Exercise 1

Submission: Wednesday, 31.01.2024.

Submission in either german or english online via moritz.ritter@stochastik.uni-freiburg.de or mailbox 3.15 at the mathematical institute. To be able to solve the exercises, you need to read the script, which can be found on the homepage, up to page 13.

Exercise 1 (Fréchet-Hoeffding bounds inequality; 4 Points). Let $C$ be a 2-dimensional copula. Prove that for every $(u, v)$ in $[0, 1]^2$,

$$
\max(u + v - 1, 0) \leq C(u, v) \leq \min(u, v).
$$

Hint: Use the monotonicity in each component for the upper bound. For the lower bound use the inclusion–exclusion principle.

Exercise 2 (Survival Copula; 4 Points). For a pair $(X, Y)$ of real-valued random variables with joint distribution function $H$, the joint survival function is given by $H(x, y) = P[X > x, Y > y]$. The marginal distribution functions of $X$ and $Y$ are denoted by $F$ and $G$, respectively. The corresponding survival functions are given by $1 - F$ and $1 - G$, respectively. Let $C$ be a copula for $(X, Y)$, i.e., $C$ is a copula such that $H(x, y) = C(F(x), G(y))$. Define the survival copula $\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v)$ and show that

$$
H(x, y) = \hat{C}(\overline{F}(x), \overline{G}(y)).
$$

Moreover, prove that $\hat{C}$ is indeed a copula.

Exercise 3 (Invariance properties; 4 Points). Let $X$ and $Y$ be random variables with distribution function $F$ and $G$, respectively. Let $F_{XY}$ be the joint distribution function of $(X, Y)$ and $C_{XY}$ be a copula for $(X, Y)$, i.e., $F_{XY}(x, y) = C_{XY}(F_X(x), F_Y(y))$. If $a$ and $b$ are strictly increasing and continuous on $\text{Ran}(X)$ and $\text{Ran}(Y)$, respectively, then $C_{XY}$ is also a copula for the random vector $(a(X), b(Y))$, i.e., the joint distribution function of $(a(X), b(Y))$ fulfills $F_{a(X),b(Y)}(x, y) = C_{XY}(F_{a(X)}(x), F_{b(Y)}(y))$. Thus, $C_{XY}$ is invariant under strictly increasing transformations of $X$ and $Y$.

Exercise 4 (Distributional transform; 4 Points). Let $X$ be a real-valued random variable with distribution function $F$, and let $V \sim U(0, 1)$ be an independent random variable uniformly distributed on $(0, 1)$. The generalized distribution function of $X$ is defined by

$$
F(x, \lambda) = P(X < x) + \lambda P(X = x) \quad \text{or equivalently} \quad F(x, \lambda) = F(x-) + \lambda (F(x) - F(x-)),
$$

and the generalized distribution transform $U$ of $X$ is given by $U = F(X, V)$.

Show that $U \sim U(0, 1)$.

Hint: Define $p = P(X < F^{-1}(\alpha))$ and $q = P(X = F^{-1}(\alpha))$ and show the equality

$$
\{U \leq \alpha\} = \{X < F^{-1}(\alpha)\} \cup \{X = F^{-1}(\alpha), p + V q \leq \alpha\}.
$$

You can use without proof that $F(F^{-1}(\alpha)) \leq \alpha$ and if $F$ is continuous at $F^{-1}(\alpha)$ that it holds that $F(F^{-1}(\alpha)) = \alpha$. 