Exercise 3

Submission: Wednesday, 14.02.2024.

Submission in either german or english online via moritz.ritter@stochastik.uni-freiburg.de or mailbox 3.15 at the mathematical institute.

Exercise 1 (4 Points). The Clayton Copula with parameter $\theta > 0$ is given by

$$C_\theta(u,v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}.$$ 

Show that the Clayton copula is an archimedian copula. Let $U_1, U_2 \sim U[0,1]$ independent random variables. Determine the function $f_\theta$ such that $(U_1, f_\theta(U_1, U_2)) \sim C_\theta$.

Exercise 2 (4 Points). Show that $\zeta_2$ in Example 2.14 is a dependence measure.

Exercise 3 (4 Points). Prove Proposition 3.4: Let $X = (X_1, \ldots, X_d)$ and $Y = (Y_1, \ldots, Y_d)$ be $d$-dimensional random vectors. Then

(i) $X \leq_{st} Y \implies X \leq_{icx} Y$, $X \leq_{wo} Y$ and $X \geq_{lo} Y$,

(ii) $X \leq_{ccx} Y \implies X \leq_{cx} Y$ and $X \leq_{dx} Y$,

(iii) $X \leq_{sm} Y \implies X \leq_{c} Y \implies X \leq_{lo} Y$ and $X \leq_{wo} Y$,

(iv) $X \leq_{sm} Y \implies X \leq_{dx} Y \implies \sum_{i=1}^{d} X_i \leq_{cx} \sum_{i=1}^{d} Y_i$.

Exercise 4 (4 Points). Prove Corollary 3.19: Let $(X,Y)$ be a bivariate random vector with continuous marginal distribution functions. If $(X,Y)$ is PLOD, then $\text{Cor}(X,Y) \geq 0$, $\tau(X,Y) \geq 0$ and $\rho_S(X,Y) \geq 0$.

Exercise 5 (4 Points; Bonus). Show that the Markov product $A \ast B$ is a bivariate copula. Moreover, show that for a bivariate copula $C$ it holds that

$$\Pi^2 * C = C * \Pi^2 = \Pi^2,$$ 

$$M^2 * C = C * M^2 = C,$$ 

$$W^2 * C = C^{\sigma_1},$$ 

$$C * W^2 = C^{\sigma_2}.$$

Exercise 6 (4 Points; Bonus).

(i) Implement the estimator $T_n(Y|X)$ for $T(Y|X)$ following Theorem 2.21.

(ii) Implement the MFOCI Algorithm for $q = 1$, i.e., a single output variable $Y$.

(iii) Create some examples similar to the slides for several sample sizes.

(iv) Compare your results with the CODEC Package from R.

All your results should be presented in a PDF document.