Improved Hoeffding–Fréchet bounds and applications to VaR estimates

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Abstract

The classical Fréchet bounds determine upper and lower bounds for the distribution function F of a random vector X, when the marginal df's F_i are fixed. As consequence these bounds imply also upper and lower bounds for the expectation $E\varphi(X)$ of a certain class of functions $\varphi(X)$. The classical examples are the Hoeffding bounds for the expectation of the product EX_1X_2 of two random variables. In this paper we review and partially elaborate on several developments of improved Hoeffding–Fréchet bounds which assume some restriction on the dependence structure additional to the information on the marginals. We describe applications of the results to obtain improved VaR bounds for the joint portfolio of risk vectors. We consider in particular improved VaR bounds in the case where information of the joint distribution function resp. on the copula is available on some subsets and the case where higher order marginal information is available.

Key-words: Hoeffding–Fréchet bounds, VaR estimates, copulas, positive dependence

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1 Hoeffding–Fréchet bounds

The classical Fréchet bounds are one of the most prominent results in stochastic ordering. They can be stated in the following form. For an n-dimensional df F holds: $F \in \mathcal{F}(F_1, \dots, F_n)$ – the Fréchet class of n-dimensional df's with marginals $F_1, \dots, F_n$ – if and only if

$$F_- \le F \le F_+, \tag{1.1}$$

where $F_+(x) := \min_{1\le i\le n} F_i(x_i)$ and $F_-(x) := \max\left\{0, \sum_{i=1}^n F_i(x_i) - (n-1)\right\}$ are the upper and lower Fréchet bounds. While F_+ is in general a df and thus $F_+ \in \mathcal{F}(F_1, \dots, F_n)$, it holds that $F_- \in \mathcal{F}(F_1, \dots, F_n)$ only for $n = 2$ and for rare cases when $n \ge 3$. These cases were characterized in Dall'Aglio (1972). F_+ is denoted the comonotonic distribution, F_- is called in case $n = 2$ the antimonotonic distribution.

The inequalities in (1.1) imply some integral inequalities. Let for a real function $\varphi = \varphi(x_1, \dots, x_n)$ of n variables

$$\begin{aligned} M(\varphi) &:= \sup\left\{ \int \varphi dP; P \in M(P_1, \dots, P_n)\right\} \\ \text{and} \quad m(\varphi) &:= \inf\left\{ \int \varphi dP; P \in M(P_1, \dots, P_n)\right\} \end{aligned} \tag{1.2}$$

denote the generalized Hoeffding–Fréchet functionals, where P_i are probability measures on some measurable spaces $(E_i, \mathcal{A}_i)$. Since Hoeffdings paper from 1940 belongs to the earliest papers on the bounds in (1.1) we call bounds of this type in the following invariably Fréchet bounds or Hoeffding–Fréchet bounds. In particular this includes the case where $(E_i, \mathcal{A}_i) = (\mathbb{R}^1, \mathcal{B}^1)$ and P_i have distribution functions F_i.

A basic consequence of the Fréchet bounds in (1.1) is the following result on the supermodular ordering of distributions. Define for $P, Q \in M(P_1, \dots, P_n)$ – i. e. P and Q have marginals $P_1, \dots, P_n$ –

$$P \le_{\mathrm{sm}} Q \text{ if } \int f dP \le \int f dQ \tag{1.3}$$

for all supermodular functions $f \in \mathcal{F}_{\mathrm{sm}}$ such that the integrals exist. Let $\le_{\mathrm{uo}}$ denote the upper orthant ordering, i. e. $P \le_{\mathrm{uo}} Q$ if $P([a,\infty)) \le Q([a,\infty))$ for all $a \in \mathbb{R}^n$. Then the following result holds

Theorem 1.1 (Supermodular order). *Let $P, Q \in M^1(\mathbb{R}^n, \mathcal{B}^n)$, then*

a) In case $n = 2$ it holds:

$$P \le_{\mathrm{sm}} Q \Leftrightarrow P \le_{\mathrm{uo}} Q \tag{1.4}$$

b) ***Lorentz Theorem****: For any $P \in M(P_1, \dots, P_n)$ holds*

$$P \le_{\mathrm{sm}} P_+, \tag{1.5}$$

where $P_+ \sim F_+$ is the comonotonic probability measure with marginals P_i.

The characterization of the supermodular ordering by the upper orthant oder $\leq_{\text{uo}}$ in a) is due to Cambanis et al. (1976). It generalizes in particular the classical Hoeffding bounds for the expectation of the product of two random variables. Let $X \sim F$, $Y \sim G$, and $U \sim U(0,1)$, then

$$EF^{-1}(U)G^{-1}(1-U) \leq EXY \leq EF^{-1}(U)G^{-1}(U). \tag{1.6}$$

Part b) was proved in Tchen (1980) by discrete approximation and reduction to the Lorentz (1953) inequalities. In Rüschendorf[1] (1979, 1983) the problem to determine the generalized Hoeffding–Fréchet functional was identified with a rearrangement problem for functions. The Lorentz Theorem was reduced to the Lorentz inequality for functions.

There is also an analogue of the Fréchet bounds in (1.1) for the survival functions $\overline{F}_i(x_i) = P(X_i \geq x_i)$ and $\overline{F}(x_1, \dots, x_n) = P(X_i \geq x_i, 1 \leq i \leq n)$

$$\overline{F}^-(x) := \Big(\sum_{i=1}^n \overline{F}_i(x_i) - (n-1)\Big)_+ \leq \overline{F}(x_1, \dots, x_n) \leq \min_{1 \leq n} \overline{F}_i(x_i) := \overline{F}^+(x). \tag{1.7}$$

This version of the Fréchet bounds leads to an ordering result for the class of Δ-monotone (also called n-increasing) functions by means of a partial integration formula

Theorem 1.2 (Δ-monotone ordering)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^1$ be Δ-monotone and assume that for any $1 \leq i \leq n$, $\lim_{x_i \to -\infty} f(x_1, \dots, x_i, \dots, x_n) = 0$, $\forall x_j$, $j \neq i$,*

a) For any $F \in \mathcal{F}(F_1, \dots, F_n)$ holds

$$\int \overline{F}^-(x) df(x) \leq \int f dF \leq \int \overline{F}^+(x) df(x). \tag{1.8}$$

b) If $F \in \mathcal{F}(F_1, \dots, F_n)$ and $\overline{G}, \overline{H}$ are decreasing functions with

$$\overline{F}^- \leq \overline{G} \leq \overline{F} \leq \overline{H} \leq \overline{F}^+, \tag{1.9}$$

then

$$\int \overline{G}(x) df(x) \leq \int f dF \leq \int \overline{H}(x) df(x). \tag{1.10}$$

For part a) see Rü (2004, Theorem 5.5). Part b) is a direct consequence of the proof of Part a) and the assumptions in Part b).

Remark 1.3. a) *Convex ordering of risk portfolios.* Taking functions of the form $\varphi(x) = \Psi(\sum_{i=1}^n x_i)$, Ψ convex, we obtain that $\varphi \in \mathcal{F}_{\text{sm}}$ and the Lorentz Theorem implies that for any random vector $X = (X_1, \dots, X_n)$, $X_i \sim F_i$ it holds that

$$\sum_{i=1}^n X_i \leq_{\text{cx}} \sum_{i=1}^n X_i^c, \tag{1.11}$$

[1] In the further text Rüschendorf is abbreviated as Rü.

where $X^c = (X_i^c)$ is a comonotonic vector with comonotonic distribution P_+; a result due to Meilijson and Nadas (1979). In risk theory this result implies that X^c is the worst case dependence structure for the joint portfolio for any convex risk measure ϱ. Further recent results on the convex ordering of risk portfolios as well as to VaR bounds of joint portfolios are described in Puccetti and Wang (2015).

b) *Parameter free price bounds under dependence constraints.* Part b) in Theorem 1.2 implies that positive or negative dependence constraints on the survival function $\overline{F}$ in terms of the upper orthant ordering $\leq_{\mathrm{uo}}$ imply directly improved parameter free bounds for the value of options defined by Δ-monotone functions.

Dependence restrictions as in (1.8) depend typically only on the copulas. They are closely related to modelling of dependence structures and to various specific constructions and bounds on copulas. A wealth of relevant material on these kind of theory and models is given in the by now classical book of Nelsen (2006) as well as in the recent book on copulas of Durante and Sempi (2016).

In the following Section 2 we discuss various forms of dual representations which are available to deal with the Hoeffding–Fréchet functional for general aggregation functions φ and also allow to include dependence information.

In Section 3 we consider the case where additional information on the dependence structure is available on some part of the domain. Finally in Section 4 we discuss additional information on the dependence structure by including second order marginal information. We discuss in particular applications to the problem of establishing VaR-bounds for the joint portfolio.

2 Dual representation of Hoeffding–Fréchet bounds

The most relevant and general information on the generalized Hoeffding–Fréchet functional is given by the dual representation of these functionals. The basic duality theorem states under some general conditions on φ equality of $M(\varphi)$ with a dual functional $U(\varphi)$. For detailed conditions see Rü (1991a, 2007):

Duality Theorem:

$$M(\varphi) = U(\varphi) := \inf\Big\{\sum_{i=1}^n \int f_i dP_i; \sum_{i=1}^n f_i(x_i) \geq \varphi(x_1,\dots,x_n)\Big\}. \tag{2.1}$$

Similarly,

$$m(\varphi) = I(\varphi) := \sup\Big\{\sum_{i=1}^n \int f_i dP_i; \sum_{i=1}^n f_i \leq \varphi\Big\}. \tag{2.2}$$

Remark 2.1. *Some history:* The duality result was proved in Rü (1979, 1981) and Gaffke and Rü (1981) including existence of solutions for the case where φ is bounded continuous. For the case of bounded measurable functions it was shown in these papers that replacing the σ-additive measures by finitely additive measures with marginals P_i and defining

$$\widetilde{M}(\varphi) := \sup\Big\{ \int \varphi d\mu; \mu \in \mathrm{ba}(P_1, \dots, P_n)\Big\}$$

one gets

$$\widetilde{M}(\varphi) = U(\varphi). \tag{2.3}$$

This is a consequence of the Hahn–Banach separation theorem combined with Riesz representation theorem. Under suitable regularity on the spaces and on φ one obtains that $\widetilde{M}(\varphi) = M(\varphi)$. These duality results were then extended to some general classes of functions based on continuity properties of the functionals U, I and on the Choquet capacity theorem in Kellerer (1984).

For a survey of these developments see Rü (1991b, 2007) or Rachev and Rü (1998). It should be noted that in case $n = 2$ the duality result in (2.1) and (2.2) was the first instance of the Kantorovich duality theorem for mass transportation for *general* functionals φ. Kantorovich (1942) had established his duality result in the case where $\varphi(x_1, x_2)$ is a metric on a compact space.

As consequence of the duality theorem some basic inequalities and bounds were obtained, as for example the following result (see Rü (1981)).

Define for $A \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$, $M(A) := M(\mathbb{1}_A)$, then:

Theorem 2.2 (Sharpness of Fréchet bounds)**.** *For any $A_i \in \mathcal{A}_i$, $1 \le i \le n$ holds*

$$M(A_1 \times \cdots \times A_n) = \min\{P_i(A_i); 1 \le i \le n\} \tag{2.4}$$

$$\text{and} \quad m(A_1 \times \cdots \times A_n) = \min\Big(\sum_{i=1}^n P_i(A_i) - (n-1)\Big)_+. \tag{2.5}$$

In particular (2.5) was the first proof of the sharpness of the lower Fréchet bounds in (1.1). An alternative proof was given in Sklar (1998) (see Nelsen (2006, Theorem 2.10.13)).

A consequence of the duality theorem (or alternatively of Strassens's Theorem) is also a formula for the maximal and minimal value of the distribution function of the sum in case $n = 2$ due to Makarov (1981) and Rü (1982):

$$\begin{aligned} M(s) &:= \sup\{P(X_1 + X_2 \le x); X_1 \sim F_i\} \\ &= \inf_{x \in \mathbb{R}} (F_1(x) + F_2(t - x)) =: F_1 \wedge F_2(t) \qquad (2.6) \\ m(s) &:= \inf\{P(X_1 + X_2 < s); X_i \sim F_i\} \\ &= 1 - \sup_{x \in \mathbb{R}} (\overline{F}_1(x) + \overline{F}_2(t - x) := 1 - \overline{F}_1 \vee \overline{F}_2(t) \qquad (2.7) \end{aligned}$$

This implies by inversion sharp bounds for the Value at Risk (VaR) in case $n = 2$.

Embrechts and Puccetti (2006a,b) relaxed the dual representation by restricting to admissible piecewise linear functions of a simple form. This way they establish the following dual bounds:

$$M(s) \leq D(s) = \inf_{u \in \overline{U}(s)} \min \left\{ \frac{\sum_{i=1}^n \int_{u_i}^{s-\sum_{j\neq i} u_j} \overline{F}_i(t)dt}{s - \sum_{i=1}^n u_i}, 1 \right\} \tag{2.8}$$

$$\text{and } m(s) \geq d(s) = \sup_{u \in \underline{U}(s)} \max \left\{ \frac{\sum_{i=1}^n \int_{u_i}^{s-\sum_{j\neq i} u_j} \overline{F}_i(t)dt}{s - \sum_{i=1}^n u_i} - d + 1, 0 \right\} \tag{2.9}$$

where $\overline{U}(s) = \big\{u \in \mathbb{R}^n; \sum_{i=1}^n u_i < s\big\}$ and $\underline{U}(s) = \big\{u \in \mathbb{R}^n; \sum_{i=1}^n u_1 > s\big\}$.

In the homogeneous case these dual bounds simplify strongly and can be shown under some mixing conditions to be sharp bounds (see Puccetti and Rü (2012)).

The method of proving the duality theorem described above is flexible enough to be able to handle also additional constraints. Some examples of additional constraints have been considered in Ramachandran and Rü (1997, 1999, 2002) who considered e. g. upper or lower bounds on the marginals, restrictions on the domain of the admissible measures or upper local bounds on the class of admissible measures. In a similar way also other types of constraints can be dealt with this approach. Consider for example an additional positive upper orthant dependence assumption and define

$$M_{\text{PUOD}}(P_1, \dots, P_n) = \{P \in M(P_1, \dots, P_n) : P \text{ is positive upper orthant dependent}\}. \tag{2.10}$$

Here P is called positive upper orthant dependent (PUOD) if $P^\perp \leq_{\text{uo}} P$, where $P^\perp = \otimes_{i=1}^n P_i$.

Let $\mathcal{F}_\Delta$ denote the cone of Δ-monotone functions, then for $P \in M(P_1, \dots, P_n)$ holds:

$$P \text{ is PUOD iff } P^\perp \leq_{\mathcal{F}_\Delta} P \text{ i. e. } \int f dP^\perp \leq \int f dP \text{ for all } f \in \mathcal{F}_\Delta. \tag{2.11}$$

For a duality statement we consider as above the modified Hoeffding–Fréchet problem with finitely additive measures

$$\widetilde{M}_{\text{PUOD}}(P_1, \dots, P_n) = \{\mu \in \text{ba}(P_1, \dots, P_n); \mu \text{ is PUOD}\} \tag{2.12}$$

and for bounded measurable φ

$$\widetilde{M}_{\text{PUOD}}(\varphi) = \sup\{\varphi d\mu; \mu \in \widetilde{M}_{\text{PUOD}}(P_1, \dots, P_n)\}. \tag{2.13}$$

There are several possible classes of dual problems. Define

$$\mathcal{H}_1 = \Big\{((f_i), g); g \in \mathcal{F}_\Delta, f_i \in \mathcal{L}^1(P_i), \sum_{i=1}^n f_i - g \geq \varphi\Big\}.$$

Theorem 2.3 (Dual representation with PUOD constraints)**.** *For any bounded measurable function φ holds:*

$$\widetilde{M}_{\mathrm{PUOD}}(\varphi) = I_1(\varphi) := \inf\Big\{\sum_{i=1}^n \int f_i dP_i - \int g d \otimes_{i=1}^n P_i; ((f_i), g) \in \mathcal{H}_1\Big\} \tag{2.14}$$

Proof. For any $\mu \in \widetilde{M}_{\mathrm{PUOD}}(P_1, \dots, P_n)$ and any $g \in \mathcal{F}_\Delta$ holds by (2.11)

$$\int g d\mu \geq \int g d \otimes_{i=1}^n P_i.$$

This implies for any $((f_i), g) \in \mathcal{H}_1$

$$\begin{aligned}\sum_{i=1}^n \int f_i dP_i - \int g d \otimes_{i=1}^n P_i &\geq \sum_{i=1}^n \int f_i dP_i - \int g d\mu \\ &= \int \Big(\sum_{i=1}^n f_i - g\Big) d\mu \geq \int \varphi d\mu.\end{aligned}$$

As consequence we get

$$\widetilde{M}_{\mathrm{PUOD}}(\varphi) \leq I_1(\varphi). \tag{2.15}$$

By Riesz representation theorem any continuous linear functional T on $B(\mathbb{R}^n, \mathcal{B}^n)$ with $Tf_i = \int f_i dP_i$ for $f_i = f_i(x_i) \in B(\mathbb{R}^1, \mathcal{B}^1)$ can be identified with an element $\widetilde{\mu} \in \widetilde{M}(P_1, \dots, P_n)$. Further it holds that

$$\widetilde{\mu} \in \widetilde{M}_{\mathrm{PUOD}}(P_1, \dots, P_n) \Leftrightarrow \widetilde{\mu} \leq I_1.$$

Therefore, the Hahn–Banach separation theorem implies that for any $\varphi \in B(\mathbb{R}^n, \mathcal{B}^n)$

$$\widetilde{M}_{\mathrm{PUOD}}(\varphi) = I_1(\varphi).$$ □

Remark 2.4. *a) Existence and extensions:* The separation theorem also implies the existence of a solution $\widetilde{\mu} \in \mathrm{ba}(P_1, \dots, P_n)$. Restricting to the class of continuous bounded functions C_b on $\mathbb{R}^n$ we obtain by Riesz representation theorem

$$M_{\mathrm{PUOD}}(\varphi) = \widetilde{M}_{\mathrm{PUOD}}(\varphi) = I_1(\varphi), \ \ \varphi \in C_b. \tag{2.16}$$

This duality result can be further extended to more general classes of functions by suitable continuity properties of the functionals M_{PUOD} and I_1 as in the simple marginal case.

b) Reduced dual problem: The duality statements in (2.14) and (2.16) give an exact upper bound for $\int \varphi dP$, $P \in M_{\mathrm{PUOD}}(P_1, \dots, P_n)$ which however as in the simple marginal case in general is not easy to evaluate. For the dual functional we can restrict to a more simple generator of $\mathcal{V}_\Delta$ by restricting to g of the form

$$g(x) = \sum_{i=1}^m \alpha_i 1_{[a_i,\infty)}(x) \quad \text{with } \alpha_i \geq 0.$$

Then the dual problem reduces to optimize

$$\sum \int f_i dP_i - \sum_{i=1}^{m} \alpha_i \overline{F^{\perp}}(a_i) \tag{2.17}$$

over all admissible duals of this form, where $\overline{F^{\perp}}(x) = \prod_{i=1}^{n} P_i([x,\infty))$ is the survival function. In particular, any admissible dual choice yields by (2.14) and (2.15) to an upper bound.

As second example we consider the case where additional to the marginals also the covariances $\sigma_{ij} = \operatorname{Cov}(X_i, X_j) = EX_iX_j - a_ia_j$, $a_i = EX_i$ are specified. In a similar way as above one gets the dual representation for the class $M_\Sigma(P_1, \dots, P_n)$ of measures P with marginals P_i and correlation matrix $\Sigma = (\sigma_{ij})$.

Theorem 2.5 (Fixed correlations). *For any $\varphi \in B(\mathbb{R}^n, \mathcal{B}^n)$ holds*

$$\widetilde{M}_\Sigma(\varphi) = I_2(\varphi) := \inf \Big\{ \sum_{i=1}^{n} \int f_i dP_i + \sum_{(i,j)} \alpha_{ij} s_{ij}; \varphi \le \sum_{i=1}^{n} f_i(x_i) + \Sigma \alpha_{ij} x_i x_j \Big\} \tag{2.18}$$

Remark 2.6. *a)* Similarly as above for $\varphi \in C_b$, $\widetilde{M}_\Sigma(\varphi) = M_\Sigma(\varphi)$ and the duality can be extended to more general classes of functions φ.

b) If we consider as in Bernard et al. (2015) $\varphi = \mathbb{1}_{\{\sum_{i=1}^{n} x_i \ge t\}}$ and assume that it is known that additional to the marginals P_i, also it is known that $\operatorname{VaR}(S_n) \le \sigma^2$, then the dual in (2.18) simplifies strongly to the form

$$\begin{aligned} I_{2,\sigma^2}(\varphi) = \inf \Big\{ & \sum_{i=1}^{n} \int f_i dP_i + \alpha(\sigma^2 - \mu^2); \\ & \varphi(x) \le \sum_{i=1}^{n} f_i(x_i) + \alpha \Big[\Big(\sum_{i=1}^{n} x_i \Big)^2 - \mu^2 \Big], \alpha \ge 0, f_i \in L^1(P_i) \Big\} \end{aligned} \tag{2.19}$$

In Bernard et al. (2015) good upper bounds for this case were given. In contrast formula (2.19) gives theoretically sharp upper bounds.

c) Model independent price bounds: In a similar way the above sketched method also applies to various other types of constraints. For robust model independent price bounds in recent years dual representations with martingale constraints have been developed (see e. g. Acciaio et al. (2013) and Beiglböck et al. (2013)). This kind of constraints is due to the fact, that reasonable pricing measures have the martingale property. Also this type of constraints can be dealt with by the above described method.

3 Improved Hoeffding–Fréchet bounds – distributional information on domains

Motivated by the problem to determine good bounds for the Value at Risk (VaR) of the joint portfolio there has been a lot of recent papers to improve the Fréchet bounds in (1.1) by including additional dependence information and as consequence to obtain improved bounds for the tail risk $P(\sum_{i=1}^n X_i \geq t)$ or on $\mathrm{VaR}_\alpha(\sum_{i=1}^n X_i)$.

For a random vector X with $X_i \sim F_i$ and distribution function $F_X = F$ we can pose positive dependence restrictions in the form that $F \leq G$ or that the survival function $\overline{F} \geq \overline{G}$ for some increasing (resp. decreasing) function G ($\overline{G}$), where G and $\overline{G}$ are bounded above and below by the Fréchet bounds in (1.1) resp. (1.7). Defining $\mathcal{A}(s) := \{u = (u_1, \ldots, u_n) \in \mathbb{R}^n : \sum_{i=1}^n u_i = s\}$ and $\bigwedge G(s) = \inf_{u \in \mathcal{A}(s)} G(u)$, $\bigvee \overline{G}(s) = \sup_{u \in \mathcal{A}(s)} \overline{G}(u)$, the following improved standard bounds on the joint portfolio have been given in several similar forms in the literature, see Williamson and Downs (1990), Embrechts et al. (2003), Rü (2005, 2013), Embrechts and Puccetti (2006b, 2010), and Puccetti and Rü (2012).

Theorem 3.1 (Improved standard risk bounds under positive dependence restriction)**.** *Let X, F, G, and $\overline{G}$ be as introduced above.*

a) If $G \leq F$, then

$$P\Big(\sum_{i=1}^n X_i \leq s\Big) \geq \bigvee G(s) \tag{3.1}$$

b) If $\overline{G} \leq \overline{F}$, then

$$P\Big(\sum_{i=1}^n X_i < s\Big) \leq 1 - \bigvee \overline{G}(s). \tag{3.2}$$

In the case where $G = F_-$ resp. $\overline{G} = \overline{F}_-$ include no further dependence information these bounds are called standard bounds. By inversion we obtain the standard bounds for VaR which depend only on the lower Fréchet copula bound $W(u) = (\sum_{i=1}^n u_i - (n-1))_+$. We denote the corresponding standard VaR bound by VaR^W.

In particular (3.1) and (3.2) give upper and lower bounds for the distribution function and thus also for the tail risk of the sum if the risk vector X is positive quadrant dependent (i. e. PUOD and PLOD).

To establish bounds for the df as in (3.1) or in (3.2) it is of course sufficient to have bounds for the copula $C = C_X$. An elaboration on the method induced by Theorem 3.1 to VaR bounds has been given in Embrechts et al. (2013, 2014). Also several alternative ways to include dependence information in order to obtain improved VaR bounds for the sum have been discussed in the recent literature. For example Bernard et al. (2016a,b) derive improved risk bounds based on additional variance or moment information. Positive and negative dependence restrictions as in (3.1) and (3.2) based on independence and positive dependence information in subgroups were considered in Bignozzi et al. (2015) and Puccetti et al. (2015). Structural information by partially specified risk factor models was investigated in Bernard et al. (2016b). A survey of these developments is given in Rü (2017).

Assuming that for a distribution function F with marginals F_i it is known that $F \leq G$ and/or that $F \geq G$ on some subset $S \subset \mathbb{R}^n$ one obtains the following improved

Hoeffding–Fréchet bounds which were given independently in Puccetti et al. (2016) and in Lux and Papapantoleon (2016).

Theorem 3.2 (Improved Hoeffding–Fréchet bounds)**.** *Let $G : \mathbb{R}^n \to \mathbb{R}$ be an increasing function with $F_- \leq G \leq F_+$ and define*

$$F^*(x) = \min\Big(\min_{1\leq i\leq n} F_i(x_i), \inf_{y\in S}\Big\{G(y) + \sum_{i=1}^n (F_i(x_i) - F_i(y_i))_+\Big\}\Big)$$
$$F_*(x) = \max\Big(0, \sum_{i=1}^n F_i(x_i) - (n-1), \sup_{y\in S}\Big\{G(y) - \sum_{i=1}^n (F_i(y_i) - F_i(x_i))_+\Big\}\Big)$$

Then for $F \in \mathcal{F}(F_1, \dots, F_n)$ holds

i) If $F(y) \leq G(y)$ for all $y \in S$, then $F(x) \leq F^*(x)$ for all $x \in \mathbb{R}^n$.

ii) If $F(y) \geq G(y)$ for all $y \in S$, then $F(x) \geq F_*(x)$ for all $x \in \mathbb{R}^n$.

iii) If $F(y) = G(y)$ for all $y \in S$, then $F_*(x) \leq F(x) \leq F^*(x)$ for all $x \in \mathbb{R}^n$.

Remark 3.3. In the case $n = 2$ the improved Hoeffding–Fréchet bounds in Theorem 3.2 are due to Rachev and Rü (1994). They were restated in the case of uniform marginals i. e. for copulas in Tankov (2011) for the case of equality constraints. In this paper also a sharpness result for increasing sets S and an application to model free pricing bounds for multi-asset options is given. In the case that S is singleton and $n = 2$ these bounds and their sharpness were shown in Nelsen et al. (2004, Theorem 3.2.2).

In Nelsen (2006) several constructions are given for copulas with given sections. An interesting construction are f. e. the diagonal copulas with prescribed diagonal section. They are however different from the upper or lower bounds F^* resp. F_* as specified in Theorem 3.2. For $n \geq 2$ and S being a singleton the improved bounds were given in Rodríguez-Lallena and Úbeda-Flores (2004) and in Sadooghi-Alvandi et al. (2013) for finite sets S.

Extensions of the sharpness result are given in Bernard et al. (2012). The paper of Bernard et al. (2013) discusses as application the case where S is the central part of the distribution.

As corollary in the case that $F_i \sim U(0,1)$, $1 \leq i \leq n$ Theorem 3.2 implies the following improved bounds for the copula of a risk vector.

Corollary 3.4 (Improved copula bounds)**.** *Let $S \subset [0,1]^n$ and let Q be a componentwise increasing function on $[0,1]^n$ such that $W(u) \leq Q(u) \leq M(u)$, $u \in [0,1]^n$. Define the bounds $A^{S,Q}, B^{S,Q} : [0,1]^n \to [0,1]$ as*

$$A^{S,Q}(u) = \min\Big(M(u), \inf_{a\in S}\Big\{Q(a) + \sum_{i=1}^n (u_i - a_i)_+\Big\}\Big)$$
$$B^{S,Q}(u) = \max\Big(W(u), \sup_{a\in S}\Big\{Q(a) - \sum_{i=1}^n (a_i - u_i)_+\Big\}\Big)$$

Then for an n-dimensional copula C, it holds

i) If $C(u) \leq Q(u)$ for all $u \in S$, then $C(u) \leq A^{S,Q}(u)$ for all $u \in [0,1]^n$.

ii) If $C(u) \geq Q(u)$ for all $u \in S$, then $C(u) \geq B^{S,Q}(u)$ for all $u \in [0,1]^n$.

iii) If $C(u) = Q(u)$ for all $u \in S$, then $B^{S,Q}(u) \leq C(u) \leq A^{S,Q}(u)$ for all $u \in [0,1]^n$.

It is shown in Puccetti et al. (2016) in several relevant examples that the improved Hoeffding–Fréchet bounds in Theorem 3.2 may lead to strongly improved VaR bounds for the joint portfolio based on the method of improved standard risk bounds in Theorem 3.1.

Some examples showing the effect of the improved Hoeffding–Fréchet bounds are discussed in Puccetti et al. (2015). The following example shows related results in a graphical way.

Example 3.1. a) *Positive dependence in the tails.* We consider the case $n = 2$ with $F_1 = F_2 = \text{Pareto}(2)$ and assume that the copula Q of the risk vector is comonotonic on the tail area $S = [0.9, 1]^2$, i. e. for a copula vector $(U_1, U_2) \sim Q$ holds

$$P(U_1 \geq u_1, U_2 \geq u_2) = \min(1 - u_1, 1 - u_2), u_i \geq 0.9.$$

This models a case where in extreme situations a strong form of positive dependence arises. As consequence of this strong positive dependence in the tails we obtain from Corollary 3.4 and Theorem 3.1 a remarkable reduction of the improved VaR bounds $\mathrm{VaR}_\alpha^{B^{S,Q}}$ for moderate and in particular for high quantile levels α (see Figure 3.1). In fact in this example the standard bounds are know to be sharp bounds.

Based on Corollary 3.4 a similar effect also holds in the case that $n \geq 2$. The assumption of comonotonicity in the tails is a strong assumption.

In Puccetti et al. (2016) it is shown that some related effects are obtained when replacing the strong positive dependence assumption in the tail in Corollary 3.4 by a weaker assumption of the form $\overline{Q}(u) = 1 - Q(u) \geq G_\vartheta(u)$, $u \in S$, where G_ϑ is a parametric class of Gumbel or of Gaussian copulas.

b) *Independent subgroups with positive internal dependence.* In this example we modify the model assumption investigated in Bignozzi et al. (2015). We consider the case that the risks are split into k independent subgroups I_j. Bignozzi et al. (2015) allow any kind of dependence within these subgroups. In comparison we assume that the risks within the subgroups are strongly positive dependent (comonotonic) in the tails, i. e., similar as in Example 3.1 a) on $[0.9, 1]^{n_i}$, where $n_i = |I_i|$.

As concrete example we consider the case where $n = 20$, with $k = 1, 10, 20$ subgroups, where the subgroup sizes are equal to $\frac{20}{k}$. We further assume that $F_i = \text{Pareto}(2) = F$, $1 \leq i \leq n$. As consequence of Theorem 3.1 and Corollary 3.4

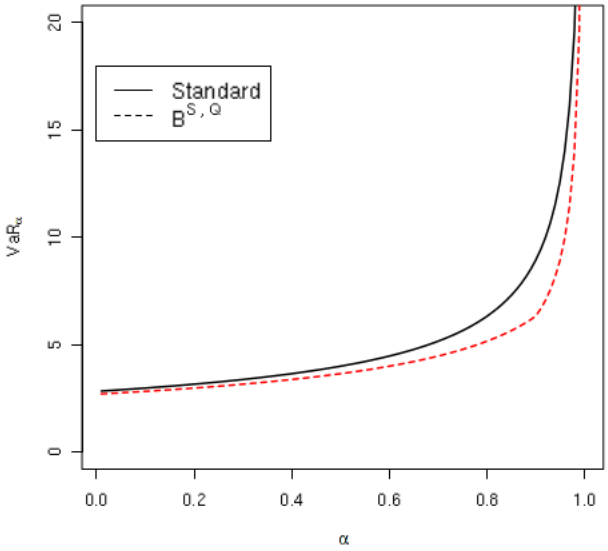


Figure 3.1 Comparison of $\mathrm{VaR}_\alpha^{B^{S,Q}}$ and the standard bound VaR_α^{S} for $n = 2$ with Pareto(2) marginals.

we obtain

$$P\Big(\sum_{i=1}^{n} X_i \le s\Big) \ge B_k^{S,Q}\Big(F\Big(\frac{s}{n}\Big), \dots, F\Big(\frac{s}{n}\Big)\Big)$$
$$= \max\Big(nF\Big(\frac{s}{n}\Big) - (n-1), \max_{a\in S}\Big\{Q(a) - \sum_{i=1}^{n}\Big(a_i - F\Big(\frac{s}{n}\Big)\Big)_+\Big\}\Big),$$

where $S = [0.9, 1]^n$ and $Q(a) := \prod_{j=1}^{k} \min_{i\in I_j} a_i$. The corresponding VaR bounds $\mathrm{VaR}_\alpha^{B_k^{S,Q}}$ are obtained by inversion and are given in Figure 3.2. In that paper also the improved standard bounds are compared with the (sharp) bounds with marginal information only. For strong enough positive dependence the improved standard bounds are sharper than the dual bounds.

The results obtained can be expected. The worst bound is the standard bound. The best bound is obtained for the case $k = 1$ of general comonotonicity in the tails. The case of 10 independent subgroups with positive tail dependence leads to a considerable reduction.

As in Example 3.1 a) Puccetti et al. (2016) describe similar effects in this example when replacing the comonotonicity assumption inside the groups by weaker Gumbel type specification in the tails.

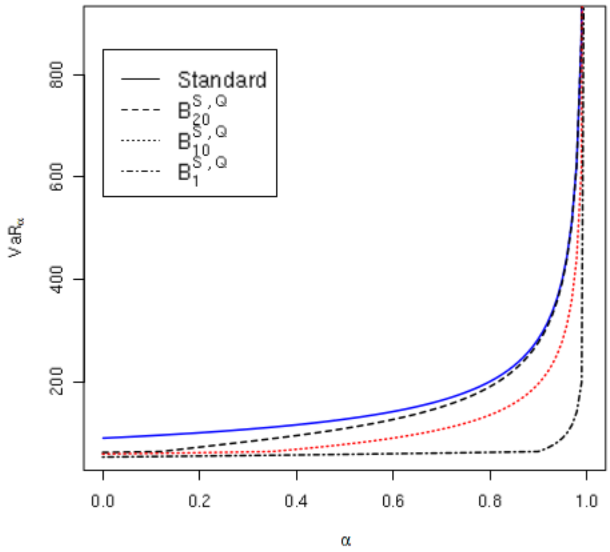


Figure 3.2 Comparison of $\mathrm{VaR}_\alpha^{B_k^{S,Q}}$ for $k = 1, 10, 20$ and standard bound VaR_α^S, $n = 20$, $F_i = \mathrm{Pareto}(2)$.

4 Higher order marginal information; comparison of various VaR bounds for the joint portfolio

If higher order marginal distributions of the risk vector X are known then it is possible to improve the Hoeffding–Fréchet bounds and as consequence of (3.1), (3.2), (2.8), and (2.9) one gets improved standard bounds for the VaR. In this section we consider the case where two dimensional marginal distributions are known. Alternative dual bounds with higher order marginals called 'reduced bounds' have been discussed in Embrechts and Puccetti (2006a), Puccetti and Rü (2012), and in Embrechts et al. (2013). As a result it was found in these papers that the additional information of higher dimensional marginals may lead to considerably improved upper VaR bounds, when the joint marginals are not 'too close' to the upper Hoeffding–Fréchet bounds.

One obtains improved Hoeffding–Fréchet bounds for the distribution function (resp. for the copula) by means of Bonferroni-type bounds (see Rü (1991a, Prop. 6)).

Proposition 4.1 (Bonferroni-type bounds). *Let C be an n-dimensional copula with bivariate marginals $C_{i,j}$ for $i \neq j$. Then*

$$\min_{i \neq j} C_{ij} \geq C \geq W_B \geq W_A \geq W, \tag{4.1}$$

where $W(u) = (\sum_{i=1}^n u_i - (n-1))_+$ is the Hoeffding–Fréchet lower bound,

$$W_A(u) = \Big(\sum_{i=1}^n u_i - (n-1) + \frac{2}{n} \sum_{i<j} (1 - u_i - u_j + C_{i,j}(u_i, u_j)) \Big)_+ \tag{4.2}$$

and $$W_B(u) = \Big(\sum_{i=1}^n u_i - (n-1) + \sup_\tau \sum_{(i,j)\in\tau} (1 - u_i - u_j + C_{i,j}(u_i, u_j)) \Big)_+, \tag{4.3}$$

the sup being taken over all spanning trees of the complete graph induced by $\{1, \ldots, n\}$.

The bound W_B is a consequence of the Bonferroni inequality from Hunter (1976) (see Rü (1991a, Prop. 6)). It improves the bound W_A arising from a Bonferroni bound of Hunter (1976) and Worsley (1982). As consequence of (3.1) and (3.2) these bounds imply improved bounds for the tail-risk and the VaR of the joint portfolio $\sum_{i=1}^n X_i$, where (X_i, X_j) have copulas $C_{i,j}$. Let

$$\begin{aligned} &\mathrm{VaR}_\alpha^W = W(F_1, \ldots, F_n)^{-1}(\alpha), \quad \mathrm{VaR}_\alpha^{W_A} = W_A(F_1, \ldots, F_n)^{-1}(\alpha) \\ \text{and} \quad &\mathrm{VaR}_\alpha^{W_B} = W_B(F_1, \ldots, F_n)^{-1}(\alpha) \end{aligned} \tag{4.4}$$

denote the upper α-quantiles of W, W_A, W_B with marginals $F_1, \ldots, F_n$. Then we obtain as consequence of (4.1)

$$\mathrm{VaR}_\alpha(S) \le \mathrm{VaR}_\alpha^{W_B} \le \mathrm{VaR}_\alpha^{W_A} \le \mathrm{VaR}_\alpha^{W}\,. \tag{4.5}$$

The upper bound $\mathrm{VaR}_\alpha^{W_A}$ has been investigated in Liu and Chan (2011). In contrast to their statement this bound is not the 'best possible upper bound' for $\mathrm{VaR}_\alpha(S)$. As their numerical results indicate the bound $\mathrm{VaR}_\alpha^{W_A}$ improves on the dual bound, which is based solely on marginal information, only for high confidence levels α and for highly positive correlated two-dimensional marginals. Correspondingly it was seen in Embrechts et al. (2013) that strong improvements of lower bounds are obtained, when the two-dimensional marginals are independent.

In the following examples we compare the Bonferroni bounds $\mathrm{VaR}_\alpha^{W_A}$ and $\mathrm{VaR}_\alpha^{W_B}$ with each other and with the standard bounds VaR_α^{W} as well as with the dual bound VaR_α^{D} arising from (2.8) for various dependence levels on the bivariate marginals.

By (3.1) we have

$$P\Big(\sum_{i=1}^n X_i \le t\Big) \ge \sup_{u \in \mathcal{U}(t)} C_L(F_1(u_1), \ldots, F_n(u_n)), \tag{4.6}$$

where C_L is either W or is one of the (improved) bounds W_A, W_B. For $u = \big(\frac{t}{n}, \ldots, \frac{t}{n}\big)$ we get the lower bound

$$P\Big(\sum_{i=1}^n X_i \le t\Big) \ge C_L\Big(F_1\Big(\frac{t}{n}\Big), \ldots, F_n\Big(\frac{t}{n}\Big)\Big). \tag{4.7}$$

In general the improvements of the Fréchet bounds as in (4.1) can be considerable. The improved standard bounds in (4.6) are not easy to determine in general in explicit form. In several cases however conditions are easy to state which allow to determine them explicitly. In general we obtain the strongest improvement of the upper bound VaR_α^{W} if the two-dimensional copulas are comonotonic.

We next state for some cases explicit solutions to (4.6). If $C_L = W$ and $F_1, \ldots, F_n$ have decreasing densities and $u^* \in U(t)$ satisfies $F_1(u_1^*) = \cdots = F_n(u_n^*)$ then $u^* =$

$(u_1^*, \dots, u_n^*)$ is uniquely determined and u^* is a solution to (4.6). If $F_1 = \cdots = F_n$ has a decreasing density, then $\left(\frac{t}{n}, \dots, \frac{t}{n}\right)$ is a solution to (4.6) and thus the bound in (4.7) coincides with that in (4.6).

More generally let

$$A = \{(F_1(u_1), \dots, F_n(u_n));\ \ u = (u_i) \in \mathcal{U}(t)\}$$

and assume that $(F_i(u_i^*))$ is a smallest element of A w.r.t. the increasing Schur convex order $\preceq_S$, then

$$\sup_{u \in \mathcal{U}(t)} W(F_1(u_1), \dots, F_n(u_n)) = W(F_1(u_1^*), \dots, F_n(u_n^*)). \tag{4.8}$$

Similarly, assuming that W_A resp W_B are Schur concave, i. e. decreasing w.r.t. the increasing Schur convex order $\preceq_S$ we obtain

$$\sup_{u \in \mathcal{U}(t)} W_A(F_1(u_1), \dots, F_n(u_n)) = W_A(F_1(u_1^*), \dots, F_n(u_n^*)) \tag{4.9}$$

$$\text{resp.}\quad \sup_{u \in \mathcal{U}(t)} W_B(F_1(u_1), \dots, F_n(u_n)) = W_B(F_1(u_1^*), \dots, F_n(u_n^*)). \tag{4.10}$$

Sufficient conditions for Schur concavity of W_A and W_B can be inferred from Chapters 3 and 4 in Marshall and Olkin (1979). For example, in the homogeneous case $C_{ij} = C_2$ for all i, j, if C_2 is concave and symmetric or more generally is Schur concave, then W_A and W_B are Schur concave.

In the following we use the vector u^* with identical components $(F_1(u_1^*), \dots, F_n(u_n^*))$ as above as a proxy for comparison of the upper bounds in (4.8)–(4.10). In particular in the case $F_1 = \cdots = F_n = F$ we use the vector $\left(F\left(\frac{t}{n}\right), \dots, F\left(\frac{t}{n}\right)\right)$. In contrast to statements in Liu and Chan (2011) this choice will not give the exact bounds in (4.8) and (4.9) (and also in (4.10)) in general.

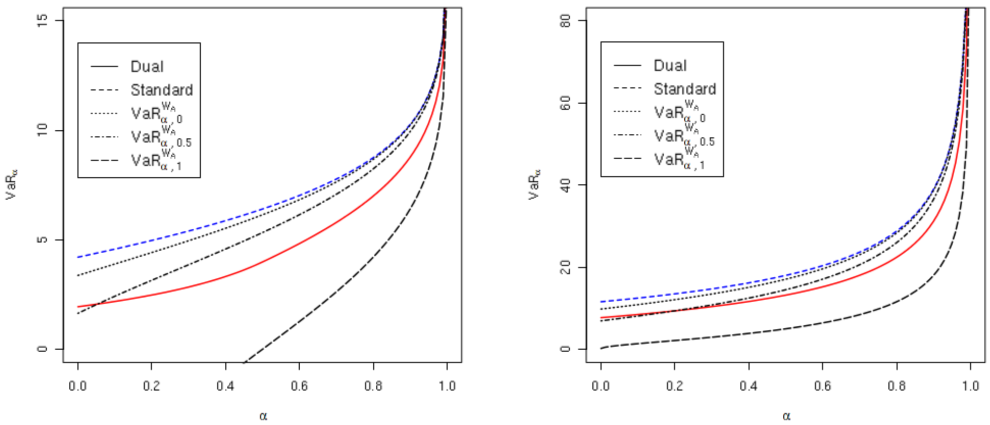
In the following examples we consider the homogeneous case where $F_i = F$ and where $C_{i,j} = C_2$ for all $i < j$. We concentrate on the approximate bounds based on u^*.

Comparison of VaR^{W_A}, standard bounds, and dual bound

In the first example we compare the standard bound, i. e. the VaR bound induced by W, the VaR bound induced by W_A and the dual bound D, which gives the optimal bound with only marginal information in this example.

Let $n = 5$ and let X_i be standard normal resp. log-normal distributed, $1 \leq i \leq 5$. Let C_2 be a Gauß-copula with correlations $\varrho = 0,\ 0.5,\ 1$. Figure 4.1 compares the $\mathrm{VaR}_{\alpha,\varrho}^{W_A}$ upper bounds with the dual bound VaR_α^D in dependence on α and ϱ for both distributions. Note that using the proxies the bounds $\mathrm{VaR}_{\alpha,\varrho}^{W_A}$ and $\mathrm{VaR}_{\alpha,\varrho}^{W_B}$ coincide in this case.

Figure 4.1 (a) shows that the $\mathrm{VaR}_{\alpha,\varrho}^{W_A}$ bound improves with increasing correlation. In particular the case $\varrho = 1$ (comonotonicity) for the two dimensional marginals gives



(a) (b)

Figure 4.1 Comparison $\mathrm{VaR}_{\alpha,\varrho}^{W_A}$, standard bound and dual bound, $n = 5$, $\varrho = 0, 0.5, 1$, in case of Gauß copula in (a) and log normal copula in (b).

better upper bounds than the case $\varrho = 0$ (independence). This kind of dependence on ϱ can also be seen directly from the definition of W_A in (4.2). Further one finds as expected, that for any ϱ the $\mathrm{VaR}_{\alpha,\varrho}^{W_A}$ bound using information on two-dimensional marginals is an improvement on the standard bound based on marginal information only.

The dual bound VaR_α^D is a strong improvement over the standard bound, both being based on marginal information only. It is known that the dual bound is optimal in this example. This example shows that the technique of standard bounds does not work well in higher dimensions.

From Figure 4.1 and Table 4.1 one sees that the dual bound VaR_α^D is even an improvement over the bounds $\mathrm{VaR}_{\alpha,\varrho}^{W_A}$ when $\varrho < 0.9$ and $\alpha \geq 0.9$, i.e. the information on two-dimensional marginal information does not lead to an improved upper bound in these cases, when using the method of improved standard bounds.

α	VaR_α^S	VaR_α^D	$\mathrm{VaR}_{\alpha,0}^{W_A}$	$\mathrm{VaR}_{\alpha,0.5}^{W_A}$	$\mathrm{VaR}_{\alpha,0.9}^{W_A}$	$\mathrm{VaR}_{\alpha,1}^{W_A}$
0.9	10.268	8.773	10.234	9.943	8.764	6.407
0.95	11.631	10.311	11.616	11.415	10.425	8.224
0.99	14.390	13.322	14.388	14.297	13.589	11.631

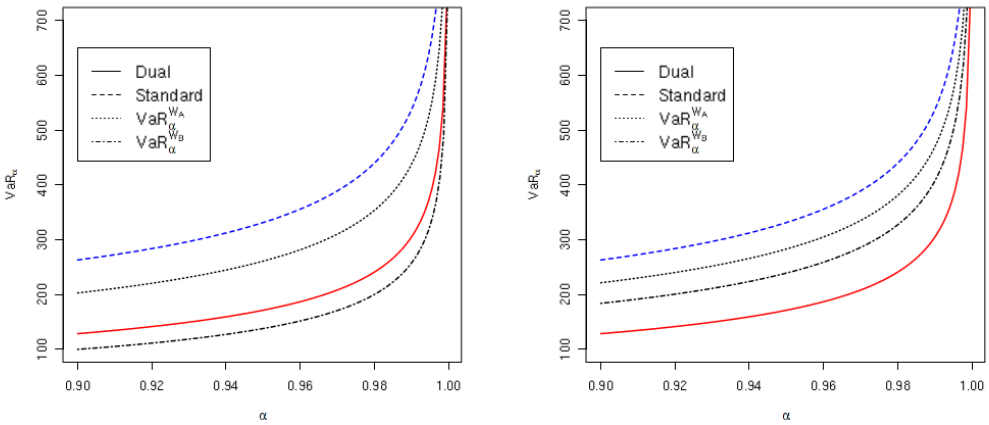
Table 4.1 Comparison of VaR^S, VaR^D and VaR^{W_A}

In Figure 4.1 (b) we see that in the case of log-normal distributions with heavy tails we obtain a similar picture of the relation between these VaR bounds.

While in this example the bounds VaR^{W_A} and VaR^{W_B} coincide when using the proxies, in the following example we show that in inhomogeneous cases the difference can be quite big so that VaR^{W_B} is a strong improvement over VaR^{W_A}.

Comparison of VaR^{W_A} and VaR^{W_B}
We consider the case $n = 20$ where the marginals X_i are log-normal distributed. We assume that $C_{i,j}(u_i, u_j)$ is a t-copula with three degrees of freedom and correlation ϱ. The risks X_i are divided into two groups of equal size 10. Within the groups the rv's are pairwise comonotone, i. e. $\varrho = \varrho_1 = 1$ and between the groups the rv's are pairwise independent, i. e. $\varrho = \varrho_2 = 0$.

In this case the sup in (4.3) is attained by the tree which uses only once the correlation $\varrho_2 = 0$. On the other hand $\mathrm{VaR}_\alpha^{W_A}$ can be seen as an average over all starwise trees which also contains trees which use several times the low correlation connections with $\varrho_2 = 0$. This construction makes the difference between both bounds in a particular way big. We find in Figure 4.2 (a) that in this case $\mathrm{VaR}_\alpha^{W_B}$ is strongly improved compared to the VaR bound $\mathrm{VaR}_\alpha^{W_A}$. For example we obtain $\mathrm{VaR}_{0.9}^{W_B} = 99.5875$ which is about 50 % better than $\mathrm{VaR}_{0.9}^{W_A} = 202.6817$. The difference between the bounds is increasing in α. For $\alpha = 0.99$ we have for example $\mathrm{VaR}_{0.99}^{W_B} = 257.1075$ an improvement of 59 % over $\mathrm{VaR}_{0.99}^{W_A} = 437.2221$. $\mathrm{VaR}_\alpha^{W_B}$ improves over the dual bound VaR_α^{D} whereas $\mathrm{VaR}_\alpha^{W_A}$ is worse than the dual bound.



(a) $\varrho_1 = 1$, $\varrho_2 = 0$ (b) $\varrho_1 = 0.9$, $\varrho_2 = 0.1$

Figure 4.2 Comparison of $\mathrm{VaR}_\alpha^{W_A}$, $\mathrm{VaR}_\alpha^{W_B}$, VaR_α^{D}, inhomogeneous case, $C_{i,j}$ t-copula

In Figure 4.2 (b) we see that under slightly weaker differences for the correlations with $\varrho_1 = 0.9$ and $\varrho_2 = 0.1$ the dual bound VaR_α^{D} is better than the Bonferroni bounds $\mathrm{VaR}_\alpha^{W_A}$ and $\mathrm{VaR}_\alpha^{W_B}$ indicating again a weakness of the method of improved standard bounds. While the Fréchet bounds for the df's improve considerably by inclusion of two dimensional marginals, the corresponding VaR bounds for the aggregated sums only improve in certain cases which exhibit strong enough positive dependence.

Remark 4.2 (Reduced bounds versus Bonferroni-type bounds). The reduced bounds in Embrechts et al. (2013) consider the case with the weaker assumption that only the non-overlapping distributions F_{12} of $(X_1, X_2), \dots, F_{2n-1,2n}$ of (X_{2n-1}, X_{2n}) are known and give a non-sharp reduction to the one-dimensional case for $Y_1 = X_1+X_2, \dots, Y_n = X_{2n-1} + X_{2n}$, which can be handled by the dual bounds. From the examples in the present and in the related papers on improved bounds, it seems reasonable to expect that the reduced bounds may be well better than the Bonferroni bounds if the positive dependence on the 2-dimensional marginals is not strong. But in grouped examples like in Example 3.1 b) or in related structured examples the Bonferroni bounds are better able to make use of this structure by the choice of a suitable dependence tree τ consisting of strongly positive dependent components. So in case that information on more than the serial two-dinensional marginals are available it seems that the Bonferroni-type bounds in combination with the improved standard bounds are better than the reduced bounds.

In the following example we compare the bounds for a set of heavy tailed marginal distributions and a different set of bivariate copulas.

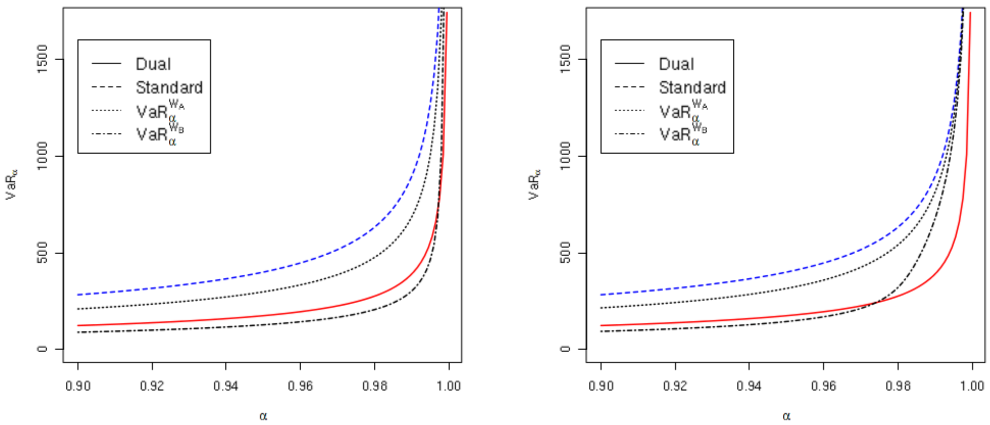
Comparison of VaR bounds for bivariate Clayton copulas
We assume that $n = 20$ and X_i are Pareto-distributed, i. e. $F_i(x) = 1 - x^{-2}$, $x \geq 1$. We assume that $C_{i,j}(u_i, u_j)$ is a Clayton copula with parameter ϑ. Note that for $\vartheta \to \infty$ the Clayton copula approaches comonotonicity while for $\vartheta \to 0$ it approaches independence. As in the third example we consider the case that the risks are divided into two groups. Within the groups the risks are approximatively comonotone (strongly positive dependent), i. e. the Clayton parameter $\vartheta = \vartheta_1$ is big. Between the groups the risks are approximatively independent, i. e. the Clayton parameter $\vartheta = \vartheta_2$ is small. This construction allows us to investigate the behaviour of the various VaR bounds in dependence of the dependence parameter ϑ of the copulas.

In Figure 4.3 and Table 4.2 we consider the choice $\vartheta_1 = 10\,000$, $\vartheta_2 = 0.1$ in 4.3a and $\vartheta = 1\,000$, $\vartheta_2 = 1$ in 4.3b. As in the case of log-normal distributions we find that the Bonferroni bound $\mathrm{VaR}_\alpha^{W_B}$ is significantly better than $\mathrm{VaR}_\alpha^{W_A}$ and in particular improves the standard bound VaR_α^{S}.

			$\vartheta_1 = 10\,000$, $\vartheta_2 = 0.1$		$\vartheta_1 = 1\,000$, $\vartheta_2 = 1$	
α	VaR_α^{S}	VaR_α^{D}	$\mathrm{VaR}_\alpha^{W_A}$	$\mathrm{VaR}_\alpha^{W_B}$	$\mathrm{VaR}_\alpha^{W_A}$	$\mathrm{VaR}_\alpha^{W_B}$
0.9	282.842	123.288	209.452	88.717	214.864	93.168
0.99	894.427	389.871	684.720	301.371	813.773	676.727
0.999	2 828.427	1 232.883	2 574.672	2 141.456	2 797.193	2 764.304

Table 4.2 Comparison of VaR bounds, $n = 20$, Pareto-marginals, Clayton copulas with parameter ϑ_1 and ϑ_2 for $\alpha \geq 0.9$

In case $\vartheta_1 = 10\,000$ and $\vartheta_2 = 0.1$ the dual bound VaR_α^{D} improves on the Bonferroni bound $\mathrm{VaR}_\alpha^{W_B}$ for $\alpha \geq 0.9975 = \alpha_0$. Experience of further examples shows that this turning point moves to smaller values of α, the smaller the dependence parameter ϑ_1 gets. For example, for $\vartheta_1 = 1\,000$ and $\vartheta_2 = 1$ the turning point is $\alpha_0 = 0.975$. For



(a) $\vartheta_1 = 10\,000$, $\vartheta_2 = 0.1$ (b) $\vartheta_1 = 1\,000$, $\vartheta_2 = 1$

Figure 4.3 Comparison of VaR bounds, $n = 20$, Pareto-marginals, bivariate Clayton copula

$\alpha > \alpha_0$ the dual bounds are better than the Bonferroni bounds if the model is in enough distance to the comonotonic case.

As general conclusion of the examples in this section we obtain that the Bonferroni bound VaR_α^B and the dual bound VaR_α^D improve upon the standard bound VaR_α^W. $\mathrm{VaR}_\alpha^{W_B}$ also improves generally on $\mathrm{VaR}_\alpha^{W_A}$. The Bonferroni bound VaR^{W_B} improves for high degree of positive dependence on the dual bound VaR^D but for weaker forms of positive dependence the dual bound may be preferable. It should be noted however that the dual bound is typically only calculable for small dimensions for inhomogeneous cases. In these cases however the rearrangement algorithm (RA) can be applied to yield sharp marginal bounds. In our applications we used proxies for the calculation of the Bonferroni bounds. These were shown above to be sharp under some conditions.

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