Exercises for the lecture „Probability Theory II“

Sheet 6

Submission deadline: Wednesday, 26.11.2025, 2 p.m. in the mailbox in the math institute.
(You may deliver the exercise solutions in pairs.)

Exercise 1 (4 points)

Let $\mathcal{C}([0,1])$ be the set of continuous functions from $[0,1]$ to $\mathbb{R}$ and $\mathscr{B}(\mathbb{R})^{[0,1]}$ the product-σ-algebra on $\mathbb{R}^{[0,1]}$. Prove the following:

(a) Singletons $\{f\}$, $f \in \mathcal{C}([0,1])$, are not measurable w.r.t. $\mathscr{B}(\mathbb{R})^{[0,1]}$.

(b) $\mathcal{C}([0,1]) \notin \mathscr{B}(\mathbb{R})^{[0,1]}$, i.e. $\mathcal{C}([0,1])$ is not measurable w.r.t. the product-σ-algebra on $\mathbb{R}^{[0,1]}$.

Exercise 2 (4 points)

Let $B = (B_t)_{t\geq 0}$ a standardized Brownian motion in $\mathcal{C}([0,\infty))$.

(a) Let $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ with $\mathcal{F}_t = \sigma(B_s : s \leq t)$ be the induced filtration and $\mu \in \mathbb{R}$. Prove that the following processes are martingales w.r.t. $\mathcal{F}$:

- (i) $(B_t)_{t\geq 0}$,
- (ii) $(B_t^2 - t)_{t\geq 0}$,
- (iii) $\left(\exp\left(\mu B_t - \frac{\mu^2}{2}t\right)\right)_{t\geq 0}$.

(b) Let $a \in \mathbb{R}\setminus\{0\}$ and $T_a := \inf\{s > 0 : B_s = a\}$ the first hitting time of a.

- (i) Prove that for every $x > 0$ we have $\mathbb{E}[e^{-xT_a}] = e^{-|a|\sqrt{2x}}$.
- (ii) Deduce that $\mathbb{E}[T_a] = \infty$.

Exercise 3 (4 points)

Let $M = (M_t)_{t\geq 0}$ be a martingale with right-continuous sample paths w.r.t. a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$. Let τ be a $\mathcal{F}$-stopping time, i.e. $\{\tau \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$. Prove that

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_0]$$

if one of the following conditions holds true:

(i) $\tau(\omega) \leq n_0$ for some $n_0 \in \mathbb{N}$ and almost all $\omega \in \Omega$.

(ii) $\mathbb{P}(\tau < \infty) = 1$, $\mathbb{E}[|M_\tau|] < \infty$ and $\mathbb{E}[M_t \mathbb{1}_{\{\tau > t\}}] \to 0$ for $t \to \infty$.

Hint: Define the approximating sequence of stopping times

$$\tau_n(\omega) := \inf\left\{\frac{k}{2^n} : \tau(\omega) \leq \frac{k}{2^n}, k \in \mathbb{N}_0\right\}$$

and use the corresponding result in discrete time (see lecture notes Probability Theory I, Theorem 5.15).

(please turn over)

Exercise 4 (4 points)

Let $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration. A random time τ is called *weak stopping time* w.r.t. $\mathcal{F}$ if $\{\tau < t\} \in \mathcal{F}_t$ for all $t \geq 0$. The filtration is called *right-continuous* if $\mathcal{F}_s = \bigcap_{t>s} \mathcal{F}_t$ for all $s \geq 0$.
Let $\mathcal{F}$ be a right-continuous filtration and τ a random time. Prove that τ is a weak stopping time w.r.t. $\mathcal{F}$ if and only if τ is a stopping time w.r.t. $\mathcal{F}$ (see Exercise 3 for the definition of a stopping time).