A Martingale Approach to Optimal Stopping

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Contents

Classical Theory

X_t diffusion with infinitesimal generator $\mathcal{D}$

$r(x,t)$ loss at (x,t)

$u(x,t) = \min_{S \geq t} E^{(x,t)} r(X_S, S)$, S stopping time

The optimal continuation region:

$$\mathcal{C}^* = \{(x,t) \mid r(x,t) > u(x,t)\}$$

The optimal stopping rule:

$$T^* = \inf\{t > 0 \mid (X_t, t) \notin \mathcal{C}^*\}.$$

Solution via free boundary problem (FBP):

$$\begin{aligned} u_t &= \mathcal{D}u && \text{in } \mathcal{C}^* \\ u &= r && \text{on } \partial\mathcal{C}^* \\ u_x &= r_x && \text{on } \partial\mathcal{C}^* \end{aligned}$$

See Shiryaev, Optimal Stopping Rules.

A stopping problem corresponds to a stationary FBP, if $\mathcal{D}u = 0$ or $\mathcal{D}u = ru$. The paper discusses the question of the structure of stopping problems with stationary FBP.

The Parking Problem

$$S_0 = -Q$$

$$S_n = \sum_{i=1}^{n} X_i - Q$$

X_i i.i.d. geometric (p)

p: probability of empty spot

Park as near as possible at "0"!

Find stopping time T^* of $S_i, i \geq 0$ with

$E \mid S_{T^*} \mid = \min_T E \mid S_T \mid$.

Solution: $T^* = \min\{n \geq 1 \mid S_n \geq -s_0\}$

with $s_0 = \min\{s \in \mathbb{N} \mid 1 - 2(1-p)^s > 0\}$

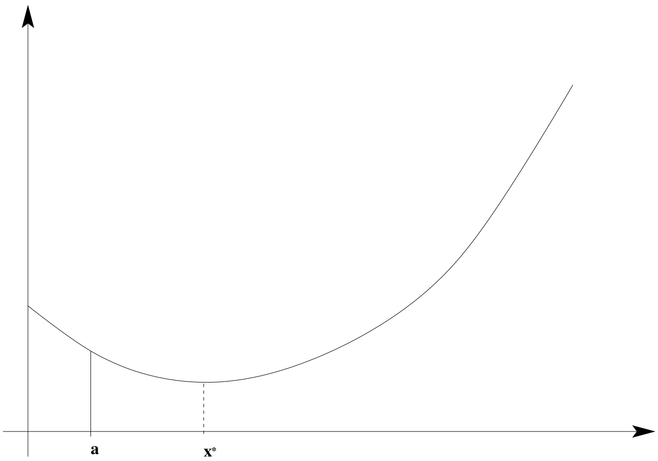
Generalized Parking Problem (GPP)

Let g be a convex function with a unique minimum at $x^* > 0$.

Assume X_i i.i.d. with $EX_i > 0$,

$$S_n = \sum_{i=1}^{n} X_i, \quad S_0 = 0.$$

Find a stopping time T^* with $Eg(S_{T^*}) = \min_T Eg(S_T)$



Solution (Keener, Lerche, Woodroofe '94):

$$T^* = \min\{n \geq 0 \mid S_n \geq a\} \text{ with } a < x^*.$$

The Main Idea: OS as GPP

Let $(Z_t, \mathcal{F}_t; t \geq 0)$ denote a continuous stochastic process.

Find a stopping time T^* with:

$$E\left(Z_{T^*} 1_{\{T^* < \infty\}}\right) = \max_T E\left(Z_T 1_{\{T < \infty\}}\right)$$

Idea:

Find a process $(X_t, \mathcal{F}_t; t \geq 0)$, and a nonnegative martingale $(M_t, \mathcal{F}_t; t \geq 0)$ with $M_0 = 1$ and a (concave) function g with unique maximum at x^* such that

$Z_t = g(X_t) M_t$ *a.s.*

Then

$$\begin{aligned} E Z_T 1_{\{T<\infty\}} &= E\left(g(X_T) M_T 1_{\{T<\infty\}}\right) \\ &\leq g(x^*) E\left(M_T 1_{\{T<\infty\}}\right) \\ &\leq g(x^*) \end{aligned}$$

With $T^* = \min\{t \geq 0 \mid X_t = x^*\}$ the inequalities become equalities, if $E\left(M_{T^*} 1_{\{T^*<\infty\}}\right) = 1$.

Optimality of Parabolic Boundaries

Let $X_t = B_t + x_0$, $t \geq 0$ B_t standard Brownian motion. For a measurable function h find a stopping time T that maximizes

$$E\left((T+1)^{-\beta} h\left(\frac{X_T}{\sqrt{T+1}}\right)\right) !$$

(Moerbeke(1974))

Let $H(x) = \int_0^\infty e^{ux-u^2/2} u^{2\beta-1}\, du$ with $\beta > 0$ and assume that there exists a unique point x^* with

$$\sup_{x\in\mathbb{R}} \frac{h(x)}{H(x)} = \frac{h(x*)}{H(x*)} = C^* \quad \text{and } 0 < C^* < \infty$$

<u>Theorem 1:</u>

Let $x_0 < x^*$. Then

$$\begin{aligned}\sup_T E\left\{(T+1)^{-\beta} h\left(\frac{X_T}{\sqrt{T+1}}\right)\right\} &= E\left\{(T^*+1)^{-\beta} h\left(\frac{X_{T^*}}{\sqrt{T^*+1}}\right)\right\} \\ &= H(x_0)C^*\end{aligned}$$

where $T^* = \inf\left\{t > 0 \mid \frac{X_t}{\sqrt{t+1}} = x^*\right\}$.

$$(t+1)^{-\beta} H\left(\frac{X_t}{\sqrt{t+1}}\right) = \int_0^\infty e^{uX_t - \frac{u^2}{2}t} e^{-\frac{u^2}{2}} u^{2\beta-1}\, du$$

is a positive martingale with starting value $H(x_0)$.
Then $M_t = (t+1)^{-\beta} H(X_t/\sqrt{t+1})/H(x_0)$ is a positive martingale with $EM_0 = 1$.

Then

$$\begin{aligned}(t+1)^{-\beta} h\left(\frac{X_t}{\sqrt{t+1}}\right) &= H(x_0)\frac{h\left(\frac{X_t}{\sqrt{t+1}}\right)}{H\left(\frac{X_t}{\sqrt{t+1}}\right)} M_t \\ &\leq H(x_0) C^* M_t\end{aligned}$$

Then

$$E\left((T+1)^{-\beta} h\left(\frac{X_T}{\sqrt{T+1}}\right)\right) \leq H(X_0) C^*$$

But $EM_{T^*} = 1$ and $P(T^* < \infty) = 1$ for
$T^* = \inf\left\{t > 0 \mid \frac{X_t}{\sqrt{t+1}} = x^*\right\}$.

<u>Special case:</u> Dvoretzky(1965), Chow-Robbins(1965)

$$h(x) = x,\ x_0 = 0,\ \beta = \frac{1}{2}$$

$$E(X_T/(T+1)) = \max \ \text{with}$$

$$T^* = \min\left\{t > 0 \mid \frac{X_t}{\sqrt{t+1}} = x^*\right\}$$

x^* is solution of $x = (1-x^2)\int_0^\infty e^{ux - u^2/2}\, du$ (Shepp 1969)

Perpetual Put Option

Samuelson(1965), McKean(1965)

$X_t = \sigma B_t + \mu t$ Brownian Motion with drift μ and variance σ^2.

Find stopping time T^* which maximizes

$$E_P e^{-rT}(K - e^{X_T})^+ 1_{\{T<\infty\}}.$$

Idea:

Find Q and g with $E_P e^{-rT}(K - e^{X_T})^+ = E_Q g(X_T)$, where $Q \ll P$ and g has a unique maximum at x^*.

Then $T^* = \min\{t \geq 0 \mid X(t) = x^*\}$, if $Q(T^* < \infty) = 1$.

Let

$$f(x) = (K - e^x)^+ \text{ and } M_t = \frac{dQ_t}{dP_t}.$$

Then

$$Ee^{-rT} f(X_T) = Ef(X_T)(e^{X_T})^{-\alpha}(e^{X_T})^{\alpha} e^{-rT}.$$

Choose $g(x) = f(x)e^{-\alpha x}$ and α such that $M_t = e^{\alpha X_T} e^{-rt}$ is a martingale. With

$$\begin{aligned} M_t &= \exp\left[\alpha(\sigma B_t) + \alpha\mu t - rt\right] \\ &= \exp\left[(\alpha\sigma)B_t - t(\alpha\sigma)^2/2\right] \end{aligned}$$

$M_0 = 1$ and M_t is a positive marginale. Then

$$\alpha^{\pm} = -\frac{\mu}{\sigma^2} \pm \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}}.$$

Let $K < 1 + (-\alpha^-)^{-1}$. Then g has a unique maximum at $x^* = \log \frac{\alpha^- K}{\alpha^- - 1} < 0$. Under Q X_t has drift

$$\alpha^- \sigma^2 + \mu = -\sigma^2 \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}} < 0.$$

This yields $Q(T^* < \infty) = 1$, if $x^* < 0$.
If $\mu < 0$ then $P(T^* < \infty) = 1$ and

$$\sup_T E_P e^{-rT}(K - e^{X_T})^+ = E_P e^{-rT^*}(K - e^{X_{T^*}})^+$$

The Repeated Significance Test is a Bayes-Test

$W(t), t \geq 0$ Brownian motion with drift θ

Testing sequentially: $H_0 : \theta < 0$ versus $H_1 : \theta > 0$

Prior: $G(d\theta) = \varphi(\tau\theta)\sqrt{\tau}d\theta$

$$R(T,\delta) = \int_{-\infty}^{0} \left(P_\theta\{\delta \text{ rejects } H_0\} + \frac{c}{2}\theta^2 E_\theta T \right) G(d\theta) + \int_{0}^{\infty} \left(P_\theta\{\delta \text{ rejects } H_1\} + \frac{c}{2}\theta^2 E_\theta T \right) G(d\theta)$$

Find (T^*, δ^*) with $R(T^*, \delta^*) = \min_{(T,\delta)} R(T, \delta)$.

$\delta^* = \delta^*_T = 1_{\{W(T)>0\}}$ $\qquad$ $T^* = \ ?$

PNAS, 83 (1986)

$$\int_{-\infty}^{0} P_\theta\{\delta \text{ rejects } H_0\} G(d\theta) + \int_0^{\infty} \{\delta \text{ rejects } H_1\} G(d\theta)$$

$$= \int G_{x,T}(-\infty, 0] 1_{\{\delta > 0\}} \overline{P}(dx)$$

$$+ \int G_{x,T}(0, \infty] 1_{\{\delta < 0\}} \overline{P}(dx)$$

$$\geq \int \min_x (G_{x,T}(-\infty, 0], G_{x,T}(0, \infty]) \overline{P}(dx)$$

$$= \int \Phi \left(-\frac{|W(T)|}{\sqrt{T+r}} \right) d\overline{P}$$

$$G = N(0, r^{-1}), \qquad \overline{P} = \int P_\theta G(d\theta)$$

$$\int \theta^2 E_\theta T G(d\theta)$$

$$= \int T \int \theta^2 G_{W(T),T}(d\theta) d\overline{P}$$

$$= \int T \left(\frac{W(T)^2}{(T+r)^2} + \frac{1}{T+r} \right) d\overline{P}$$

$$= \int (T+r) \left(\frac{W(T)^2}{(T+r)^2} + \frac{1}{T+r} \right) d\overline{P} - 1$$

$$= \int \frac{W(T)^2}{T+r} d\overline{P}$$

$$G_{W(T),T} = N\left(\frac{W(T)}{T+r}, \frac{1}{T+r} \right)$$

Representation of the risk:

$$R(T, \delta_T^*) = \int g\left(\frac{W(T)^2}{T+r}\right) d\overline{P}$$

with $g(x) = \Phi(-\sqrt{x}) + cx/2$

g is convex with unique minimum x^*

$$R(T, \delta_T^*) = \int g\left(\frac{W(T)^2}{T+r}\right) d\overline{P} \geq g(x^*)$$

Let $T^* = \min\{t > 0 \mid W(t)^2/(t+r) = x^*\}$

Since $\overline{P}\{T^* < \infty\} = 1$ it follows

$$R(T^*, \delta_{T^*}^*) = g(x^*).$$

The same type of argument holds for the SPRT.

The representation is $R(T, \delta_T^*) = \int g(|\theta|\,|W(T)|) d\overline{Q}$

with $\overline{Q} = \frac{1}{2}(P_\theta + P_{-\theta}), \theta > 0$.

Disruption Problem

Dissertation of Shiryaev (1961)

Observations: $W(t) = B(t) + \theta(t-\tau)^+$ with

$B(t), t \geq 0$ standard Brownian motion,

$\theta > 0$ fixed

Filtration: $\mathcal{F}_t = \sigma(W(s); 0 \leq s \leq t)$

Change-point: τ random time,

with distribution $\pi = p\delta_0 + (1-p)F$,

where $F(t) = 1 - e^{-\lambda t}$

Risk: $R(T) = P_\pi(T < \tau) + cE_\pi(T-\tau)^+$

Find T^* with $R(T^*) = \min\limits_T R(T)$

<u>Theorem 2:</u>

$T^* = \min\{t > 0 \mid \pi_t \geq p^*\}$ with $\pi_t = P(\tau \leq t \mid \mathcal{F}_t)$

Here p^* is the unique solution of $G(p) = p$ and $G'(p) = -1$, where G is the positive (and finite at 0) solution of

$$\frac{\theta}{2} x^2(1-x^2)G''(x) + \lambda(1-x)G'(x) = -x$$

$$\begin{aligned} R(T) &= P(T < \tau) + cE(T-\tau)^+ \\ &= E\left[(1-\pi_T) + c\int_0^T \pi_s ds\right] \end{aligned}$$

Find f with

$$E\left(\int_0^T \pi_s ds\right) = Ef(\pi_T) - f(p)$$

Then $g(x) = (1-x) + cf(x)$ yields

$$R(T) = \int g(\pi_T) dP$$

But this holds with $f = G$!

g is convex with a unique minimum at p^*.

This insight openes a new direction to Bayes tests of power one for change point problems. Cusum and Mixture stopping rules can be derived as Bayes tests. (Beibel 1996, 1997), (Beibel - Lerche 2003).

Disruption Problem: The Representation

Let $\pi_t = P(\tau < t \mid \mathcal{F}_t)$.

π_t is a diffusion with

$$d\pi_t = \lambda(1-\pi_t)dt + \theta\pi_t(1-\pi_t)d\overline{W_t}$$

where $\overline{W_t}$ is a Standard Brownian motion.

Ito's formula yields:

$$\begin{aligned} dG(\pi_t) &= G'(\pi_t)d\pi_t + \frac{1}{2}G''(\pi_t)(d\pi_t)^2 \\ &= G'(\pi_t)\lambda(1-\pi_t)dt + \theta\pi_t(1-\pi_t)d\overline{W_t} \\ &\quad + \frac{1}{2}G''(\pi_t)\theta^2\pi_t^2(1-\pi_t)^2 dt \end{aligned}$$

If G satisfies the equation

$$\frac{\theta^2}{2}x^2(1-x)^2G''(x) + \lambda(1-x)G'(x) = x$$

then

$$G(\pi_t) - G(\pi_0) = \int_0^t \pi_s ds + \int_0^t \theta\pi_s(1-\pi_s)d\overline{W_s}$$

$$\Rightarrow E\left[G(\pi_T) - G(\pi_0)\right] = E\int_0^T \pi_s ds$$

Stopping of Diffusions with Random Discounting

$B_t, t \geq 0 \quad SBM$

X_t diffusion with $X_0 = x$ and

$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$

Find stopping time T^* of X with

$$E\Big(e^{-A(T)}h(X_T)1_{\{T<\infty\}}\Big) = \max$$

$A(s)$: additive stochastic process adapted to $\mathcal{F}^X$

$$A(s+t) = A(s) + A(t) \circ \theta_s$$

Example: $r(x) \geq 0, \alpha > 0$

$$E\left(\exp\left\{-\int_0^T r(B_t)dt\right\}(B_T)^\alpha 1_{\{T<\infty\}}\right) = \max$$

How to choose the martingales?

$$\psi_+(x) \;=\; \begin{cases} E_x\left(e^{-A(T_{x_0})}\mathbf{1}_{\{T_{x_0}<\infty\}}\right) & \text{for } x \le x_0 \\ \left[E_{x_0}\left(e^{-A(T_x)}\mathbf{1}_{\{T_x<\infty\}}\right)\right]^{-1} & \text{for } x \ge x_0 \end{cases}$$

$$\psi_-(x) \;=\; \begin{cases} \left[E_{x_0}\left(e^{-A(T_x)}\mathbf{1}_{\{T_x<\infty\}}\right)\right]^{-1} & \text{for } x \le x_0 \\ E_x\left(e^{-A(T_{x_0})}\mathbf{1}_{\{T_{x_0}<\infty\}}\right) & \text{for } x \ge x_0 \end{cases}$$

$$\begin{aligned} M_t^{(+)} &= e^{-A(t)}\psi_+(X_t) \\ M_t^{(-)} &= e^{-A(t)}\psi_-(X_t) \end{aligned} \quad \text{are u.i. martingales with}$$

$$E_x(M_{T_b}^{(+)}\mathbf{1}_{\{T_b<\infty\}}) = \psi_+(x) \quad \text{for } b \ge x$$

$$E_x(M_{T_a}^{(-)}\mathbf{1}_{\{T_a<\infty\}}) = \psi_-(x) \quad \text{for } x \ge a$$

Play the trick with $h(x)/\psi_{\pm}(x)$, where $\psi_{\pm}(x_0) = 1$

$$X(0) = x_0, \ x_0 \in I$$

Assume $\psi_+(x_0) = \psi_-(x_0) = 1$

Then

$$e^{-A_t} h(X_t) = M_t \frac{h(X_t)}{p\psi_+(X_t) + (1-p)\psi_-(X_t)}$$

with $M_t = e^{-A_t}\left(p\psi_+(X_t) + (1-p)\psi_-(X_t)\right)$**.**

M_t **is a positive local martingale hence**

$$E(M_T 1_{\{T<\infty\}}) \le 1$$

<u>Problem:</u>

Maximize $\frac{h(x)}{p\psi_+(x)+(1-p)\psi_-(x)}$ **over all** $x \in I$ **with proper** p**.**

Distinguish the following Cases

1) $$\sup_{x \geq x_0, x \in I} (h(x)/\psi_+(x)) = \infty$$

2) $$\sup_{x \leq x_0, x \in I} (h(x)/\psi_-(x)) = \infty$$

3) $$0 < C^* = \sup_{x \in I} \frac{h(x)}{\psi_+(x)} = \sup_{x \geq x_0} \frac{h(x)}{\psi_+(x)}$$

4) $$0 < C^* = \sup_{x \in I} \frac{h(x)}{\psi_-(x)} = \sup_{x \leq x_0, x \in I} \frac{h(x)}{\psi_-(x)}$$

5) $$0 < \sup_{x \geq x_0, x \in I} (h(x)/\psi_+(x)) < \infty$$

$$0 < \sup_{x \leq x_0} (h(x)/\psi_-(x)) < \infty$$

and

$$\sup_{x \leq x_0} \frac{h(x)}{\psi_+(x)} > \sup_{x \geq x_0} \frac{h(x)}{\psi_+(x)} \text{ and}$$

$$\sup_{x \geq x_0} \frac{h(x)}{\psi_-(x)} > \sup_{x \leq x_0} \frac{h(x)}{\psi_-(x)}$$

Case 3/4

<u>Theorem 3:</u>

If $0 < C^* = \sup_{x \in I} \frac{h(x)}{\psi_\pm(x)} = \sup_{x \geq x_0, x \in I} \frac{h(x)}{\psi_\pm(x)} < \infty$

Then

$$\sup_T E_{x_0} \left\{ e^{-A_T} h(X_T) 1_{\{T<\infty\}} \right\} = C^* \qquad (+)$$

If there exists a point $x^* \gtreqless x_0$ with $C^* = h(x^*)/\psi_\pm(x^*)$, then the supremum in (+) is attained by

$$T^* = \inf \{ t \geq 0 \mid X_t = x^* \} .$$

Of course

$$E_{x_0} e^{-A_{T^*}} h(X_{T^*}) = E_{x_0} g(X_{T^*}) M_{T^*} = C^*.$$

Theorem 4:

Let p^* be such that

$$0 < \sup_{x \geq x_0, x \in I} \frac{h(x)}{p^* \psi_+(x) + (1-p^*)\psi_-(x)}$$
$$= \sup_{x \leq x_0, x \in I} \frac{h(x)}{p^* \psi_+(x) + (1-p^*)\psi_+(x)}$$

then $\sup_T E_{x_0}\left(e^{-A_T} h(X_T) 1_{\{T<\alpha\}}\right) = C^*$.

If there exist points $x_1 > x_0$ **and** $x_2 < x_0$ **such that**

$$\frac{h(x_1)}{p^* \psi_+(x) + (1-p^*)\psi_-(x)}$$
$$= \frac{h(x_2)}{p^* \psi_+(x) + (1-p^*)\psi_-(x)} = C^*,$$

then the supremum is attained for

$T^* = \inf\{t > 0 \mid X_t = x_1, X_2 = x_2\}$.

Generalized Parking Problem: Discrete Case

$X_1, X_2, \ldots$ i.i.d. with $EX_i > 0$,

$$S_n = \sum_{i=1}^{n} X_i, \quad S_0 = 0.$$

Find stopping time T^* with $Eg(S_{T^*}) = \min_T Eg(S_T)$

Solution: $T^* = \min\{n \geq 0 \mid S_n \geq a\}$

with

$$a = sup\{x \mid H^+g(x) < g(x)\}$$

$$H^+g(x) := \int g(x+y)H^+(dy)$$

$$H^+(y) := P(S_\eta \leq y)$$

$$\eta := \min\{n > 0 \mid S_n > 0\}$$

H^+ : the distribution of the first ladder height S_η.

Let $K(x) = \int_0^x \frac{1 - H^+(y)}{\gamma_1}\, dy$

with $\gamma_i = \int y^i \, dH^+(y)$, $i \in \mathbb{N}$.

Theorem 5:

If $Kg(x) < \infty$ for all $0 \leq x < \infty$, then $Kg(x)$ is minimized at $x = a$.

Example 1:

If $g(x) = |x - b| \, \forall \, x \in \mathbb{R} \Rightarrow b - a = \mathrm{med}(K)$

Example 2:

If $g(x) = (x - b)^2 \, \forall \, x \in \mathbb{R}$

$\Rightarrow b - a = \mathrm{mean}(K) = \gamma_2/\gamma_1$

Example 3:

If $g(x) = e^{-x} + cx \, \forall \, x \in \mathbb{R}, \; 0 < c < 1$

$\Rightarrow b = \log(1/c)$.

If $\int x^2 H^+(dx) < \infty$ and if $\kappa := \int_0^\infty e^{-x} K(dx)$

$\Rightarrow Kg(x) = \kappa e^{-x} + c(x + \frac{\gamma_2}{2\gamma_1})$ and is minimized when $x = \log(\kappa/c)$.

Related Literature

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