Random metric measure spaces

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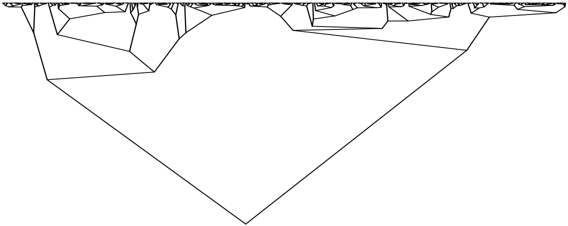
Tree-like metric spaces

- In population models, relationships between individuals are modelled with trees
- Coalescent trees: Genealogies of a constant size population
- Brownian Continuum Random Tree: limit object of a critical branching process
- Main question: what is a good way to encode trees? Especially in the case of infinite populations?

Tree-like metric spaces

- Example: Kingman coalescent:



- Coalescent trees are ultrametric

$$r(u, v) \vee r(v, w) \geq r(u, w)$$

Tree-like metric spaces

- A tree is a metric space (X, r):

$$r(u, v) = 2 \cdot \text{time to the common ancestor of } u, v$$

- In order to be able to 'pick' individuals from a population, consider a probability measure on X.
- Metric measure space: $\mathcal{X} = (X, r, \mu)$ where $r(.,.)$: metric on X, μ: probability measure on X.
- $\mathbb{M}$: the space of (isometry classes) of complete and separable metric measure spaces

Questions

- What does convergence of metric measure spaces mean?
- Is their a characterization of weak convergence of random metric measure spaces?

Philosophy

- Kolmogoroff, Aldous,...
- $\mathcal{X}, \mathcal{X}_1, \mathcal{X}_2, \ldots$.

$$\mathcal{X}_n \xrightarrow{n\to\infty} \mathcal{X}$$

if all finite dimensional distributions converge

- Here:

$$\text{Finite dimensional distributions} \;=\; \text{finite sampled subspaces}$$

Polynomials

- For $\phi : \mathbb{R}^{\binom{n}{2}} \to \mathbb{R}$,

$$\Phi\big((X, r, \mu)\big) := \int \mu^{\otimes n}(dx_1, \ldots, dx_n)\phi\big((r(x_i, x_j))_{1 \le i < j \le n}\big)$$

  polynomial of degree n
- Examples: length, diameter of sample-subtree of n points
- Important fact: polynomials separate points
- Gromov-weak topology: $\mathcal{X}_n \xrightarrow{n\to\infty} \mathcal{X}$ iff

$$\Phi(\mathcal{X}_n) \xrightarrow{n\to\infty} \Phi(\mathcal{X}) \text{ for all polynomials } \Phi$$

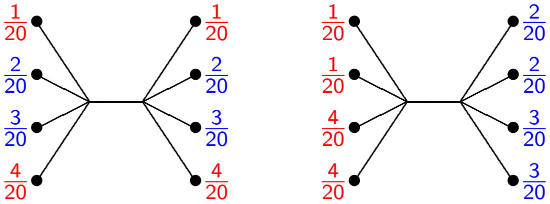
Random Distance Distribution

- Let $\mathcal{X} = (X, r, \mu)$. Every $x \in X$ defines the distribution of distances $\mu_x := r(x, .)_*\mu$. Call

$$\widehat{\mu} := (\mu_.)_*\mu \in \mathcal{M}_1(\mathcal{M}_1(\mathbb{R}_+))$$

the random distance distribution of $\mathcal{X}$.

- The distance distribution does not characterize $\mathcal{X}$.



Distance distribution and modulus of mass distribution

Let $\mathcal{X} = (X, r, \mu) \in \mathbb{M}$.

- What do distances of two typical points in $\mathcal{X}$ look like?

$$w_{\mathcal{X}} := r_*\mu^{\otimes 2}, \text{ i.e. } w_{\mathcal{X}}(\cdot) := \mu^{\otimes 2}\{(x, x') : r(x, x') \in \cdot\}$$

- Which mass do thin points have?

$$v_\delta(\mathcal{X}) := \inf\Big\{\varepsilon > 0 : \mu\{x \in X : \mu(B_\varepsilon(x)) \leq \delta\} \leq \varepsilon\Big\}, \quad \delta > 0$$

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$$w_{\mathcal{X}} = \int \widehat{\mu}(d\nu)\nu$$

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$$= \inf\left\{\varepsilon > 0 : \hat{\mu}_{\mathcal{X}}\{\nu \in \mathcal{M}_1([0, \infty)) : \nu([0, \varepsilon]) \leq \delta\} \leq \varepsilon\right\}.$$

Polish

- **Theorem:** [Greven, P, Winter] The space $\mathbb{M}$, equipped with the Gromov-weak topology, is Polish.

- So, $\mathbb{M}$ is accessible to the notion of weak convergence

Pre-compact sets

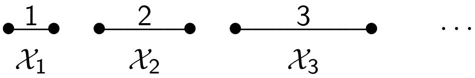
- **Theorem:** [Greven, P, Winter] A sequence $\mathcal{X}_1, \mathcal{X}_2, \dots$ is pre-compact iff:
  - Distances do not explode:

$$\text{The family } \{w_{\mathcal{X}_1}, w_{\mathcal{X}_2}, w_{\mathcal{X}_3}, \dots\} \text{ is tight}$$

  and
  - Thin points are uniformly rare:

$$\lim_{\delta \to 0} \limsup_{n \to \infty} v_\delta(\mathcal{X}_n) = 0$$

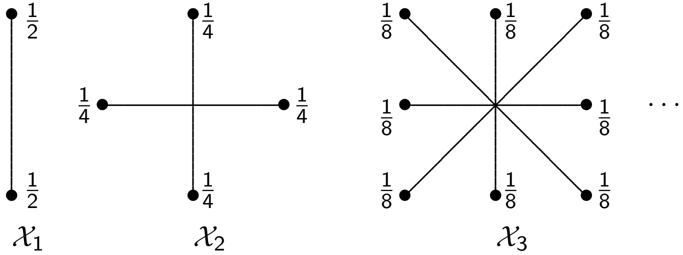
Counterexample I



Indeed,

$w_{\mathcal{X}_n} = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_n$ is not tight

Counterexample II



Indeed,

$$v_\delta(\mathcal{X}_n) = \begin{cases} 0, & \text{for } 2^{-n} > \delta, \\ 1, & \text{for } 2^{-n} \le \delta, \text{ i.e. } n \ge \log_2(1/\delta), \end{cases}$$

Example III

- Restricted doubling property with doubling constant C:

$$\mu_X(B_{2\varepsilon}(x)) \le C \cdot \mu_X(B_\varepsilon(x)) \qquad (x \in X, \varepsilon > 0)$$

- **Proposition:** Let $\mathcal{X}_1, \mathcal{X}_2, \dots$ have the restricted doubling property with doubling constant C. The sequence is pre-compact iff $\{w_{\mathcal{X}_1}, w_{\mathcal{X}_2}, \dots\}$ is tight
- Let $\mathrm{diam}(\mathcal{X}_n)$ be bounded. For $\varepsilon > 0$ and some large $N = N_\varepsilon$

$$\mu(B_\varepsilon(x)) \ge \frac{1}{C}\mu(B_{2\varepsilon}(x)) \ge \dots \ge \frac{1}{C^N}\mu(B_{2^{N}\varepsilon}(x)) = \frac{1}{C^N}$$

Set $\delta := \frac{1}{C^N}$

$$\mu\{x : \mu(B_\varepsilon(x)) > \delta\} = 1 > 1 - \varepsilon \quad \Rightarrow \quad v_\delta(\mathcal{X}_n) < \varepsilon$$

Random metric measure spaces

- **Proposition:** A sequence $\mathbb{P}_1, \mathbb{P}_2, \ldots$ of distributions on $\mathbb{M}$ converges weakly iff:
  - The sequence $\mathbb{P}_1, \mathbb{P}_2, \ldots$ is tight and
  - All polynomials converge, i.e.

$$\mathbb{E}_1\big[\Phi\big], \mathbb{E}_2\big[\Phi\big], \ldots \text{ converges in } \mathbb{R}$$

Tightness

- **Theorem:** [Greven, P, Winter] A sequence $\mathbb{P}_1, \mathbb{P}_2, \ldots$ of distributions on $\mathbb{M}$ is tight iff

$$(w_\cdot)_*\mathbb{P}_1, (w_\cdot)_*\mathbb{P}_2, \ldots \text{ is tight in } \mathcal{M}_1(\mathbb{R})$$

and

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mathbb{E}_n\big[v_\delta(\mathcal{X})\big] = 0$$

Example: Λ-coalescent measure trees

- Introduced by Pitman, Sagitov
- Start a process ξ in the partition $\{\{1\},\{2\},\dots\}$
- In ξ, from any b blocks, k merge at rate

$$\lambda_{b,k} := \int_0^1 \Lambda(dx)\, x^{k-2}(1-x)^{b-k}$$

for some non-negative, finite measure Λ on $[0,1]$

- $\Lambda = \delta_0$: Kingman coalescent
- Distribution on path space: $\mathbb{P}^\Lambda$.

Example: Λ-coalescent measure trees

- Define the (completion of the) metric space

$$r^{\xi}(i,j) := \inf\left\{t \geq 0 : \; i \sim_{\xi(t)} j\right\}.$$

- Define the random metric measure spaces

$$\mathbb{P}^{\Lambda,n} := (H_n)_* \mathbb{P}^{\Lambda} \quad \text{for} \quad H_n : \xi \mapsto \left(\mathbb{N}, r^{\xi}, \mu_n := \frac{1}{n}\sum_{i=1}^{n} \delta_i\right)$$

Example: Λ-coalescent measure trees

- **Theorem:** [Greven, P, Winter] The family $\{\mathbb{P}^{\Lambda,n};\, n \in \mathbb{N}\}$ converges if and only if

$$\int_0^1 \Lambda(dx)\, x^{-1} = \infty. \qquad (*)$$

- Let $f = \{f(\pi) : \pi \in \xi(\varepsilon)\}$ be the ranked rearrangement of

$$\tilde{f}(\pi) := \lim_{n\to\infty} \frac{1}{n} \#\{j \in \{1, \ldots, n\} : j \in \pi\}$$

- [Pitman '99] $(*)$ is equivalent to the dust-free property

$$\sum_i f(\pi_i) = 1 \quad \Longleftrightarrow \quad \mathbb{P}^{\Lambda}\{\tilde{f}(\xi(\varepsilon)^1) = 0\} = 0$$

Example: Λ-coalescent measure trees

- $w_{\mathcal{X}}$: Coalescence rate for any pair is $\lambda_{2,2} > 0$. So, expected time to coalescence bounded.
- v_δ: Due to exchangeability

$$\mathbb{P}^{\Lambda,n}\{v_\delta(H_n(\xi)) \geq \varepsilon\} = \mathbb{P}^{\Lambda}\{\underbrace{\mu_n(B_\varepsilon(1))}_{\xrightarrow{n\to\infty} \tilde{f}\left((\xi(\varepsilon))^1\right)} \leq \delta\}.$$

Hence,

$$\lim_{\delta\to 0}\lim_{n\to\infty}\mathbb{P}^{\Lambda,n}\{v_\delta(H_n(\xi)) \geq \varepsilon\} = \mathbb{P}^{\Lambda}\{\tilde{f}((\xi(\varepsilon))^1) = 0\}.$$

Summary

- Metric measure spaces are useful in the context of (infinite) genealogical trees
- Metric measure spaces form a 'nice' space
- Pre-compactness results, as well as characterization of weak convergence can be given explicitely