Comparison and extension of VaR bounds for joint portfolios

Dennis Manko and Ludger Rüschendorf
University of Freiburg

Abstract

The paper is concerned with the comparison and extension of VaR bounds for joint portfolios. The aim is to improve VaR bounds based on marginal information only by including dependence information by higher order marginals, by some positive or negative dependence information in the tails, or in some central part of the distribution or by including some copula information. Various methods to include this information are introduced and the magnitude of reduction of the VaR bounds is illustrated in a series of examples.

1 Introduction

To establish reliable bounds for the VaR of a joint portfolio is a relevant subject in connection with the amount of risk capital in the Basel II/III regulations for the finance sector as well as with the solvency regulations for the insurance sector. A series of results and different methods have been established in the last 10 years to find good bounds for the VaR based on available information on the dependence structure of the portfolio. Some general descriptions of these developments can be found in Puccetti and Rüschendorf (2012, 2014), Embrechts et al. (2013, 2014) and in Rüschendorf (2013). The methods described there concern in particular the standard bounds and the dual bounds in the case of marginal information and the rearrangement algorithm (RA) to calculate these bounds.

For the case with dependence information improved standard bounds are available. Effects of this dependence information on the reduction of the VaR bounds are described in Bignozzi et al. (2015) and in Bernard et al. (2013a). Some higher order marginal information has been investigated in Embrechts and Puccetti (2010), Puccetti and Rüschendorf (2012), Embrechts et al. (2013) and in Puccetti et al. (2015). The reduction of VaR bounds by inclusion of additional second or higher order moment information was described in Bernard et al. (2013b) and in Bernard et al. (2015).

In Section 3 of this paper we compare the influence of higher order marginals as given by various types of Bonferroni inequalities on the improved standard bounds with the ‚nearly‘ optimal dual bounds based on marginal information only and describe which of these bounds is giving the best bound in dependence on the dimension n of the portfolio and on the confidence level. The findings are illustrated in several examples.

In Section 4 we consider the case where bounds on the distribution function of the joint portfolio are known on a given subset of the domain. This allows to include positive or negative dependence information in the tail of the distribution. We determine based on this information bounds on the distribution function and thus can apply the improved standard bounds. We describe the effects of this additional information on the VaR bounds. We also include some cases of additional copula information on the model. In particular we consider the case of independence of subgroups combined with positive or negative copula information within the subgroups and describe the various effects in several illustrative examples. This model is a modification of the model assumptions made in Bignozzi et al. (2015).

2 Improved standard and dual bounds with dependence information

For a risk vector $X = (X_1, \dots, X_n)$ with $X_i \sim F_i$ it is a classical problem to find good (best possible) bounds for the distribution function and tail probability of the joint portfolio $S = \sum_{i=1}^n X_i$. Define

$$\begin{aligned} M(s) &= \sup\{P(S \geq s);\ X_i \sim F_i, 1 \leq i \leq n\} \\ \text{and} \quad m(s) &= \inf\{P(S \geq s);\ \ X_i \sim F_i, 1 \leq i \leq n\}. \end{aligned} \tag{2.1}$$

So called 'standard bounds' were derived for (2.1) in several ways in the literature, see Frank et al. (1987), Denuit et al. (1999), Embrechts et al. (2003), Rüschendorf (2005) and Puccetti and Rüschendorf (2012). As result one obtains the following **Standard bounds:**

$$\max\Big(\bigvee_{i=1}^n \overline{F}_i(t) - (n-1), 0\Big) \leq m(t) \leq M(t) \leq \min\Big(\bigwedge_{i=1}^n \overline{F}_i(t), 1\Big), \tag{2.2}$$

where the infimal and supremal convolutions are defined as

$$\bigwedge_{i=1}^n \overline{F}_i(t) = \inf\Big\{\sum_{i=1}^n \overline{F}_i(u_i);\ \ u \in \mathcal{U}(t)\Big\},\ \ \bigvee_{i=1}^n \overline{F}_i(t) = \sup\Big\{\sum_{i=1}^n \overline{F}_i(u_i);\ \ u \in \mathcal{U}(t)\Big\},$$

and where

$$\mathcal{U}(t) = \Big\{u = (u_1, \dots, u_n) \in \mathbb{R}^n;\ \sum_{i=1}^n u_i = t\Big\},\ \text{and } \overline{F}_i(t) = P(X_i \geq t) = 1 - F_i(t-).$$

The standard bounds have been improved under additional positive or negative dependence restrictions on the distribution functions. Let $\overline{F}_X(x) = P(X \geq x)$, $x \in \mathbb{R}^n$ denote the tail probability and assume that H is a decreasing function on $\mathbb{R}^n$, G an increasing function on $\mathbb{R}^n$ such that

$$\begin{aligned} &\max\Big(\Big\{\sum_{i=1}^n F_i(x_i) - (n-1)\Big\}, 0\Big) \leq G(x), \\ &\qquad\qquad H(x) \leq \max\Big(\sum_{i=1}^n \overline{F}_i(x_i) - (n-1), 0\Big). \end{aligned} \tag{2.3}$$

Then the following improved standard bounds have been given in various forms in Williamson and Downs (1990), Denuit et al. (1999), Embrechts et al. (2003), Rüschendorf (2005) and Embrechts and Puccetti (2006).

Improved standard bounds:

a) If H is decreasing, satisfies (2.3) and if $\overline{F}_X(x) \geq H(x), \quad \forall x$, then

$$P\Big(\sum_{i=1}^n X_i \geq t\Big) \geq \bigvee_{i=1}^n H(t). \tag{2.4}$$

b) If G is increasing, satisfies (2.3) and if $F_X(x) \geq G(x), \quad \forall x$, then

$$P\Big(\sum_{i=1}^n X_i > t\Big) \leq 1 - \bigvee_{i=1}^n G(t). \tag{2.5}$$

In the case that H resp. G is the lower Fréchet bound, i. e.

$$H(x) = \max\Big(\sum_{i=1}^n \overline{F}_{X_i}(x_i) - (n-1), 0\Big), \text{ resp. } G(x) = \max\Big(\sum_{i=1}^n F_i(x_i) - (n-1), 0\Big).$$

(2.4) and (2.5) are identical to the standard bounds. In the particular case where X is positive orthant dependent (POD) (2.4) and (2.5) lead to the bounds

$$\begin{aligned} \bigvee\Big(\prod_{j=1}^n \overline{F}_i\Big)(t) &\leq P\Big(\sum_{i=1}^n X_i \geq t\Big) \leq P\Big(\sum_{i=1}^n X_i > t\Big) \\ &\leq 1 - \bigvee\Big(\prod_{j=1}^n F_i\Big)(t). \end{aligned} \tag{2.6}$$

Similar inequalities also hold for monotone increasing aggregation functions $\Psi(X)$ replacing the sum $S = \sum_{i=1}^n X_i$, where the inf(sub)-convolutions are replaced by the Ψ-convolutions (see Puccetti and Rüschendorf (2012)).

To determine sharp upper and lower bounds of $P(\sum_{i=1}^n X_i \geq t)$ there are exact dual representations which however are difficult to evaluate in general. Embrechts and Puccetti (2006) restricted the class of admissible dual functions to admissible piecewise linear dual functions and as a result got the following

dual bounds:

$$M(s) \leq D(s) = \inf_{u \in \overline{\mathcal{U}}(s)} \min\left\{\frac{\sum_{i=1}^n \int_{u_i}^{s-\sum_{j\neq i} u_j} \overline{F}_i(t)\, dt}{s - \sum_{i=1}^n u_i}, 1\right\}, \tag{2.7}$$

where $\overline{\mathcal{U}}(s) = \{u \in \mathbb{R}^n; \sum_{i=1}^n u_i < s\}$. A similar lower bound $d(s)$ is also given. In the homogeneous case where $F_i = F$, $1 \leq i \leq n$, the dual bound simplifies to

$$D(s) = \inf_{t < \frac{s}{n}} \frac{n \int_t^{s-(n-1)t} \overline{F}(u)\, du}{s - nt}. \tag{2.8}$$

It was shown in Puccetti and Rüschendorf (2013) that in the homogeneous case $D(s)$ is a sharp bound if F has a decreasing density on $t \geq t_0$ for some $t_0 \in \mathbb{R}$. This implies for $\alpha \geq \alpha_0$, that

$$\overline{\text{VaR}}_\alpha(S) = D^{-1}(1-\alpha). \tag{2.9}$$

3 Improved Hoeffding–Fréchet bounds with higher order marginals

If higher order marginal distributions of the risk vector X are known then it is possible to improve the Hoeffding–Fréchet bounds and as consequence of (2.4) and (2.7) one gets improved standard bounds for the VaR. In this section we consider the case where two dimensional marginal distributions are known. Alternative dual bounds with higher order marginals have been discussed in Embrechts and Puccetti (2006) and in Embrechts et al. (2013). As a result it was found in these papers that the additional information of higher dimensional marginals may lead to considerably improved upper VaR bounds, when the joint marginals are not 'too close' to the upper Hoeffding–Fréchet bounds.

One obtains improved Hoeffding–Fréchet bounds for the distribution function (resp. for the copula) by means of Bonferroni-type bounds (see Rüschendorf (1991, Prop. 6)).

Proposition 3.1 (Bonferroni-type bounds)**.** *Let C be an n-dimensional copula with bivariate marginals $C_{i,j}$ for $i \neq j$. Then*

$$C \geq W_B \geq W_A \geq W, \tag{3.1}$$

where $W(u) = (\sum_{i=1}^n u_i - (n-1))_+$ is the Hoeffding–Fréchet lower bound,

$$W_A(u) = \Big(\sum_{i=1}^n u_i - (n-1) + \frac{2}{n} \sum_{i<j} (1 - u_i - u_j + C_{i,j}(u_i, u_j)) \Big)_+ \tag{3.2}$$

$$\text{and} \quad W_B(u) = \Big(\sum u_i - (u-1) + \sup_\tau \sum_{(i,j)\in\tau} (1 - u_i - u_j + C_{i,j}(u_i, u_j)) \Big)_+, \tag{3.3}$$

the sup being taken over all spanning trees of the complete graph induced by $\{1, \ldots, n\}$.

The bound W_B is a consequence of the Bonferroni inequality from Hunter (1976) (see Rüschendorf (1991, Prop. 6)). It improves the bound W_A arising from a Bonferroni bound of Hunter (1976) and Worsley (1982). As consequence of (2.4) and (2.5) these bounds imply improved bounds for the tail-risk and the VaR of the joint portfolio $\sum_{i=1}^n X_i$, where (X_i, X_j) have copulas $C_{i,j}$. Let

$$\begin{aligned} \mathrm{VaR}_\alpha^W &= W(F_1, \ldots, F_n)^{-1}(\alpha), \quad \mathrm{VaR}_\alpha^{W_A} = W_A(F_1, \ldots, F_n)^{-1}(\alpha) \\ \text{and} \quad \mathrm{VaR}_\alpha^{W_B} &= W_B(F_1, \ldots, F_n)^{-1}(\alpha) \end{aligned} \tag{3.4}$$

denote the upper α-quantiles of W, W_A, W_B with marginals $F_1, \ldots, F_n$. Then we obtain as consequence of (3.1)

$$\mathrm{VaR}_\alpha(S) \leq \mathrm{VaR}_\alpha^{W_B} \leq \mathrm{VaR}_\alpha^{W_A} \leq \mathrm{VaR}_\alpha^{W}. \tag{3.5}$$

The upper bound $\mathrm{VaR}_\alpha^{W_A}$ has been investigated in Liu and Chan (2011). In contrast to their statement this bound is not the 'best possible upper bound' for $\mathrm{VaR}_\alpha(S)$. As their numerical results indicate the bound $\mathrm{VaR}_\alpha^{W_A}$ improves on the dual bound, which is based solely on marginal information, only for high confidence levels α and for highly positive correlated two-dimensional marginals. Correspondingly it was seen in Embrechts et al. (2013) that strong improvements of lower bounds are obtained, when the two-dimensional marginals are independent.

In the following examples we compare the Bonferroni bounds $\mathrm{VaR}_\alpha^{W_A}$ and $\mathrm{VaR}_\alpha^{W_B}$ with each other and with the standard bounds VaR_α^{W} as well as with the dual bound VaR_α^{D} for various dependence levels on the bivariate marginals.

By (2.5) we have

$$P\Big(\sum_{i=1}^n X_i \leq t\Big) \geq \sup_{u\in\mathcal{U}(t)} C_L(F_1(u_1), \ldots, F_n(u_n)), \tag{3.6}$$

where C_L is either W or is one of the (improved) bounds W_A, W_B. For $u = \big(\frac{t}{n}, \ldots, \frac{t}{n}\big)$ we get the lower bound

$$P\Big(\sum_{i=1}^n X_i \leq t\Big) \geq C_L\Big(F_1\Big(\frac{t}{n}\Big), \ldots, F_n\Big(\frac{t}{n}\Big)\Big). \tag{3.7}$$

In general the improvements of the Fréchet bounds as in (3.1) can be considerable. The improved standard bounds in (3.6) are not easy to determine in general in explicit form. In several cases however conditions are easy to state which allow to determine them explicitly. In general we obtain the strongest improvement of the upper bound VaR_α^W if the two-dimensional copulas are comonotonic.

We next state for some cases explicit solutions to (3.6). If $C_L = W$ and $F_1, \dots, F_n$ have decreasing densities and $u^* \in U(t)$ satisfies $F_1(u_1^*) = \cdots = F_n(u_n^*)$ then $u^* = (u_1^*, \dots, u_n^*)$ is uniquely determined and u^* is a solution to (3.6). If $F_1 = \cdots = F_n$ has a decreasing density, then $\left(\frac{t}{n}, \dots, \frac{t}{n}\right)$ is a solution to (3.6) and thus the bound in (3.7) coincides with that in (3.6).

More generally let

$$A = \{(F_1(u_1), \dots, F_u(u_n));\ \ u = (u_i) \in \mathcal{U}(t)\}$$

and assume that $u^* = (u_i^*)$ is a largest element of A w.r.t. the increasing Schur convex order $\preceq_S$, then

$$\sup_{u \in \mathcal{U}(t)} W(F_1(u_1), \dots, F_n(u_n)) = W(F_1(u_1^*), \dots, F_n(u_n^*)). \tag{3.8}$$

Similarly, assuming that W_A resp W_B are increasing w.r.t. the increasing Schur convex order $\preceq_S$ we obtain

$$\sup_{u \in \mathcal{U}(t)} W_A(F_1(u_1), \dots, F_n(u_n)) = W_A(F_1(u_1^*), \dots, F_u(u_n^*)) \tag{3.9}$$

$$\text{resp.} \quad \sup_{u \in \mathcal{U}(t)} W_B(F_1(u_1), \dots, F_n(u_n)) = W_B(F_1(u_1^*), \dots, F_u(u_n^*)). \tag{3.10}$$

In the following we use the vector u^* with identical components $(F_1(u_1^*), \dots, F_n(u_n^*))$ as above as a proxy for comparison of the upper bounds in (3.8)–(3.10). In particular in the case $F_1 = \cdots = F_n = F$ we use the vector $\left(F\left(\frac{t}{n}\right), \dots, F\left(\frac{t}{n}\right)\right)$. In contrast to statements in Liu and Chan (2011) this choice will not give the exact bounds in (3.8) and (3.9) (and also in (3.10)) in general.

In the following examples we consider the homogeneous case where $F_i = F$ and where $C_{i,j} = C_2$ for all $i < j$. We concentrate on the approximate bounds based on u^*.

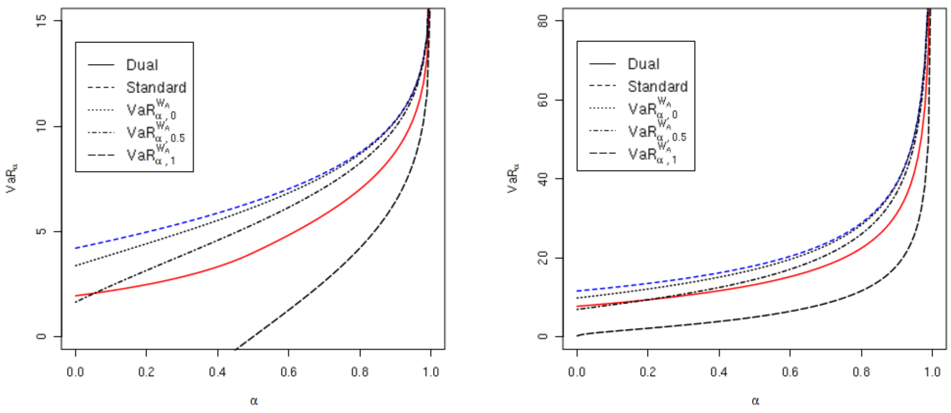
Comparison of VaR^{W_A}, standard bounds, and dual bound

In the first example we compare the standard bound, the VaR bound induced by W_A and the dual bound D, which gives the optimal bound with marginal information only in this example.

Let $n = 5$ and let X_i be standard normal resp. log-normal distributed, $1 \leq i \leq 5$. Let C_2 be a Gauß-copula with correlations $\varrho = 0,\ 0.5,\ 1$. Figure 3.1 compares the $\mathrm{VaR}_{\alpha,\varrho}^{W_A}$ upper bounds with the dual bound VaR_α^D in dependence on α and ϱ for both distributions. Note that using the proxies the bounds $\mathrm{VaR}_{\alpha,\varrho}^{W_A}$ and $\mathrm{VaR}_{\alpha,\varrho}^{W_B}$ coincide in this case.

Figure 3.1 a) shows that the $\mathrm{VaR}_{\alpha,\varrho}^{W_A}$ bound improves with increasing correlation. In particular the case $\varrho = 1$ (comonotonicity) for the two dimensional marginals gives better upper bounds than the case $\varrho = 0$ (independence). This kind of dependence on ϱ can also be seen directly from the definition of W_A in (3.2). Further one finds as expected, that for any ϱ the $\mathrm{VaR}_{\alpha,\varrho}^{W_A}$ bound using information on two-dimensional marginals is an improvement on the standard bound based on marginal information only.

The dual bound VaR_α^D is a strong improvement over the standard bound, both being based on marginal information only. It is known that the dual bound is optimal



a) b)

Figure 3.1 Comparison $\mathrm{VaR}^{W_A}_{\alpha,\varrho}$, standard bound and dual bound, $n = 5$, $\varrho = 0,\ 0.5,\ 1.$

in this example. This example shows that the technique of standard bounds does not work well in higher dimensions.

From Figure 3.1 and Table 3.1 one sees that the dual bound VaR^D_α is even an improvement over the bounds $\mathrm{VaR}^{W_A}_{\alpha,\varrho}$ when $\varrho < 0.9$ and $\alpha \geq 0.9$, i.e. the information on two-dimensional marginal information does not lead to an improved upper bound in these cases, when using the method of improved standard bounds.

α	VaR^S_α	VaR^D_α	$\mathrm{VaR}^{W_A}_{\alpha,0}$	$\mathrm{VaR}^{W_A}_{\alpha,0.5}$	$\mathrm{VaR}^{W_A}_{\alpha,0.9}$	$\mathrm{VaR}^{W_A}_{\alpha,1}$
0.9	10.268	8.773	10.234	9.943	8.764	6.407
0.95	11.631	10.311	11.616	11.415	10.425	8.224
0.99	14.390	13.322	14.388	14.297	13.589	11.631

Table 3.1 Comparison of VaR^S, VaR^D and VaR^{W_A}

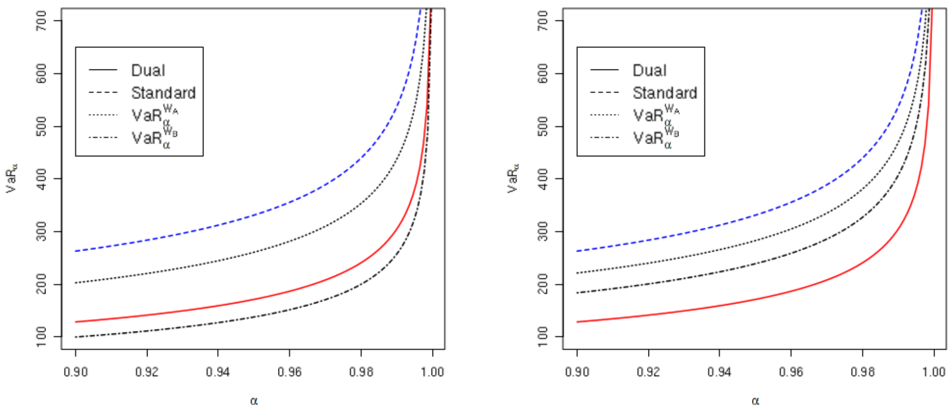
In Figure 3.1 b) we see that in the case of log-normal distributions with heavy tails we obtain a similar picture of the relation between these VaR bounds.

While in this example the bounds VaR^{W_A} and VaR^{W_B} coincide when using the proxies, in the following example we show that in inhomogeneous cases the difference can be quite big so that VaR^{W_B} is a strong improvement over VaR^{W_A}.

Comparison of VaR^{W_A} and VaR^{W_B}

We consider the case $n = 20$ where the marginals X_i are log-normal distributed. We assume that $C_{i,j}(u_i, u_j)$ is a t-copula with three degrees of freedom and correlation ϱ. The risks X_i are divided into two groups of equal size 10. Within the groups the rv's are pairwise comonotone, i. e. $\varrho = \varrho_1 = 1$ and between the groups the rv's are pairwise independent, i. e. $\varrho = \varrho_2 = 0$.

In this case the sup in (3.3) is attained by the tree which uses only once the correlation $\varrho_2 = 0$. On the other hand $\mathrm{VaR}^{W_A}_\alpha$ can be seen as an average over all starwise trees which also contains trees which use several times the low correlation connections with $\varrho_2 = 0$. This construction makes the difference between both bounds in a particular way big. We find in Figure 3.2 a) that in this case $\mathrm{VaR}^{W_B}_\alpha$



a) $\varrho_1 = 1$, $\varrho_2 = 0$ b) $\varrho_1 = 0.9$, $\varrho_2 = 0.1$

Figure 3.2 Comparison of $\mathrm{VaR}_\alpha^{W_A}$, $\mathrm{VaR}_\alpha^{W_B}$, VaR_α^{D}, inhomogeneous case $C_{i,j}$ t-copula

is strongly improved compared to the VaR bound $\mathrm{VaR}_\alpha^{W_A}$. For example we obtain $\mathrm{VaR}_{0.9}^{W_B} = 99.5875$ which is about 50 % better than $\mathrm{VaR}_{0.9}^{W_A} = 202.6817$. The difference between the bounds is increasing in α. For $\alpha = 0.99$ we have for example $\mathrm{VaR}_{0.99}^{W_B} = 257.1075$ an improvement of 59 % over $\mathrm{VaR}_{0.99}^{W_A} = 437.2221$. $\mathrm{VaR}_\alpha^{W_B}$ improves over the dual bound VaR_α^{D} whereas $\mathrm{VaR}_\alpha^{W_A}$ is worse than the dual bound.

In Figure 3.2 b) we see that under slightly weaker differences for the correlations with $\varrho_1 = 0.9$ and $\varrho_2 = 0.1$ the dual bound VaR_α^{D} is better than the Bonferroni bounds $\mathrm{VaR}_\alpha^{W_A}$ and $\mathrm{VaR}_\alpha^{W_B}$ indicating again a weakness of the method of improved standard bounds. While the Fréchet bounds for the df's improve considerably by inclusion of two dimensional marginals, the corresponding VaR bounds for the aggregated sums only improve in certain cases which exhibit strong enough positive dependence.

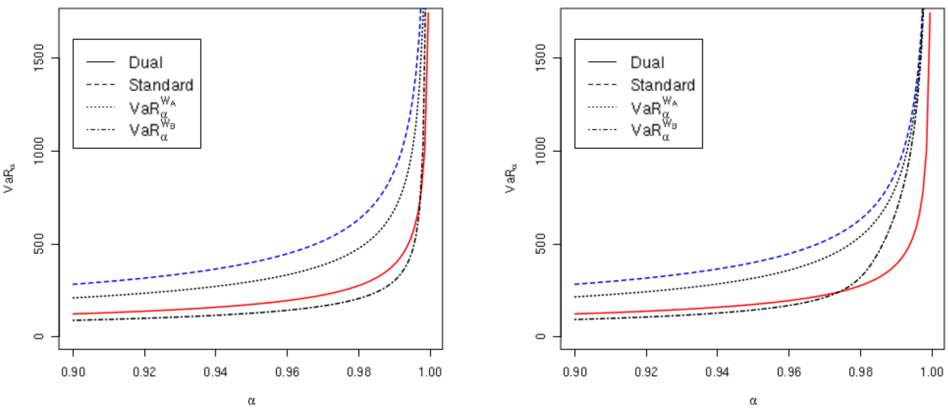
In the following example we compare the bounds for a set of heavy tailed marginal distributions and a different set of bivariate copulas.

Comparison of VaR bounds with bivariate Clayton copula

We assume that $n = 20$ and X_i are Pareto(2)-distributed, i. e. $F_{X_i}(x) = 1 - x^{-2}$, $x \geq 1$. We assume that $C_{i,j}(u_i, u_j)$ is a Clayton copula with parameter Θ. Note that for $\Theta \to \infty$ the Clayton copula approaches comonotonicity while for $\Theta \to 0$ it approaches independence. As in the third example we consider the case that the risks are divided into two groups. Within the groups the risks are approximatively comonotone (strongly positive dependent), i. e. the Clayton parameter $\Theta = \Theta_1$ is big. Between the groups the risks are approximatively independent, i. e. the Clayton parameter $\Theta = \Theta_2$ is small. This construction allows us to investigate the behaviour of the various VaR bounds in dependence of the dependence parameter Θ of the copulas.

In Figure 3.3 and Table 3.2 we consider the choice $\Theta_1 = 10\,000$, $\Theta_2 = 0.1$ in a) and $\Theta = 1\,000$, $\Theta_2 = 1$ in b). As in the case of log-normal distributions we find that the Bonferroni bound $\mathrm{VaR}_\alpha^{W_B}$ is significantly better than $\mathrm{VaR}_\alpha^{W_A}$ and in particular improves the standard bound VaR_α^{S}.

In case $\Theta_1 = 10\,000$ and $\Theta_2 = 0.1$ the dual bound VaR_α^{D} improves on the Bonferroni bound $\mathrm{VaR}_\alpha^{W_B}$ for $\alpha \geq 0.9975 = \alpha_0$. Experience of further examples shows that this turning point moves to smaller values of α, the smaller the dependence parameter Θ_1 gets. For example, for $\Theta_1 = 1\,000$ and $\Theta_2 = 1$ the turning point is



a) $\Theta_1 = 10\,000$, $\Theta_2 = 0.1$ b) $\Theta_1 = 1\,000$, $\Theta_2 = 1$

Figure 3.3 Comparison of VaR bounds, $n = 20$, Pareto(2)-marginals, bivariate Clayton copula

			$\Theta_1 = 10\,000$, $\Theta_2 = 0.1$		$\Theta_1 = 1\,000$, $\Theta_2 = 1$	
α	VaR_α^S	VaR_α^D	$\mathrm{VaR}_\alpha^{W_A}$	$\mathrm{VaR}_\alpha^{W_B}$	$\mathrm{VaR}_\alpha^{W_A}$	$\mathrm{VaR}_\alpha^{W_B}$
0.9	282.842	123.288	209.452	88.717	214.864	93.168
0.99	894.427	389.871	684.720	301.371	813.773	676.727
0.999	2 828.427	1 232.883	2 574.672	2 141.456	2 797.193	2 764.304

Table 3.2 Comparison of VaR bounds, $n = 20$, Pareto(2)-marginals, Clayton copulas with parameter Θ_1 and Θ_2 for $\alpha \geq 0.9$

$\alpha_0 = 0.975$. For $\alpha > \alpha_0$ the dual bounds are better than the Bonferroni bounds if the model is in enough distance to the comonotonic case.

As general conclusion of the examples in this section we obtain that the Bonferroni bound VaR_α^B and the dual bound VaR_α^D improve upon the standard bound VaR_α^S. $\mathrm{VaR}_\alpha^{W_B}$ also improves generally on $\mathrm{VaR}_\alpha^{W_A}$. The Bonferroni Bound VaR^{W_B} improves for high degree of positive dependence on the dual bound VaR^D but for weaker forms of positive dependence the dual bound may be preferable. It should be noted however that the dual bound is typically only calculable for small dimensions for inhomogeneous cases. In these cases however the rearrangement algorithm can be applied to yield sharp marginal bounds. In our applications we used proxies for the calculation of the Bonferroni bounds. These were shown above to be sharp under some conditions.

4 Improved Hoeffding–Fréchet bounds under dependence restrictions

For a random vector $X = (X_1, \dots, X_n)$ with marginals $F_1, \dots, F_n$ we assume that some additional dependence information is known which are described in the following way. We assume that $G : \mathbb{R}^n \to \mathbb{R}^1$ is an increasing function such that

$$\underline{F}(x) \leq G(x) \leq \overline{F}(x), \quad \forall x \in \mathbb{R}^n, \tag{4.1}$$

where $\underline{F}(x) = \left(\sum_{i=1}^n F_i(x) - (n-1)\right)_+$ and $\overline{F}(x) = \min_{1 \leq i \leq n} F_i(x_i)$ are the lower and upper Fréchet bounds. We assume that for some subset $S \subset \mathbb{R}^n$ it is known that $F = F_X \le G$ or that $F \ge G$ or that $F = G$ on S. Under this dependence assumption one can determine improved Hoeffding–Fréchet bounds on F. These improved bounds can be used to derive by the method of improved standard bounds in (2.4) and (2.5) improved VaR bounds for the aggregated portfolio.

In the case $n = 2$ improved Hoeffding–Fréchet bounds have been derived in Rachev and Rüschendorf (1994). These bounds were rederived in the case of uniform marginals, (i. e. for copulas) in Tankov (2011) for the case of equality constraints, where also a sharpness result for increasing sets S is given. Also an application to model free pricing bounds for multi-asset options is given there. For the case $|S| = 1$ and $n = 2$ sharpness of this bound is shown in Nelsen et al. (2004). Extensions of the sharpness result are in Bernard et al. (2012). Bernard et al. (2013a) discuss as application the case where S is the central part of the distribution. As to be expected in this case one only obtains improvements of the VaR bounds for small values of α (see Figure 4.1).

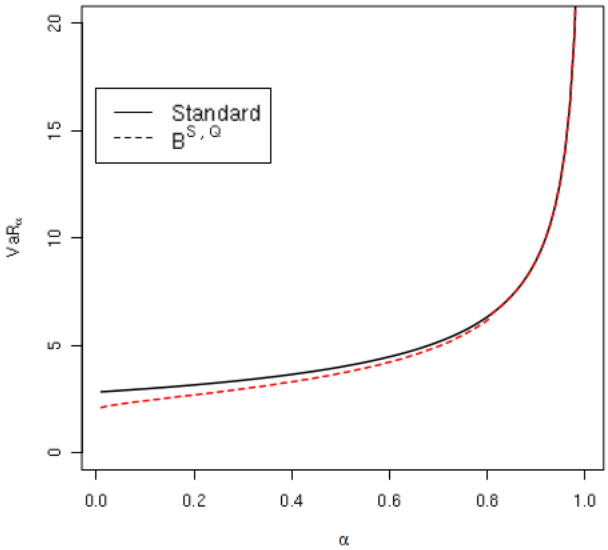


Figure 4.1 Comparison of $\mathrm{VaR}_\alpha^{B^{S,Q}}$ and VaR_α^S, $F_i = \text{Pareto}(2)$, $i = 1, 2$.

To any function G as above we define

$$F^*(x) = \min\Big(\min_{1\le i\le n} F_i(x), \inf_{y\in S}\Big\{G(y) + \sum_{i=1}^n (F_i(x_i) - F_i(y_i))_+\Big\}\Big) \tag{4.2}$$

and

$$F_*(x) = \max\Big(\sum_{i=1}^n F_i(x_i) - (n-1), \sup_{y<\in S}\Big\{G(y) - \sum_{i=1}^n (F_i(y_i) - F_i(x_i))_+\Big\}\Big). \tag{4.3}$$

The following results extends the improved Hoeffding–Fréchet bounds from the two-dimensional case to general $n \ge 2$.

Theorem 4.1 (Improved Hoeffding–Fréchet bounds with dependence restrictions). *Let $S \subset \mathbb{R}^n$ and let G be an increasing real function on $\mathbb{R}^n$ satisfying* (4.1)*. Further let $F \in \mathcal{F}(F_1, \dots, F_n)$ be a distribution function with marginals F_i, then*

(i) If $F(y) \le G(y)$ for all $y \in S$, then $F(x) \le F^*(x)$, for all $x \in \mathbb{R}^n$.

(ii) If $F(y) \ge G(y)$ for all $y \in S$, then $F(x) \ge F_*(x)$, for all $x \in \mathbb{R}^n$.

(iii) If $F(y) = G(y)$ for all $y \in S$, then $F_*(x) \le F(x) \le F^*(x)$ for all $x \in \mathbb{R}^n$.

Proof. (i) Let $X = (X_1, \dots, X_n)$ be a random vector with df F and w. l. g. let $y \in S$ satisfy $y_i \le x_i$ for $1 \le i \le n$. Then using the assumption $F(y) \le G(y)$ for $y \in S$ we obtain

$$\begin{aligned} F(x) &= P(X_1 \le x_1, \dots, X_n \le x_n) \\ &= P(X_1 \le y_1, X_2 \le x_2, \dots, X_n \le x_n) \\ &\quad + P(y_1 < X_1 \le x_1, X_2 \le x_2, \dots, X_n \le x_n) \\ &= P(X_1 \le y_1, X_2 \le y_2, X_3 \le x_3, \dots, X_n \le x_n) \\ &\quad + P(y_1 < X_1 \le x_1, X_2 \le x_2, \dots, X_n \le x_n) \\ &\quad + P(X_1 \le y_1, y_2 < X_2 \le x_2, X_3 \le x_3, \dots, X_n \le x_n) \\ &\;\vdots \\ &= P(X_1 \le y_1, \dots, X_n \le y_n) + P(y_1 < X_1 \le x_1, X_2 \le x_2, \dots, X_n \le x_n) \\ &\quad + \dots + P(X_1 \le y_1, \dots, X_{n-1} \le y_{n-1}, y_n < X_n \le x_n) \\ &\le G(y) + \sum_{i=1}^n (F_i(x_i) - F_i(y_i)). \end{aligned}$$

This implies by the classical Hoeffding–Fréchet bounds that $F \le F^*$.

(ii) $F(y) \ge G(y)$ is equivalent to $\overline{F}(y) \le \overline{G}(y)$ where $\overline{F}(y) = 1 - F(y)$. This implies as in (i)

$$\begin{aligned} \overline{F}(x) &\le \min\Big(\min_{i=1,\dots,n} \overline{F_i}(x_i), \inf_{y\in S} \Big\{ \overline{G}(y) + \sum_{i=1}^n (\overline{F_i}(x_i) - \overline{F_i}(y_i))_+ \Big\}\Big) \\ &= \min\Big(\min_{i=1,\dots,n} (1 - F_i(x_i)), \inf_{y\in S} \Big\{ 1 - G(y) + \sum_{i=1}^n (F_i(y_i) - F_i(x_i))_+ \Big\}\Big) \\ &= -\max\Big(\max_{i=1,\dots,n} (F_i(x_i) - 1), \sup_{y\in S} \Big\{ G(y) - 1 - \sum_{i=1}^n (F_i(y_i) - F_i(x_i))_+ \Big\}\Big), \end{aligned}$$

and, therefore, obtain

$$\begin{aligned} F(x) &\ge \max\Big(\sum_{i=1}^n F_i(x_i) - (n-1), \sup_{y\in S} \Big\{ G(y) - \sum_{i=1}^n (F_i(y_i) - F_i(x_i))_+ \Big\}\Big) \\ &= F_*(x). \end{aligned}$$

(iii) is a consequence of (i) and (ii). □

Remark 4.2. *1) If X is positive lower orthant dependent (PLOD), i.e. it holds $F(x) \ge \prod_{i=1}^n F_i(x_i) =: G(x)$ for all $x \in S = \mathbb{R}^n$, then*

$$\sup_{y\in\mathbb{R}^n} \Big\{ \prod_{i=1}^n F_i(y_i) - \sum_{i=1}^n (F_i(y_i) - F_i(x_i))_+ \Big\} = \prod_{i=1}^n F_i(x_i) = G(x) \tag{4.4}$$

and as consequence $F_*(x) = \prod_{i=1}^n F_i(x_i) = G(x)$ coincides with G and is a sharp lower bound. Similarly if $G \in \mathcal{F}(F_1, \dots, F_n)$ and $S = \mathbb{R}^n$, then the improved Hoeffding–Fréchet bounds coincide with G and are sharp, i. e. $F^* = G$ under condition (i) and $F_* = G$ under condition (ii).

2) In the particular case where $F_i \sim U[0,1]$, $1 \le i \le n$, Theorem 4.1 implies improved bounds for the copulas.

Corollary 4.3 (Improved copula bounds)**.** *Let $S \subset [0,1]^n$ and let Q be an increasing function on $[0,1]^n$ such that*

$$\Big(\sum_{i=1}^n u_i - (n-1)\Big)_+ \leq Q(u) \leq \min\{u_i, 1 \leq i \leq n\}. \tag{4.5}$$

Define

$$A^{S,Q}(u) := \min\Big(\min_{i=1,\dots,n} u_i, \min_{a\in S}\Big\{Q(a) + \sum_{i=1}^n (u_i - a_i)_+\Big\}\Big), \tag{4.6}$$

$$B^{S,Q}(u) := \max\Big(\sum_{i=1}^n u_i - (n-1), \max_{a\in S}\Big\{Q(a) - \sum_{i=1}^n (a_i - u_i)_+\Big\}\Big), \tag{4.7}$$

then for any copula C with $C(a) = Q(a)$ for $a \in S$ holds

$$B^{S,Q}(u) \leq C(u) \leq A^{S,Q}(u), \quad \forall u \in [0,1]^n \tag{4.8}$$

and equality holds in (4.8) *for $u \in S$.*

As is clear from Remark 4.2 the improved Hoeffding–Fréchet bounds may be considerable improvements of the classical Hoeffding–Fréchet bounds and thus may lead to strongly improved VaR bounds for the aggregated risk by the method of improved standard bounds. The degree of improvement depends on the dependence information described by S and G.

Known central domain

The first example is motivated by Bernard et al. (2013a) and Bernard and Vanduffel (2015). For a portfolio it is assumed that the distribution is known by statistical analysis in the central domain of the distribution while generally only marginals are known. How much does the knowledge of the central part contribute to reduce VaR bounds?

As model example to investigate this effect we consider the case $n = 2$ with $F_X = F_Y = F$ a Pareto(2)-distribution. Let the central part S of the copula be given as $S = [0, 0.9]^2$ and assume that Q is given as product copula on S, i. e. $Q(a, b) = ab$, $(a, b) \in S$, i. e. on the central part of the distribution the risks are independent.

In this case the bound $B^{S,Q}$ is a sharp bound for the joint distribution function $F_{(X,Y)}$ (see Bernard et al. (2012, 2013a)).

As consequence we obtain from the method of improved standard bounds in Section 3

$$\begin{aligned} P(X+Y \leq s) &\geq \sup_{(u,v)\in\mathcal{U}(s)} B^{S,Q}(F(u), F(v)) \\ &= B^{S,Q}\Big(F\Big(\frac{s}{2}\Big), F\Big(\frac{s}{2}\Big)\Big), \end{aligned} \tag{4.9}$$

where $\mathcal{U}(s) = \{(u,v) \in \mathbb{R}^2 : u + v = s\}$. By inversion this implies improved bounds for the VaR of the aggregated risk $X + Y$, which we call $\mathrm{VaR}_\alpha^{B^{S,Q}}$.

Figure 4.1 shows that one gets a strict improvement of the standard bound VaR_α^S only for small levels $\alpha \leq 0.82$. This is no surprise since the central domain S does not bear much information on large quantiles of the sum, so that $B^{S,Q}$ is close to the Hoeffding–Fréchet bound in the case of large quantiles $\alpha \geq 0.9$.

Based on Theorem 4.1 and Corollary 4.3 a similar effect can be found also for high dimensional portfolios, i. e. the knowledge of the central part of the distribution only helps to improve VaR bounds for moderate quantiles α but not for large quantiles.

Positive dependence in the tails

In this example we consider as in the fourth example in Section 3 the case $n = 2$, $F_1 = F_2 = \text{Pareto}(2)$ where $S = [0.9, 1]^2$. We assume that in the extreme tail domain the copula Q is comonotonic, i. e. for a copula vector $(U_1, U_2) \sim Q$ holds

$$P(U_1 \geq u_1, U_2 \geq u_2) = \min(1 - u_1, 1 - u_2), \quad u_i \geq 0.9.$$

This models a case where in extreme situations a strong form of positive dependence arises. As consequence of this strong positive dependence in the tails we obtain from Corollary 4.3 and Theorem 4.1 a remarkable reduction of the improved VaR bounds $\text{VaR}_\alpha^{B^{S,Q}}$ for moderate and in particular for high quantile levels α (see Figure 4.2).

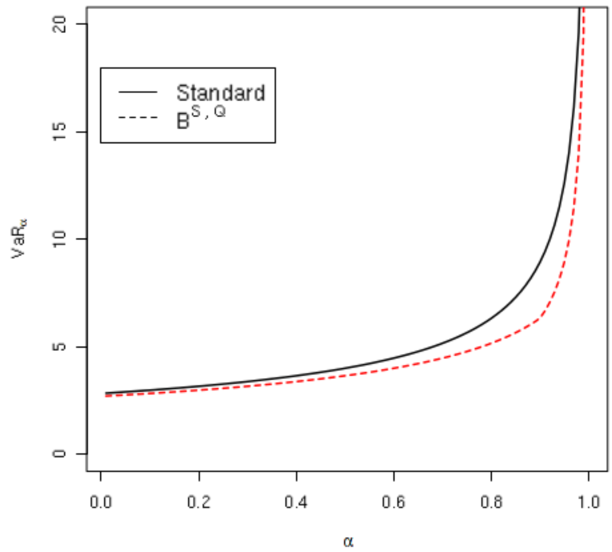


Figure 4.2 Comparison of $\text{VaR}_\alpha^{B^{S,Q}}$ and the standard bound VaR_α^S for $n = 2$ with Pareto(2) marginals.

Based on Corollary 4.3 a similar effect also holds in the case that $n \geq 2$. The assumption of comonotonicity in the tails is a strong assumption. It is expected that one obtains somewhat reduced effects of a similar form, when weakening the positive dependence assumption in the tails by a weaker assumption of the form $\overline{Q}(u) = 1 - Q(u) \geq G(u)$ for $u \in S$, where S is a tail area and G is a decreasing function ensuring positive dependence in the tail. A more detailed investigation of this situation is planned for future research.

Independent subgroups with positive internal dependence

In this example we modify the model assumption investigated in Bignozzi et al. (2015). We consider the case that the risks are split into k independent subgroups I_j. Bignozzi et al. (2015) allow any kind of dependence within these subgroups. In comparison we assume that the risks within the subgroups are strongly positive dependent (comonotonic) in the tails, i. e., similar as in the fourth example in Section 3 on $[0.9, 1]^{n_i}$, where $n_i = |I_i|$.

As concrete example we consider the case where $n = 20$, with $k = 1, 10, 20$ subgroups, where the subgroup sizes are equal to $\frac{20}{k}$. We further assume that $F_i = \text{Pareto}(2) = F$, $1 \leq i \leq n$. As consequence of Theorem 4.1 and Corollary 4.3

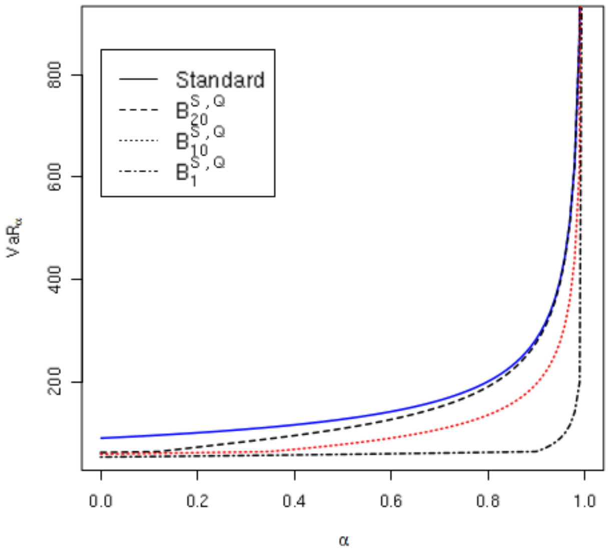


Figure 4.3 Comparison of $\mathrm{VaR}_\alpha^{B_k^{S,Q}}$ for $k = 1, 10, 20$ and standard bound VaR_α^S, $n = 20$, $F_i = \mathrm{Pareto}(2)$.

we obtain

$$P\Big(\sum_{i=1}^n X_i \le s\Big) \ge B_k^{S,Q}\Big(F\Big(\frac{s}{n}\Big), \dots, F\Big(\frac{s}{n}\Big)\Big)$$
$$= \max\Big(nF\Big(\frac{s}{n}\Big) - (n-1), \max_{a\in S}\Big\{Q(a) - \sum_{i=1}^n \Big(a_i - F\Big(\frac{s}{n}\Big)\Big)_+\Big\}\Big),$$

where $S = [0.9, 1]^n$ and $Q(a) := \prod_{j=1}^k \min_{i\in I_j} a_i$. The corresponding VaR bounds $\mathrm{VaR}_\alpha^{B_k^{S,Q}}$ are obtained by inversion and are given in Figure 4.3.

The results obtained can be expected. The worst bound is the standard bound. The best bound is obtained for the case $k = 1$ of general comonotonicity in the tails. The case of 10 independent subgroups with positive tail dependence leads to a considerable reduction. As in the previous example it is of interest to describe this kind of positive dependence effects under weaker assumptions on the used notion of positive dependence. This will be subject of a further study.

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