Vorlesung: Prof. Dr. Thorsten Schmidt
Exercise: Dr. Tolulope Fadina

http://www.stochastik.uni-freiburg.de/lehre/2015WiSe/inhalte/2015WiSeStochProz

## Exercise 12

**Submission: 02-02-2016**

**Problem 1** (4 Points)**.** Let $G$ be the generator of a Feller semigroup. If $f \in Dom(G)$ and its bounded, show that

$$\left( f(X_t) - f(X_0) - \int_0^t Gf(X_s)ds \right)_{t \geq 0}$$

is a martingale. If, in particular $Gf = 0$, then $f(X_t)$ is a martingale.
Hint: If $f \in Dom(G)$, then

$$P_t f - f = \int_0^t P_s G f ds = \int_0^t G P_s f ds$$

.

**Problem 2** (4 Points)**.** Let $X = (X_t)_{t \geq 0}$ be a Markov process with generator $G$. Show that for $f \in Dom(G)$ with $Gf \in C_b$ and $\lambda \geq 0$,

$$\left( e^{-\lambda t} f(X_t) + \int_0^t e^{-\lambda s}(\lambda f(X_s)) - Gf(X_s) ds \right)_{t \geq 0}$$

is a martingale.
Hint: Use product rule and the Chapman Kolmogorov equation.

**Problem 3** (4 Points)**.** The following problem illustrates the important steps in the proof of the Burkholder-Davis-Gundy inequality. Let $W = (W_t)_{t \geq 0}$ be a Brownian motion and $X$ a measurable, adapted process satisfying $\mathbb{E}\left[\int_0^T |X_t|^{2m} dt\right] < \infty$ for some real numbers $T > 0$ and $m \geq 1$. Show that

$$\mathbb{E}\left[\left| \int_0^T X_t dW_t \right|^{2m}\right] \leq (m(2m-1))^m T^{m-1} \mathbb{E}\left[\int_0^T |X_t|^{2m} dt\right].$$

Hint: Consider the martingale $M_t = \int_0^t X_s dW_s$, for $0 \leq t \leq T$, and apply the Ito's rule to the submartingale $|M_t|^{2m}$.

**Problem 4** (4 Points)**.** Let $M = (M^{(1)}, \cdots, M^{(d)})$ be a vector of continuous, local martingales and denote

$$\|M\|_t^* := \max_{0 \leq s \leq t} \|M_s\|, \qquad A_t := \sum_{i=1}^d \langle M^i \rangle_t, \quad 0 \leq t < \infty.$$

Show that for any $m > 0$, there exist (universal) positive constants $\lambda_m, \gamma_m$ such that

$$\lambda_m \mathbb{E}\left[A_T^m\right] \leq \mathbb{E}\left[(\|M\|_T^*)^{2m}\right] \leq \gamma_m \mathbb{E}\left[A_T^m\right]$$

holds for every stopping time $T$.
Use the 1-dimensional Burkholder-Davis-Gundy inequality for the proof.