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http://www.stochastik.uni-freiburg.de/lehre/2015WiSe/inhalte/2015WiSeStochProz

Exercise 10

Submission: 12-01-2016

Problem 1 (4 Points)**.** Let $(X_t)_{t\geq 1}$ be sequence of independently identically distributed random variables with $P(X_1 = 1) = p$ and $P(X_1 = -1) = q = 1 - p$. Define

$$S(t) = \sum_{i=1}^{t} X_i, \quad t \geq 1, \qquad S_0 = 0.$$

Show that

$$P(S_t = j | S_0 = i) = \binom{t}{\frac{t+j-i}{2}} p^{\frac{t+j-i}{2}} \cdot q^{\frac{t-j+i}{2}}$$

If $t + j - i$ is an even non-negative integer, and

$$P(S_t = j | S_0 = i) = 0 \quad \text{otherwise.}$$

Hint: Use induction. Note that

$$\binom{t}{\frac{t+j-i}{2}} p^{\frac{t+j-i}{2}} = 0$$

if $|j - i| \geq t + 1$.

Problem 2 (4 Points)**.** Verify that

$$X(t) = \Phi(t)\left(X(0) + \int_0^t \Phi(t)^{-1}(s)a(s)ds + \int_0^t \Phi(t)^{-1}(s)\sigma(s)dW(s)\right), \qquad t \geq 0, \tag{1}$$

solves the stochastic differential equation

$$\begin{aligned} dX(t) &= (A(t)X(t) + a(t))dt + \sigma(t)dW(t) \\ X_0 &= \xi \end{aligned} \tag{2}$$

where W is a Brownian motion independent of ξ, $A(t)$, $a(t)$ and $\sigma(t)$ are non-random, measurable and locally bounded. Assuming $\Phi(t) = A(t)\Phi(t)$, $\Phi(0) = 1$, has a unique (absolutely continuous) solution defined for $0 \leq t < \infty$. Hint: Use the Itô formula.

Problem 3 (4 Points)**.** If $a(t) = 0$, $A(t) = -\alpha < 0$, and $\sigma(t) = \sigma > 0$ in (2), ((2) becomes the Ornstein-Uhlenbeck Stochastic differential equation, see Exercise 8-Problem 3), and the solution to the SDE is

$$X(t) = X(0)\exp(-\alpha t) + \sigma \exp(-\alpha t)\int_0^t \exp(\alpha s)dW(s) \qquad t \geq 0.$$

If $\mathbb{E}(X_0^2) < \infty$, compute

(*a*) the expectation: $\mathbb{E}(X_t)$.

(*b*) the variance: $Var(X_t)$.

(b) the covariance function: $c(X_s, X_t)$.

Problem 4 (2 Points). (Brownian Bridge) Show that X_t defined by

$$X(t) = a(1 - \frac{t}{T}) + b\frac{t}{T} + (T - t)\int_0^t \frac{dW(s)}{T - s}, \qquad 0 \le t < T, \tag{3}$$

solves the stochastic differential equation

$$\begin{aligned} dX(t) &= \frac{b - X(t)}{T - t} dt + dW(t) \\ X_0 &= a \end{aligned} \tag{4}$$

for given real numbers a, b, $T > 0$.
Hint: Use the Itô formula. if $A(t) = \frac{-1}{T-t}$, $a(t) = \frac{b}{T-t}$, and $\sigma(t) = 1$, then, $\Phi(t) = 1 - \frac{t}{T}$ in (2) and (2) becomes (4).

Problem 5 (4 Points). Show that the process

$$Y(t) = \begin{cases} (T - t)\int_0^t \frac{dW_s}{T-s}; & 0 \le t < T \\ \\ 0; \quad t = T. \end{cases}$$

is continuous, has zero mean, its Gaussian, with covariance function

$$c(s, t) = (s \wedge t) - \frac{st}{T}; \quad 0 \le s, t \le T.$$

Problem 6 (4 Points). Let $(X_t)_{t \ge 0}$ be a Brownian motion and

$$Y(t) = e^{-t/2} X(e^t - 1).$$

Show that $(Y_t)_{t \ge 0}$ is a Gaussian process and a Markov process.