1. **Doob-Meyer Decomposition**
   Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t=0,\dots,T})$ be a filtered probability space with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. Let (Y_t) be an integrable, adapted process and denote by $Y = M - A$ its Doob-Meyer decomposition. For every stopping time τ, show that $Y^\tau = M^\tau - A^\tau$ is the Doob-Meyer decomposition of the stopped process Y^τ.
   Points for Question 1: 4

2. **Essential Supremum**
   Let $\Omega = [0,1] \subseteq \mathbb{R}$, and let $\mathcal{F} = \mathcal{B}(\Omega)$ be the Borel σ-algebra on Ω. Equip $(\Omega, \mathcal{F})$ with $\mathbb{P} := \lambda$, the Lebesgue measure. Set $\Phi := \{\mathbb{1}_{\{x\}} : x \in \Omega\}$.
   (a) Compute the pointwise supremum $\sup\{\phi(\omega) : \phi \in \Phi\}$ for $\omega \in \Omega$. (2)
   (b) Compute the essential supremum of Φ. (2)
   Points for Question 2: 4

3. **Embedding in continuous time**
   Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t=0,\dots,T})$ be a filtered probability space, and let $(X_t)_{t=0,\dots,T}$ be a process. We define the function $Y\colon [0,T] \times \Omega \to \mathbb{R}$ by
   $$Y(t,\omega) := X_n(\omega), \quad t \in [n, n+1).$$
   Further, we define a family of σ-fields $(\hat{\mathcal{F}}_t)_{t\in[0,T]}$ by
   $$\hat{\mathcal{F}}_t := \mathcal{F}_n, \quad t \in [n, n+1).$$
   (a) Show that, for fixed $\omega \in \Omega$, the function (1)
   $$[0,T] \ni t \mapsto Y(t,\omega)$$
   is right-continuous.
   (b) Compute, for $t \in (0,T]$, the quantity (1)
   $$Y_t - \lim_{s\uparrow t} Y_s.$$
   (c) We define the filtration $\mathbb{F}_+$ by (1)
   $$\mathcal{F}_{t+} := \bigcap_{\epsilon>0} \hat{\mathcal{F}}_{t+\epsilon}, \quad t \in \{0,\dots,T\}.$$
   Compare $\mathbb{F}_+$ and $(\mathcal{F}_t)$.
   (d) We define the filtration $\mathbb{F}_-$ by (1)
   $$\mathcal{F}_{t-} := \bigcup_{s<t} \hat{\mathcal{F}}_s, \quad t \in \{0,\dots,T\}.$$
   Compare $\mathbb{F}_-$ and $(\mathcal{F}_t)$.
   Points for Question 3: 4

You can achieve a total of **12** points for this sheet.