Exercises for the lecture „Probability Theory II“

Sheet 5

Submission deadline: Wednesday, 19.11.2025, 2 p.m. in the mailbox in the math institute.
(You may deliver the exercise solutions in pairs.)

Exercise 1 (4 points)

Prove the following:

(a) The space $(\mathcal{C}([0,\infty), d)$ endowed with the metric $d(f,g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(f,g)}{1+d_k(f,g)}$, where $d_k(f,g) = \sup_{x\in[0,k]} |f(x) - g(x)|$ is Polish.

(b) The space $(\mathcal{C}_c([0,\infty), d_{\sup})$ of continuous functions on $[0,\infty)$ with compact support and endowed with the supremum metric is separable. Is it also complete?

Exercise 2 (4 points)

Prove that every Polish space $\mathcal{E}$ is Borel isomorphic to a Borel subset of $[0,1]$, i.e. there exists a Borel measurable injection $\Phi : \mathcal{E} \to [0,1]$ with $\Phi(\mathcal{E}) \in \mathscr{B}([0,1])$ and Borel measurable inverse mapping $\Phi^{-1} : \Phi(\mathcal{E}) \to \mathcal{E}$.

Exercise 3 (4 points)

Let μ be a finite Borel measure on a Polish space $\mathcal{E}$. Prove that for every $B \in \mathscr{B}(\mathcal{E})$ and every $\varepsilon > 0$ there exists a compact set $K \in \mathscr{B}(\mathcal{E})$ with $K \subset B$ such that $\mu(B \setminus K) < \varepsilon$. In particular, every Borel measure is tight.

Exercise 4 (4 points)

Let $(\mathcal{C}([0,1]), d_{\sup})$ be the set of continuous functions from $[0,1]$ to $\mathbb{R}$ endowed with the supremum metric. Prove the following:

(a) The finite-dimensional projections $\pi_{t_1,\dots,t_k} : \mathcal{C}([0,1]) \to \mathbb{R}^k$ with $f \mapsto (f(t_1), \dots, f(t_k))$ are continuous and measurable.

(b) We have $\mathscr{B}(\mathcal{C}([0,1])) = \mathscr{B}(\mathbb{R})^{[0,1]} \cap \mathcal{C}([0,1])$.