UNI FREIBURG

Stochastische Prozesse

WS 15/16

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http://www.stochastik.uni-freiburg.de/lehre/2015WiSe/inhalte/2015WiSeStochProz

Exercise 6

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Problem 1 (4 Points). (*a*) Let X be a submartingale, $n \in \mathbb{Z}^+$ and let $\lambda > 0$. Show that

$$\lambda \mathbb{P}\left(\max_{1\le i\le n} |X_i| \ge 3\lambda\right) \le 4\mathbb{E}[|X_0|] + 3\mathbb{E}[|X_n|].$$

(*b*) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of integrable random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which converges weakly in $L^1(\mathbb{P})$ to an integrable random variable X. Show that for each σ-field $\mathcal{G} \subset \mathcal{F}$, the sequence $\mathbb{E}[X_n|\mathcal{G}]$ converges to $\mathbb{E}[X|\mathcal{G}]$ weakly in $L^1(\mathbb{P})$

Problem 2 (4 Points). (*a*) Let $X = (X_t)_{0\le t<\infty}$ be a local martingale and τ is a stopping time. Show that $Y_t = X_{t\wedge\tau}$ is also a local martingale.

(*b*) $X = (X_n)_{n\in\mathbb{N}}$ be i.i.d with $\mathbb{P}(X_1 = 1) = p$ and $\mathbb{P}(X_1 = -1) = q = 1 - p$. Furthermore,

$$S_n = \sum_{i=1}^{n} X_i$$

and

$$\tau = \inf\{n \ge 1 : S_n \ge b\} \tag{1}$$

where $b \in \mathbb{N}$. For $\{\cdots\} = \emptyset$ in (1) set $\tau = \infty$ and on $\{\tau = \infty\}$

$$S_\tau = \lim_{n\to\infty} S_n,$$

if the limit exists. Show that

$$\mathbb{P}(\tau < \infty) = \left(\frac{p}{q}\right)^b \quad \text{for } p < q.$$

Problem 3 (4 Points). Let $X = (X_t)_{0\le t<\infty}$ be a right-continuous martingale with respect to $\mathcal{F}_t$. X is said to be square integrable if $\mathbb{E}[X_t^2] < \infty$ and $X_0 = 0$ a.s., and we write $X \in \mathcal{M}_2$.
Let X be a process in $\mathcal{M}_2$ or in $\mathcal{M}_{loc}$, and we assume its quadratic variation $\langle X\rangle$ is integrable. i.e., $\mathbb{E}[\langle X\rangle_\infty] < \infty$. Show that

(*a*) X is a martingale

(*b*) X and submartingale X^2 are both uniformly integrable, in particular

$$X_\infty = \lim_{t\to\infty} X_t$$

exists almost surely and

$$\mathbb{E}[X_\infty^2] = \mathbb{E}[\langle X\rangle_\infty]$$

Hint: Conditions $(a)-(d)$ are equivalent: (a)X is uniformly integrable family of random variables, (b) X converges in L^1 as $t \to \infty$, (c) X converges almost surely to an integrable variable X_∞, such that X_t is a martingale (respectively submartingale), (d) there exists an integrable random variable Y such that $X_t = \mathbb{E}[Y|\mathcal{F}_t]$ $\mathbb{P}$-a.s. for every $t \geq 0$. Note: conditions $(a) - (c)$ also holds for non-negative right-continuous submartingale X.
If $X \in \mathcal{M}_{loc}$ and τ is a stopping time of $\mathcal{F}_t$, then $\mathbb{E}[X_\tau^2] \leq \mathbb{E}[\langle X \rangle_\tau]$, where

$$X_\infty^2 = \underline{\lim}_{t\to\infty} X_t^2.$$

Problem 4 (4 Points). (a) Show that for any optional time τ and predictable process X, the random variable $X_\tau \mathbf{1}_{\{\tau<\infty\}}$ is $\mathcal{F}_{\tau-}$-measurable.

(b) Let $A \in \mathcal{V}$. Show that there exist a unique pair (B, C) of adapted increasing processes such that $A = B - C$ and $Var(A) = B + C$.
Hint: If A is predictable, B, C and $Var(A)$ are also predictable.