}) ds \\ &\quad - \int_0^t \int_{\mathbb{R}} \left( e^{\gamma(s, T_{n-2})x} - 1 - \gamma(s, T_{n-2})x \right) \tilde{\nu}^{T_{n-1}}(ds, dx) \end{aligned}$$

and

$$\begin{aligned} \int_0^t \alpha^d(s, T_{n-2}, T_{n-1}) ds &= -\frac{1}{2} \int_0^t \gamma^d(s, T_{n-2}) c_s \gamma^d(s, T_{n-2}) ds \\ &\quad - \int_0^t \int_{\mathbb{R}} \left( e^{\gamma^d(s, T_{n-2})x} - 1 - \gamma^d(s, T_{n-2})x \right) \tilde{\nu}^{T_{n-1}}(ds, dx) \end{aligned}$$

and can express  $S^d(T_{n-2})$  as a stochastic exponential

$$S_t^d(T_{n-2}) = S_0^d(T_{n-2}) \mathcal{E}_t(V^d(T_{n-2}))$$

where

$$\begin{aligned} V_t^d(T_{n-2}) &= \int_0^t \gamma^d(s, T_{n-2}) \sqrt{c_s} d\tilde{W}_s^{T_{n-1}} \\ &\quad + \int_0^t \int_{\mathbb{R}} \left( e^{\gamma^d(s, T_{n-2})x} - 1 \right) (\mu^L - \tilde{\nu}^{T_{n-1}})(ds, dx). \end{aligned}$$

The dynamics under  $\tilde{P}_{T_{n-1}}$  is then given by

$$\begin{aligned} dS_t^d(T_{n-2}) &= S_{t-}^d(T_{n-2}) \left( \gamma^d(t, T_{n-2}) \sqrt{c_t} d\tilde{W}_t^{T_{n-1}} \right. \\ &\quad \left. + \int_{\mathbb{R}} \left( e^{\gamma^d(t, T_{n-2})x} - 1 \right) (\mu^L - \tilde{\nu}^{T_{n-1}})(dt, dx) \right). \end{aligned} \quad (3.10)$$

In order to specify the coefficients of the process  $Z_1^d(\cdot, T_{n-1})$  in (3.8) we note that

$$Z_1^d(t, T_{n-1}) = S_t^d(T_{n-1}) + Z_1^d(t, T_n) = S_t^d(T_{n-1}) + \frac{1}{\delta}. \quad (3.11)$$

Then by using (3.4) and (3.8), we obtain

$$\begin{aligned} Z_1^d(t-, T_{n-1}) &\left( \varphi_1^d(t, T_{n-1}) d\tilde{W}_t^{T_n} + \int_{\mathbb{R}} \psi_1^d(t, x, T_{n-1}) (\mu^L - \tilde{\nu}^{T_n})(dt, dx) \right) \\ &= S_{t-}^d(T_{n-1}) \left( \gamma^d(t, T_{n-1}) \sqrt{c_t} d\tilde{W}_t^{T_n} + \int_{\mathbb{R}} \left( e^{\gamma^d(t, T_{n-1})x} - 1 \right) (\mu^L - \tilde{\nu}^{T_n})(dt, dx) \right). \end{aligned}$$

By [Jacod and Shiryaev \(2003, Theorem II.2.34 and Ch. I.4b\)](#) we have

$$Z_1^d(t-, T_{n-1}) \varphi_1^d(t, T_{n-1}) = S_{t-}^d(T_{n-1}) \gamma^d(t, T_{n-1}) \sqrt{c_t} \quad (3.12)$$

$$Z_1^d(t-, T_{n-1}) \psi_1^d(t, x, T_{n-1}) = S_{t-}^d(T_{n-1}) \left( e^{\gamma^d(t, T_{n-1})x} - 1 \right). \quad (3.13)$$

For the next step of the backward induction we consider the forward swap rates  $S(T_{n-3}, \delta)$  and  $S^d(T_{n-3}, \delta)$  which are given in the form

$$S_t(T_{n-3}, \delta) = \frac{F_t(T_{n-3}, K') + F_t(T_{n-2}, K') + F_t(T_{n-1}, K')}{\delta B_t^d(T_{n-2}) + \delta B_t^d(T_{n-1}) + \delta B_t^d(T_n)} + K' \quad (3.14)$$

and

$$S_t^d(T_{n-3}) = \frac{B_t^d(T_{n-3}) - B_t^d(T_n)}{\delta B_t^d(T_{n-2}) + \delta B_t^d(T_{n-1}) + \delta B_t^d(T_n)}.$$

Both are local martingales under the swap measure  $\tilde{P}_{T_{n-2}}$ . We formulate the expression (3.14) as

$$\begin{aligned} S_t(T_{n-3}, \delta) &= \frac{\frac{F_t(T_{n-1}, K')}{\delta B_t^d(T_{n-1}) + \delta B_t^d(T_n)}}{\delta \frac{B_t^d(T_{n-2})}{\delta B_t^d(T_{n-1}) + \delta B_t^d(T_n)} + 1} + \frac{\frac{F_t(T_{n-2}, K')}{\delta B_t^d(T_{n-1}) + \delta B_t^d(T_n)}}{\delta \frac{B_t^d(T_{n-2})}{\delta B_t^d(T_{n-1}) + \delta B_t^d(T_n)} + 1} \\ &\quad + \frac{\frac{F_t(T_{n-3}, K')}{\delta B_t^d(T_{n-1}) + \delta B_t^d(T_n)}}{\delta \frac{B_t^d(T_{n-2})}{\delta B_t^d(T_{n-1}) + \delta B_t^d(T_n)} + 1} + K'. \end{aligned} \quad (3.15)$$

The dynamics of the local  $\tilde{P}_{T_{n-1}}$ -martingales

$$Z_2(t, T_j) = \frac{F_t(T_j, K')}{\delta B_t^d(T_{n-1}) + \delta B_t^d(T_n)}$$

and

$$Z_2^d(t, T_j) = \frac{B_t^d(T_j)}{\delta B_t^d(T_{n-1}) + \delta B_t^d(T_n)}$$

which appear in (3.15) can be written in the general form

$$dZ_2(t, T_j) = Z_2(t-, T_j) \left( \varphi_2(t, T_j) d\tilde{W}_t^{T_{n-1}} + \int_{\mathbb{R}} \psi_2(t, x, T_j) (\mu^L - \tilde{\nu}^{T_{n-1}})(dt, dx) \right) \quad (3.16)$$

and

$$dZ_2^d(t, T_j) = Z_2^d(t-, T_j) \left( \varphi_2^d(t, T_j) d\tilde{W}_t^{T_{n-1}} + \int_{\mathbb{R}} \psi_2^d(t, x, T_j) (\mu^L - \tilde{\nu}^{T_{n-1}})(dt, dx) \right) \quad (3.17)$$

for some functions  $\varphi_2^d, \psi_2^d, \varphi_2, \psi_2$ . Analogous to (3.9) we define the forward swap measure associated with date  $T_{n-2}$  by setting its Radon-Nikodym density as

$$\frac{d\tilde{P}_{T_{n-2}}}{d\tilde{P}_{T_{n-1}}} = \mathcal{E}_{T_{n-2}}(M_2^d)$$

where

$$M_2^d(t) = \int_0^t \frac{\delta Z_2^d(s-, T_{n-2}) \varphi_2^d(s, T_{n-2})}{1 + \delta Z_2^d(s-, T_{n-2})} d\tilde{W}_s^{T_{n-1}} \\ + \int_0^t \int_{\mathbb{R}} \frac{\delta Z_2^d(s-, T_{n-2}) \psi_2^d(s, x, T_{n-2})}{1 + \delta Z_2^d(s-, T_{n-2})} (\mu^L - \tilde{\nu}^{T_{n-1}})(ds, dx).$$

Then the Brownian motion for the date  $T_{n-2}$  is

$$\tilde{W}_t^{T_{n-2}} = \tilde{W}_t^{T_{n-1}} - \int_0^t \frac{\delta Z_2^d(s-, T_{n-2}) \varphi_2^d(s, T_{n-2})}{1 + \delta Z_2^d(s-, T_{n-2})} ds$$

and the  $\tilde{P}_{T_{n-2}}$ -compensator of  $\mu^L$  is

$$\tilde{\nu}^{T_{n-2}}(dt, dx) = \left( 1 + \frac{\delta Z_2^d(t-, T_{n-2}) \psi_2^d(t, x, T_{n-2})}{1 + \delta Z_2^d(t-, T_{n-2})} \right) \tilde{\nu}^{T_{n-1}}(dt, dx).$$

The swap rates  $S_t(T_{n-3}, \delta)$  and  $S_t^d(T_{n-3})$  are modelled in an analogous way as before.

In order to obtain the coefficients in (3.17) we consider the relation

$$\begin{aligned} Z_2^d(t, T_{n-2}) &= S_t^d(T_{n-2}) + Z_2^d(t, T_n) \\ &= S_t^d(T_{n-2}) + \frac{Z_1^d(t, T_n)}{\delta Z_1^d(t, T_{n-1}) + 1} \\ &= S_t^d(T_{n-2}) + \frac{Z_1^d(t, T_n)}{\delta S_t^d(T_{n-1}) + \delta Z_1^d(t, T_n) + 1}. \end{aligned} \quad (3.18)$$

First we apply [Eberlein and Liinev \(2007, Lemma A.1.\)](#) to the second process on the right hand side of equation (3.18). In that Lemma we set  $G = Z_1^d(\cdot, T_n) = \frac{1}{\delta}$  and  $H = \delta Z_1^d(\cdot, T_{n-1})$ . Note that the processes  $Z_1^d(\cdot, T_n)$  and  $Z_1^d(\cdot, T_{n-1})$  are completely characterised because all coefficients are specified in the previous step (in particular see the relations (3.12) and (3.13)). We observe that the related probability measure as well as the corresponding process and the compensator derived in the lemma are  $\tilde{P}_{T_{n-1}}$ ,  $\tilde{W}^{T_{n-1}}$  and  $\tilde{\nu}^{T_{n-1}}$ . Then we obtain

$$\begin{aligned} d \left( \frac{Z_1^d(t, T_n)}{\delta Z_1^d(t, T_{n-1}) + 1} \right) &= - \frac{S_{t-}^d(T_{n-1}) \gamma^d(t, T_{n-1}) \sqrt{c_t}}{(1 + \delta Z_1^d(t-, T_{n-1}))^2} d\tilde{W}_t^{T_{n-1}} \\ &\quad + \int_{\mathbb{R}} \left[ \frac{\frac{1}{\delta}}{\delta(Z_1^d(t-, T_{n-1}) + S_{t-}^d(T_{n-1})(e^{\gamma^d(t, T_{n-1})x} - 1)) + 1} \right. \\ &\quad \left. - \frac{\frac{1}{\delta}}{\delta Z_1^d(t-, T_{n-1}) + 1} \right] (\mu^L - \tilde{\nu}^{T_{n-1}})(dt, dx). \end{aligned}$$

By using relations (3.10), (3.17) and (3.18) as well as Jacod and Shiryaev (2003, Theorem II.2.34 and Chapter I.4b), we obtain

$$\begin{aligned} Z_2^d(t-, T_{n-2})\varphi_2^d(t, T_{n-2}) &= S_{t-}^d(T_{n-2})\gamma^d(t, T_{n-2})\sqrt{c_t} - \frac{S_{t-}^d(T_{n-1})\gamma^d(t, T_{n-1})\sqrt{c_t}}{(1 + \delta Z_1^d(t-, T_{n-1}))^2} \\ Z_2^d(t-, T_{n-2})\psi_2^d(t, x, T_{n-2}) &= S_{t-}^d(T_{n-2}) \left( e^{\gamma^d(t, T_{n-2})x} - 1 \right) \\ &+ \frac{\frac{1}{\delta}}{\delta(Z_1^d(t-, T_{n-1}) + S_{t-}^d(T_{n-1})(e^{\gamma^d(t, T_{n-1})x} - 1)) + 1} - \frac{\frac{1}{\delta}}{\delta Z_1^d(t-, T_{n-1}) + 1}. \end{aligned}$$

Now we consider the general step of the backward procedure. For  $m \in \{2, \dots, n-1\}$  the forward swap rates  $S_t(T_{n-1}, \delta), \dots, S_t(T_{n-m}, \delta)$  and  $S_t^d(T_{n-1}), \dots, S_t^d(T_{n-m})$  as well as the forward swap measure  $\tilde{P}_{T_{n-(m-1)}}$  are given. In particular the dynamics of  $S_t^d(T_{n-m})$  is

$$\begin{aligned} dS_t^d(T_{n-m}) &= S_{t-}^d(T_{n-m}) \left( \gamma^d(t, T_{n-m})\sqrt{c_t}d\tilde{W}_t^{T_{n-(m-1)}} \right. \\ &\quad \left. + \int_{\mathbb{R}} \left( e^{\gamma^d(t, T_{n-m})x} - 1 \right) (\mu^L - \tilde{\nu}^{T_{n-(m-1)}})(dt, dx) \right). \end{aligned}$$

We will consider

$$S_t^d(T_{n-(m+1)}) = \frac{B_t^d(T_{n-(m+1)}) - B_t^d(T_n)}{\sum_{j=0}^m \delta B_t^d(T_{n-j})}$$

and

$$S_t(T_{n-(m+1)}, \delta) = \sum_{l=1}^{m+1} \frac{\frac{F_t(T_{n-l}, K')}{\sum_{j=0}^{m-1} \delta B_t^d(T_{n-j})}}{\delta \frac{B_t^d(T_{n-m})}{\sum_{j=0}^{m-1} \delta B_t^d(T_{n-j})} + 1} + K'. \quad (3.19)$$

and the local  $\tilde{P}_{T_{n-(m-1)}}$ -martingales

$$Z_m(t, T_j) = \frac{F_t(T_j, K')}{\sum_{j=0}^{m-1} \delta B_t^d(T_{n-j})}$$

and

$$Z_m^d(t, T_j) = \frac{B_t^d(T_j)}{\sum_{j=0}^{m-1} \delta B_t^d(T_{n-j})}.$$

The dynamics are given by

$$\begin{aligned} dZ_m(t, T_j) &= Z_m(t-, T_j) \left( \varphi_m(t, T_j)d\tilde{W}_t^{T_{n-(m-1)}} \right. \\ &\quad \left. + \int_{\mathbb{R}} \psi_m(t, x, T_j)(\mu^L - \tilde{\nu}^{T_{n-(m-1)}})(dt, dx) \right) \end{aligned}$$

and

$$\begin{aligned} dZ_m^d(t, T_j) &= Z_m^d(t-, T_j) \left( \varphi_m^d(t, T_j)d\tilde{W}_t^{T_{n-(m-1)}} \right. \\ &\quad \left. + \int_{\mathbb{R}} \psi_m^d(t, x, T_j)(\mu^L - \tilde{\nu}^{T_{n-(m-1)}})(dt, dx) \right) \end{aligned}$$

for some functions  $\varphi_m^d, \psi_m^d, \varphi_m, \psi_m$ .

The forward swap measure associated with date  $T_{n-m}$  is given by its Radon-Nikodym density

$$\frac{d\tilde{P}_{T_{n-m}}}{d\tilde{P}_{T_{n-(m-1)}}} = \mathcal{E}_{T_{n-m}}(M_m^d)$$

where

$$\begin{aligned} M_m^d(t) &= \int_0^t \frac{\delta Z_m^d(s-, T_{n-m}) \varphi_m^d(s, T_{n-m})}{1 + \delta Z_m^d(s-, T_{n-m})} d\tilde{W}_s^{T_{n-(m-1)}} \\ &\quad + \int_0^t \int_{\mathbb{R}} \frac{\delta Z_m^d(s-, T_{n-m}) \psi_m^d(s, x, T_{n-m})}{1 + \delta Z_m^d(s-, T_{n-m})} (\mu^L - \tilde{\nu}^{T_{n-(m-1)}})(ds, dx). \end{aligned}$$

The  $\tilde{P}_{T_{n-m}}$ -Brownian motion is

$$\tilde{W}_t^{T_{n-m}} = \tilde{W}_t^{T_{n-(m-1)}} - \int_0^t \frac{\delta Z_m^d(s-, T_{n-m}) \varphi_m^d(s, T_{n-m})}{1 + \delta Z_m^d(s-, T_{n-m})} ds \quad (3.20)$$

and the corresponding compensator is

$$\tilde{\nu}^{T_{n-m}}(dt, dx) = \left( 1 + \frac{\delta Z_m^d(t-, T_{n-m}) \psi_m^d(t, x, T_{n-m})}{1 + \delta Z_m^d(t-, T_{n-m})} \right) \tilde{\nu}^{T_{n-(m-1)}}(dt, dx). \quad (3.21)$$

The forward swap rates  $S_t(T_{n-(m+1)}, \delta)$  and  $S_t^d(T_{n-(m+1)})$  are modelled as

$$\begin{aligned} S_t(T_{n-(m+1)}, \delta) &= S_0(T_{n-(m+1)}, \delta) \exp \left( \int_0^t \alpha(s, T_{n-(m+1)}, T_{n-m}) ds \right. \\ &\quad \left. + \int_0^t \gamma(s, T_{n-(m+1)}) d\tilde{L}_s^{T_{n-m}} \right) \end{aligned}$$

and

$$\begin{aligned} S_t^d(T_{n-(m+1)}) &= S_0^d(T_{n-(m+1)}) \exp \left( \int_0^t \alpha^d(s, T_{n-(m+1)}, T_{n-m}) ds \right. \\ &\quad \left. + \int_0^t \gamma^d(s, T_{n-(m+1)}) d\tilde{L}_s^{T_{n-m}} \right) \end{aligned}$$

where

$$\tilde{L}_t^{T_{n-m}} = \int_0^t \sqrt{c_s} d\tilde{W}_s^{T_{n-m}} + \int_0^t \int_{\mathbb{R}} x (\mu^L - \tilde{\nu}^{T_{n-m}})(ds, dx).$$

We choose the drift terms as

$$\begin{aligned} \int_0^t \alpha(s, T_{n-(m+1)}, T_{n-m}) ds &= -\frac{1}{2} \int_0^t \gamma(s, T_{n-(m+1)}) c_s \gamma(s, T_{n-(m+1)}) ds \\ &- \int_0^t \int_{\mathbb{R}} \left( e^{\gamma(s, T_{n-(m+1)})x} - 1 - \gamma(s, T_{n-(m+1)})x \right) \tilde{\nu}^{T_{n-m}}(ds, dx) \end{aligned} \quad (3.22)$$

and

$$\begin{aligned} \int_0^t \alpha^d(s, T_{n-(m+1)}, T_{n-m}) ds &= -\frac{1}{2} \int_0^t \gamma^d(s, T_{n-(m+1)}) c_s \gamma^d(s, T_{n-(m+1)}) ds \\ &- \int_0^t \int_{\mathbb{R}} \left( e^{\gamma^d(s, T_{n-(m+1)})x} - 1 - \gamma^d(s, T_{n-(m+1)})x \right) \tilde{\nu}^{T_{n-m}}(ds, dx). \end{aligned}$$

Then we derive the coefficients of the process  $Z_m^d(\cdot, T_{n-m})$  by considering the relation

$$Z_m^d(t, T_j) = \frac{Z_{m-1}^d(t, T_j)}{\delta Z_{m-1}^d(t, T_{n-(m-1)}) + 1}$$

and observing that

$$\begin{aligned} Z_m^d(t, T_{n-m}) &= S_t^d(T_{n-m}) + Z_m^d(t, T_n) \\ &= S_t^d(T_{n-m}) + \frac{Z_{m-1}^d(t, T_n)}{\delta Z_{m-1}^d(t, T_{n-(m-1)}) + 1}, \end{aligned}$$

where the processes  $S^d(T_{n-m})$ ,  $Z_{m-1}^d(\cdot, T_{n-(m-1)})$  and  $Z_{m-1}^d(\cdot, T_n)$  and therefore their coefficients are completely specified in the previous step. By applying [Eberlein and Liinev \(2007, Lemma A.1.\)](#) we obtain the relation

$$\begin{aligned} Z_m^d(t-, T_{n-m}) \varphi_m^d(t, T_{n-m}) &= S_{t-}^d(T_{n-m}) \gamma^d(t, T_{n-m}) \sqrt{c_t} + \frac{Z_{m-1}^d(t-, T_n) \varphi_{m-1}^d(t, T_n)}{\delta Z_{m-1}^d(t, T_{n-(m-1)}) + 1} \\ &- \frac{\delta Z_{m-1}^d(t-, T_{n-(m-1)}) \varphi_{m-1}^d(t, T_{n-(m-1)}) Z_{m-1}^d(t-, T_n)}{(\delta Z_{m-1}^d(t, T_{n-(m-1)}) + 1)^2} \end{aligned}$$

and

$$\begin{aligned} Z_m^d(t-, T_{n-m}) \psi_m^d(t, x, T_{n-m}) &= S_{t-}^d(T_{n-m}) \left( e^{\gamma^d(t, T_{n-m})x} - 1 \right) \\ &+ \frac{Z_{m-1}^d(t-, T_n) (1 + \psi_{m-1}^d(t, x, T_n))}{\delta Z_{m-1}^d(t-, T_{n-(m-1)}) (1 + \psi_{m-1}^d(t, x, T_{n-(m-1)})) + 1} \\ &- \frac{Z_{m-1}^d(t-, T_n)}{\delta Z_{m-1}^d(t-, T_{n-(m-1)}) + 1}. \end{aligned} \quad (3.23)$$

Hence the coefficient functions can be deduced successively.



## 4. VALUATION OF SWAPTIONS

We denote by  $\mathcal{T}_i^\delta$  for  $i \in \{0, \dots, n-1\}$  the subset of tenor time points  $\mathcal{T}_i^\delta = \{T_i, \dots, T_n\} \subset \mathcal{T}^\delta$ . The value of a swaption at date  $t \in [0, T_i)$  is given by

$$\text{Swpt}(t, \mathcal{T}_i^\delta, \kappa) = D_{i,n}(t) \cdot \mathbb{E}_{\tilde{P}_{T_{i+1}}} \left[ (S_{T_i}(T_i, \delta) - \kappa)^+ | \mathcal{F}_t \right] \quad (4.1)$$

where  $D_{i,n}(t) = \sum_{j=i+1}^n B_t^d(T_j)\delta$  and

$$S_t(T_i, \delta) = S_0(T_i, \delta) \exp \left( \int_0^t \alpha(s, T_i, T_{i+1}) ds + \int_0^t \gamma(s, T_i) d\tilde{L}_s^{T_{i+1}} \right)$$

(see [Grbac and Runggaldier \(2015, section 1.4.7\)](#)). The expectation in (4.1) can be efficiently approximated by using a Fourier based approach. For this approach two ingredients are necessary: the extended moment generating or characteristic function of the properly transformed stochastic integral and the Fourier transform of the payoff function. The latter is derived by elementary integration whereas for the former a suitable formula (see e.g. [Eberlein and Kallsen \(2019, Proposition 3.54\)](#)) is only available when the integrator of the stochastic integral has independent increments. As the random terms in (3.20) and (3.21) show, independence of increments of the initial driving process  $\tilde{L}^{T_n}$  is lost during the backward induction. It can be re-established by freezing the random terms by their starting values. As we will use purely discontinuous Lévy processes  $\tilde{L}^{T_n}$  in the implementation later, we consider here only the part which is relevant in this case, namely the compensator. By freezing the random coefficients in (3.21) in each induction step we obtain recursively the approximation

$$\begin{aligned} \tilde{\nu}^{T_{i+1}}(dt, dx) &\approx \prod_{j=i+1}^{n-1} \left( 1 + \frac{\delta Z_{n-j}^d(0, T_j) \psi_{n-j}^d(0, x, T_j)}{1 + \delta Z_{n-j}^d(0, T_j)} \right) \tilde{\nu}^{T_n}(dt, dx) \\ &=: \hat{\nu}^{T_{i+1}}(dt, dx). \end{aligned} \quad (4.2)$$

The explicit form of any term  $Z_{n-j}^d(0, T_j) \psi_{n-j}^d(0, x, T_j)$  can be deduced successively by using the relationship (3.23). Define

$$X_t^i = \int_0^t \gamma(s, T_i) d\tilde{L}_s^{T_{i+1}}.$$

The extended moment generating function of  $X_t^i$  (where it exists) can then be approximated by

$$\begin{aligned} m_{X_t^i}(z) &= \mathbb{E}_{\tilde{P}_{T_{i+1}}} \left[ e^{z X_t^i} \right] \\ &\approx \exp \left( \int_0^t \int_{\mathbb{R}} \left[ e^{z \gamma(s, T_i) x} - 1 - z \gamma(s, T_i) x \right] \hat{\nu}^{T_{i+1}}(ds, dx) \right) \\ &=: \hat{m}_{X_t^i}(z) \end{aligned}$$

for  $z \in \mathbb{C}$ .

For any payoff function  $f$  let  $g$  denote the dampened payoff function defined by

$$g(x) = e^{-Rx} f(x)$$

for some  $R \in \mathbb{R}$ . We denote the extended Fourier transform of a function  $g$  by  $\bar{g}$ . In order to derive the pricing formula for swaptions we consider the payoff function  $f(x) = (e^x - \kappa)^+$  with strike  $\kappa > 0$ . Let  $z \in \mathbb{C}$  with  $\text{Im}(z) > 1$ , then the Fourier transform of the payoff function is

$$\bar{f}(z) = \frac{\kappa^{1+iz}}{iz(1+iz)}.$$

We easily obtain for  $R > 1$  that  $g \in L^1_{\text{bc}}(\mathbb{R}) \cap L^2(\mathbb{R})$ . The weak derivative of  $g$  is given by

$$\partial g(x) = \begin{cases} e^{-Rx}(e^x - Re^x + R\kappa), & \text{if } x > \ln \kappa \\ 0, & \text{if } x < \ln \kappa. \end{cases}$$

As  $\partial g \in L^2(\mathbb{R})$  we have  $g \in H^1(\mathbb{R})$  such that we can conclude that  $\bar{g} \in L^1(\mathbb{R})$ . Consequently conditions (C1) and (C3) of Theorem 2.2 in [Eberlein, Glau, and Papantoleon \(2010\)](#) are satisfied.

Hence, according to this theorem the time-0 value of the swaption can be approximated by

$$\text{Swpt}(0, \mathcal{F}_i^\delta, \kappa) \approx D_{i,n}(0) \frac{e^{-Rd_i}}{\pi} \int_0^\infty \text{Re} \left( e^{-iud_i} \hat{m}_{X_{T_i}^i}(R+iu) \bar{f}(iR-u) \right) du \quad (4.3)$$

where  $d_i = -\ln S_0(T_i, \delta) - \int_0^{T_i} \hat{\alpha}(s, T_i, T_{i+1}) ds$  and  $\hat{\alpha}(s, T_i, T_{i+1})$  is derived from [\(3.22\)](#) by replacing the compensator in the same way as in [\(4.2\)](#).

## 5. MODEL CALIBRATION

**5.1. Initial Curves.** In order to derive the initial discount curve  $B_0^d(\cdot)$  we follow the bootstrapping method given in [Gerhart and Lütkebohmert \(2018\)](#) (see also [Henrard \(2014\)](#) for other bootstrapping approaches). More specifically, we use the market quotes of overnight indexed swap rates that are provided for increasing maturities. Then we successively derive the bond prices  $B_0^d(\cdot)$  from the representation [\(3.1\)](#). Of course, the quantities  $Z_m^d(0, \cdot)$  are then implicitly given by the bootstrapped discount curve. Other model inputs are the current values of the tenor-dependent swap rates  $S_0(T_i, \delta)$ . Unfortunately these are not quoted for the required maturities and, therefore, have to be bootstrapped (see also [Gerhart and Lütkebohmert \(2018\)](#)). We obtain the missing values by successively using the relation [\(3.2\)](#). Note that the discount curve  $B_0^d(\cdot)$  constructed above is used in this step.

The bootstrapping approach is paired with cubic splines interpolation. This guarantees enough smoothness of the curves. Furthermore, the bootstrapped initial values for tradable assets are a priori arbitrage free as they are derived from risk neutral pricing formulas. The monotonicity resulting from the interpolation via cubic splines guarantees arbitrage free values between the considered maturities.

**5.2. Model Specification.** We specify the driving process  $\tilde{L}^{T_n}$  under  $\tilde{P}_{T_n}$  as a generalized hyperbolic Lévy process with parameters  $\alpha, \beta, \delta, \lambda$  and  $\mu$  (for a detailed treatment see for example [Eberlein \(2009\)](#) or [Eberlein and Kallsen \(2019, section 2.4.9\)](#)) which have to satisfy  $0 \leq |\beta| < \alpha, \delta > 0, \lambda \in \mathbb{R}$  and  $\mu \in \mathbb{R}$ . The last parameter  $\mu$  does not enter into the valuation formulas and is chosen such that the expectation of  $\tilde{L}_1^{T_n}$  is equal to zero. The extended moment generating function  $m_{GH}(u)$  is of the form

$$m_{GH}(z) = e^{\mu z} \left( \frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + z)^2} \right)^{\frac{\lambda}{2}} \frac{K_\lambda(\delta \sqrt{\alpha^2 - (\beta + z)^2})}{K_\lambda(\delta \sqrt{\alpha^2 - \beta^2})}$$

where  $\text{Re}(z) \in (-\alpha - \beta, \alpha - \beta)$  and  $K_a$  denotes the modified Bessel function of the third kind with index  $a$ . We emphasize that only parameters which lead to a Lévy measure  $\tilde{F}^{T_n}$  that satisfies Assumption (EM) are admissible. Generalized hyperbolic Lévy processes are purely discontinuous processes and the density of the Lévy measure  $\tilde{F}^{T_n}$  is of the form

$$g_{GH}(x) = \frac{e^{\beta x}}{|x|} \left( \int_0^\infty \frac{e^{-\sqrt{2y+\alpha^2}|x|}}{\pi^2 y (J_{|\lambda|}^2(\delta \sqrt{2y}) + Y_{|\lambda|}^2(\delta \sqrt{2y}))} dy + \mathbf{1}_{\{\lambda \geq 0\}} \lambda e^{-\alpha|x|} \right)$$

where  $J_a$  and  $Y_a$  are the modified Bessel functions of the first and second kind with index  $a$ . Recall that  $\tilde{\nu}^{T_n}(dt, dx) = \tilde{F}^{T_n}(dx)dt$ . The explicit form of  $\hat{\nu}^{T_{i+1}}(dt, dx)$  can then be derived by using (4.2) together with the relation (3.23). From this we obtain the function  $\hat{\alpha}(s, T_i, T_{i+1})$ .

The volatility structure is chosen to be  $\gamma^d(t, T) = \sigma_d e^{-a_d(T-t)}$  and  $\gamma(t, T) = \sigma_k e^{-a_k(T-t)}$  for  $t \leq T$ . The parameters are restricted to an admissible set that guarantees that the volatility structures are bounded (cf. assumption (VOL)).

**5.3. Calibration.** A swaption contract is specified by its strike rate  $\kappa$  and expiry date  $T_i$  with corresponding tenor structure  $\mathcal{T}^i = \{T_i, \dots, T_n\}$  based on some tenor  $\delta = T_{j+1} - T_j$  and maturity  $T_n$ . For the calibration, we use market quotes of swaptions for a number of strike rates and expiry dates on August 8, 2017. More specifically, we consider various swaption contracts for the set of expiry dates  $\mathcal{T} = \{4\text{yrs}, 5\text{yrs}, 6\text{yrs}, 7\text{yrs}\}$  with corresponding strike rates  $\mathcal{K}_{4\text{yrs}} = \{0.016457, 0.018957, 0.021457\}$ ,  $\mathcal{K}_{5\text{yrs}} = \{0.017717, 0.020217, 0.022717\}$ ,  $\mathcal{K}_{6\text{yrs}} = \{0.018947, 0.021447, 0.023947\}$  and  $\mathcal{K}_{7\text{yrs}} = \{0.020088, 0.022588, 0.025088\}$ . The tenor length is six months and we have a fixed time horizon  $T^* = T_n = 10\text{yrs}$ . The swaption market quotes are provided by Bloomberg and are given in form of the implied volatilities (in bps) calculated based on the Bachelier model using OIS discounting. From these quotes we can derive the market prices of the swaptions by using the Bachelier pricing formula.

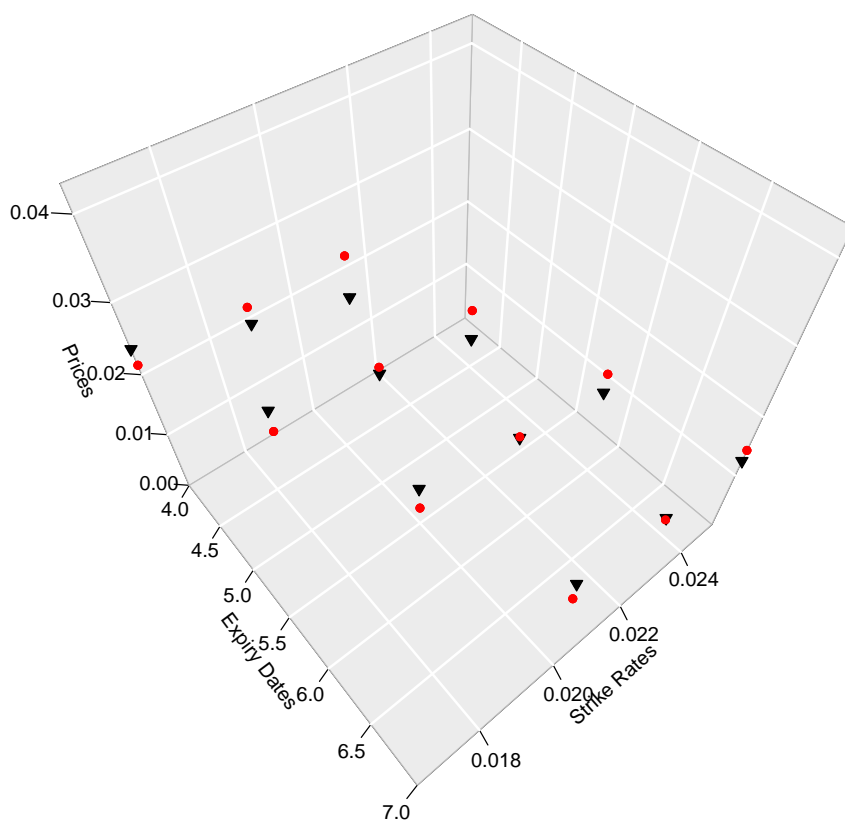
In the calibration procedure we derive swaption model prices by using formula (4.3). Let  $\Theta$  be the set of admissible model parameters. We minimise the sum of the squared relative errors between model and market swaption prices

$$\sum_{T \in \mathcal{T}, K \in \mathcal{K}} \left( \frac{\text{swaption model price}(\vartheta, T, K) - \text{swaption market price}(T, K)}{\text{swaption market price}(T, K)} \right)^2$$

with respect to  $\vartheta \in \Theta$  where we set  $\mathcal{K} = \mathcal{K}_{4\text{yrs}} \cup \mathcal{K}_{5\text{yrs}} \cup \mathcal{K}_{6\text{yrs}} \cup \mathcal{K}_{7\text{yrs}}$ . This optimisation is done by using a randomised Powell algorithm (see [Powell \(1978\)](#)).

The differences between the model and market prices specify the accuracy of the calibration.

FIGURE 1. Calibration results of model prices (points) to market prices (triangles) on August 8, 2017.



5.4. **Results.** In Figure 1 we display the market prices as triangles and the corresponding calibrated model prices as points. The calibrated model parameters are given in Table 1. As we can see the model is able to fit the market prices with adequate accuracy.

TABLE 1. Calibrated model parameters on August 8, 2017.

$\alpha$	$\beta$	$\delta$	$\lambda$
6.4471	2.0896	4.0572	-1.0595
$a_d$	$\sigma_d$	$a_k$	$\sigma_k$
0.5292	-2.2289	0.6495	2.8566

## 6. CONCLUSION

In developing models for interest rate markets one has considerable freedom to choose the basic quantities whose dynamics are specified. The approach in this paper is based on modeling the forward swap rates. This choice is natural in terms of the availability of reliable market data. Another important reason for this choice is a convenient valuation formula for swaptions which can then be used for calibration purposes.

We are convinced that the results obtained will remain relevant also after a discontinuation of LIBOR after 2021. The replacement of LIBOR by alternative reference rates such as secured overnight rates has been intensively discussed in the past years. While these rates are less exposed to manipulations, they do not represent the costs of unsecured borrowing over a fixed term. The pricing of various interest rate derivative contracts, however, relies on such forward looking rates. The corresponding term and spread adjustments to secured overnight rates are currently being discussed (see e.g. [International Swaps and Derivatives Association \(2019\)](#)). This will again give rise to multiple curve term structures.

## ACKNOWLEDGEMENT

This research was supported by the German Research Foundation through the project LU1186/4-1. Eva Lütkebohmert gratefully acknowledges this financial support.

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