

Exercises for the lecture „Probability Theory II“

Sheet 7

Submission deadline: Wednesday, 03.12.2025, 2 p.m. in the mailbox in the math institute.
(You may deliver the exercise solutions in pairs.)

Exercise 1 (4 points)

Let $B = (B_t)_{t \geq 0}$ be a standardized Brownian motion with natural filtration $(\mathcal{F}_t)_{t \geq 0}$ and define $L := \sup\{0 \leq s \leq 1 : B_s = 0\}$.

(a) Prove that for $0 < a < b$,

$$\mathbb{P}(B_s \neq 0 \text{ for all } s \in [a, b]) = \int \mathbb{P}(\tilde{\tau}_{-y} > b - a) d\mathbb{P}^{B_a}(y),$$

where $\tilde{\tau}_s := \inf\{u \geq 0 : \tilde{B}_u = s\}$ for a standardized Brownian motion $(\tilde{B}_t)_{t \geq 0}$ that is independent of $\mathcal{F}_{a+} = \bigcap_{h>0} \mathcal{F}_{a+h}$.

(b) Using part (a), prove that

$$\mathbb{P}(L \leq s) = \frac{2}{\pi} \arcsin(\sqrt{s}).$$

HINT: First show that $\mathbb{P}(L \leq s) = \frac{1}{\pi} \int_{1-s}^{\infty} \sqrt{\frac{(r+s)^2}{rs}} \frac{s}{(r+s)^2} dr$, where you can use that the distribution of $\tilde{\tau}_s$ is known from the lecture. Then, the transformation $t(r) = \sqrt{\frac{s}{r+s}}$ is useful.

Exercise 2 (4 points)

Let $B = (B_t)_{t \geq 0}$ be a standardized Brownian motion.

(a) Prove that B has almost surely for any $\varepsilon > 0$ at least one zero in $(0, \varepsilon)$.

(b) Let $A(\omega) := \{t \in [0, \infty) : B_t(\omega) = 0\}$. Use part (a) and the strong Markov property to prove that A is almost surely a closed set without isolated points.

Exercise 3 (4 points)

Let $B = (B_t)_{t \geq 0}$ be a standardized Brownian motion and $T_x := \inf\{s \geq 0 : B_s = x\}$ for $x \geq 0$. Prove that $(T_x)_{x \geq 0}$ is a process with independent and stationary increments.

Exercise 4 (4 points)

Let $B = (B_t)_{t \geq 0}$ be a standardized Brownian motion on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Prove the following:

(a) The mapping $[0, 1] \times \Omega \rightarrow \mathbb{R}$ with $(t, \omega) \mapsto B_t(\omega)$ is measurable w.r.t. $\mathcal{B}([0, 1]) \otimes \mathcal{A}$.

(b) We have $\lambda(\{t \in [0, 1] : B_t(\omega) = 0\}) = 0$ $\mathbb{P}$ -almost surely.