UNI FREIBURG

Stochastische Prozesse **WS 15/16**

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http://www.stochastik.uni-freiburg.de/lehre/2015WiSe/inhalte/2015WiSeStochProz

Exercise 4

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Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F}$. A positive $\mathbb{F}$-adapted process λ is given. We denote

$$\Lambda_t := \int_0^t \lambda_s ds, \quad t \geq 0.$$

We assume there exist a random variable Θ constructed on Ω independent of $\mathcal{F}_\infty$, with the exponential law of parameter 1. i.e.,

$$P\{\Theta \geq t\} = \exp(-t).$$

We define the random time τ as the first time when the process Λ_t is above the random level Θ. i.e.,

$$\tau = \inf\{t \geq 0 : \Lambda_t \geq \Theta\}.$$

Note: $\{\tau \geq s\} = \{\Lambda_s \leq \Theta\}$. We assume $\Lambda_t < \infty$, for all t, and $\Lambda_\infty = \infty$.

Problem 1 (4 Points). (*a*) Show that a random variable Θ with exponential distribution satisfies

$$\mathbb{P}\{\Theta > t + s \mid \Theta > s\} = \mathbb{P}\{\Theta > t\}, \quad \text{for } 0 \leq s \leq t.$$

(*b*) Let $(X_t)_{t\geq 0}$ be a Poisson process with parameter $\lambda = 1$. We set $Y_t = X_{\Lambda_t}$.
Show that

$$Y_t - \Lambda_t$$

is a martingale.

Problem 2 (4 Points). (*a*) Show that the conditional distribution of τ given the σ-algebra $\mathcal{F}_t$, for $t \geq s$ is

$$P\{\tau > s \mid \mathcal{F}_t\} = \exp(-\Lambda_s).$$

Hint: $(\Lambda_t)_{t\geq 0}$ is an increasing and $\mathcal{F}_t$-adapted process.

(*b*) If $t < s$, show that the conditional distribution of τ given the σ-algebra $\mathcal{F}_t$ is

$$P\{\tau > s \mid \mathcal{F}_t\} = \mathbb{E}[\exp(-\Lambda_s) \mid \mathcal{F}_t].$$

Problem 3 (4 Points). Let $D_t = \mathbb{1}_{\{\tau \leq t\}}$ and $\mathbb{D}_t = \sigma(D_s; s \leq t)$. We introduce the smallest right-continuous filtration $\mathbb{G}$ which contains $\mathbb{F}$ and turns τ to a stopping time. $\mathbb{G}_t = \mathcal{F}_t \vee \mathbb{D}_t$. Let Y be an integrable random variable. Show that

$$\mathbb{1}_{\{\tau>t\}}\mathbb{E}[Y \mid \mathbb{G}_t] = \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{E}[Y\mathbb{1}_{\{\tau>t\}} \mid \mathcal{F}_t]}{\mathbb{E}[\mathbb{1}_{\{\tau>t\}} \mid \mathcal{F}_t]} = \mathbb{1}_{\{\tau>t\}} \exp(\Lambda_t)\mathbb{E}[Y\mathbb{1}_{\{\tau>t\}} \mid \mathcal{F}_t].$$

Hint: From the Monotone class theorem, any $\mathbb{G}_t$-measurable random variable Y_t satisfies

$$\mathbb{1}_{\{\tau>t\}}Y_t = \mathbb{1}_{\{\tau>t\}}y_t$$

where y_t is an $\mathcal{F}_t$-measurable random variable.

Problem 4 (4 Points). Let $D_t = \mathbb{1}_{\{\tau \le t\}}$ and $\mathbb{D}_t = \sigma(D_s; s \le t)$. We introduce the smallest right-continuous filtration $\mathbb{G}$ which contains $\mathbb{F}$ and turns τ to a stopping time. $\mathbb{G}_t = \mathcal{F}_t \vee \mathbb{D}_t$. If Y is an integrable $\mathcal{F}_T$-measurable random variable. Show that, for $t < T$,

$$\mathbb{E}[Y \mathbb{1}_{\{T<\tau\}} \mid \mathbb{G}_t] = \mathbb{1}_{\{\tau > t\}} \exp(\Lambda_t) \mathbb{E}[Y \exp(-\Lambda_T) \mid \mathcal{F}_t].$$