Exercises for the lecture „Probability Theory II“

Sheet 4

Submission deadline: Wednesday, 12.11.2025, 2 p.m. in the mailbox in the math institute.
(You may deliver the exercise solutions in pairs.)

Exercise 1 (6 points)

Let $(N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$ and $(D_n)_{n\in\mathbb{N}}$ a sequence of independent and identically distributed random variables that is independent of $(N_t)_{t\geq 0}$. Then the process $(Y_t)_{t\geq 0}$ given by

$$Y_t := \sum_{k=1}^{N_t} D_k$$

is called a *compound Poisson process*.

(a) Prove that $(Y_t)_{t\geq 0}$ has stationary and independent increments.

(b) Assume $\mathbb{E}[|D_1|] < \infty$. Determine $\mathbb{E}[Y_t]$.

(c) Assume $\mathbb{E}[D_1^2] < \infty$. Determine $\operatorname{Var}(Y_t)$ and $\operatorname{Cov}(Y_s, Y_t)$.

(d) Determine the characteristic function Φ_{Y_t} of Y_t.

(e) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables that are independent of $(N_t)_{t\geq 0}$ and consider $D_n = \mathbb{1}_{\{X_n\in A\}}$ for some Borel set $A \subset \mathbb{R}$ and the associated compound Poisson process $Y_t(A) = \sum_{k=1}^{N_t} \mathbb{1}_{\{X_n\in A\}}$. Determine the law of $Y_t(A)$.

Exercise 2 (6 points)

Let I be an index set and $\mathbb{R}^I$ endowed with the product topology, i.e. the smallest topology such that all projections $p_i : \mathbb{R}^I \to \mathbb{R}$ are continuous.

(a) Prove that

$$\bigotimes_{i\in I} \mathscr{B}(\mathbb{R}) = \bigcup_{\substack{J\subset I \\ J \text{ countable}}} \left\{ B_J := A_J \times \mathbb{R}^{I\setminus J} : A_J \in \bigotimes_{j\in J} \mathscr{B}(\mathbb{R}) \right\}.$$

Hint: Prove that the right-hand side is a σ-algebra.

(b) Let I be at most countable. Prove that

$$\mathscr{B}(\mathbb{R}^I) = \bigotimes_{i\in I} \mathscr{B}(\mathbb{R}).$$

(c) Show that $\mathscr{B}(\mathbb{R}^I) \neq \bigotimes_{i\in I} \mathscr{B}(\mathbb{R})$ for uncountable I.

(please turn over)

Exercise 3 (4 points)

For each $n \in \mathbb{N}$, let (E_n, d_n) be a Polish space. Prove that $E := \prod_{n\in\mathbb{N}} E_n$ endowed with the product topology is Polish.

HINT: First, prove that $d(x,y) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(x,y)}{1+d_k(x,y)}$ defines a metric on $E \times E$. Then show that topology induced by d is the product topology.