Homework accompanying the lecture „Basics in Applied Mathematics“

Homework 10

Hand in: Tuesday, 26.11.2024, after the lecture in the mailbox at the Math Institut
(Don't forget to put your name on your homework.
Please hand in your solutions in groups of two.)

Exercise 1 (4 points)

Let $Q : C^0([a,b]) \to \mathbb{R}$ be a quadrature formula with quadrature points $(x_i)_{i=0,1,\dots,n}$ and weights $(w_i)_{i=0,1,\dots,n}$, which is exact of degree n.

a) Show that

$$w_i = \int_a^b L_i(x)\,dx,$$

for $i = 0, 1, \dots, n$ with the Lagrange basis polynomials $(L_i)_{i=0,1,\dots,n}$ defined by the support points $(x_i)_{i=0,1,\dots,n}$.

b) Show that in the case of exactness of degree $2n$, $w_i > 0$ for $i = 0, 1, \dots, n$.

Exercise 2 (4 points)

Let $\langle f, g\rangle_\omega = \int_a^b f(x)g(x)\omega(x)\,dx$ for continuous functions $f, g\colon [a,b] \to \mathbb{R}$. The function ω is given such that $\langle \cdot, \cdot\rangle_\omega$ defines a scalar product.
Prove the following theorem:

There exist orthogonal polynomials $(\pi_j)_{j=0,1,\dots,n}$ such that $\pi_j \in \mathcal{P}_j$ and $\langle \pi_j, \pi_k\rangle_\omega = \delta_{jk}$ for all $0 \le j, k \le n$ with $j \ne k$. In particular, $\langle \pi_j, p\rangle_\omega = 0$ for all $p \in \mathcal{P}_{j-1}$ and the polynomials form a basis of $\mathcal{P}_n$.

Exercise 3 (4 points)

a) Calculate three steps of Newton's method for the function $f(x) = arctan(x)$ with the starting values $x_0 = 1; 3/2; 2$.

b) Repeat the calculations for the damped Newton's method

$$x_{k+1} = x_k - \omega f(x_k)/f'(x_k)$$

with the damping parameters $\omega = 1/2; 3/4$.

Programming exercise 4 (4 points)

Implement Newton method for finding the zeros of a function $f : \mathbb{R} \to \mathbb{R}$ and test it with the function $f(x) = \exp(x) + x^2 - 2$, the starting value $x_0 \in \{-1, 0, 1\}$ and the termination criterion $|x_{k+1} - x_k| \le 10^{-12}$. Terminate the Newton method with 100 iterations if the termination criterion is not reached. Compare the iteration numbers and the number of decimal places retained from step to step.