Semimartingale Modelling
in
Finance

Dissertation
zur Erlangung des Doktorgrades
der Mathematischen Fakultät
der Albert-Ludwigs-Universität Freiburg i. Br.

vorgelegt von

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Datum der Promotion: 27.4.1998

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To Birgit Schoen-Kallsen

and

to Ernst Kallsen

Preface

IUCUNDI ACTI LABORES.[1]

Last autumn the average newspaper reader was most likely confronted with the existence of something called *financial mathematics*. Its short period of fame was due to Robert Merton and Myron Scholes receiving the Nobel prize in economics for their work on the pricing and hedging of stock options. But in fact, since the famous article by Black & Scholes (1973) and its reformulation by Harrison & Pliska (1981) in terms of martingale theory, many papers have been written about the implications of different market models on derivative prices and hedging portfolios. Most of these approaches rely heavily on specific assumptions concerning the distribution of the underlying securities price processes. This fact makes adaptation to more general situations and comparison between models difficult. Our goal is to present a new formalism for derivative hedging and pricing which meets the three following demands as far as possible:

1. It shall not be restricted too closely to a specific distribution hypothesis, but instead be applicable to a large class of underlying securities price processes.

2. In cases where market completeness is not given, the additional assumptions necessary to determine strategies and prices shall be economically meaningful.

3. The derived formulae shall be numerically tractable.

In order to achieve the generality, we are striving for, we express diverse models for the underlyings in a uniform manner. This is done in terms of *semimartingale characteristics* and *martingale problems*. These are intuitive notions that have not yet sufficiently found their way into applications. To overcome this gap we present these concepts here and we also state a new (though classical in spirit) existence and uniqueness result for martingale problems. Modelling dynamical phenomena by martingale problems should be considered a stochastic counterpart of ordinary differential equations. Therefore, it is by no means restricted to financial applications and the title of this thesis could as well have been *Semimartingale Modelling and Finance*. We have nevertheless chosen the preposition *in*, since the financial aspect is expounded upon and cannot be fully appreciated without the general mathematical framework.

[1] Cicero, de finibus 2.32.105

Let me thank those to whom I am indebted: first of all my parents and my wife Birgit; moreover my grammar school maths teacher Erik Christensen; Murad Taqqu from Boston University for his hospitality and encouraging spirit; Martin Beibel, Barbara Grünewald, Uli Keller, Sebastian Raible and Siegfried Trautmann for fruitful discussions; David Kenfield for removing many misuses of the English language before I put new ones in; Jean Jacod for his comprehensive Springer Lecture Notes; and the governmental organisations DAAD and DFG for financial support. Last but not least I want to thank my advisor Ernst Eberlein for all his support and confidence.

Contents

Chapter 1

Introduction

1.1 Objective

As an investor in a securities market you may be facing two questions. How shall you compose your portfolio? What is a good probabilistic model for the market enabling you, for instance, to estimate your value at risk? To tackle these problems we propose proceeding in three steps.

Firstly, one divides the securities of interest into underlyings and derivatives. The assignment of any asset to either group may be quite arbitrary. The only condition we impose is that – roughly speaking – the value of any derivative is, at a certain future time, uniquely determined by the present or past values of the underlyings. Usually we treat stocks, short-term fixed income investments etc. as underlyings, while futures, options, zero-coupon bonds etc. are considered derivatives of these assets. Now one needs a good probabilistic model for the underlyings, including all unknown parameters that have to be statistically estimated. In a second step, one extends this statistical model for the underlyings to the whole market, including the derivatives. In this enlarged model one computes optimal trading or hedging strategies. The results from the second step are usually not given in closed form. Hence, step three is to evaluate the formulas by means of numerical algorithms.

This thesis deals with how to perform the second step. For the construction of an appropriate formalism we are guided by three goals:

1. *Generality:* The statistical setting for the underlyings in the first step will usually be given by econometricians and/or statisticians. They work hard at improving the models for financial data, including correlation of different securities and analysis of high frequency data. Therefore, we want our approach to be applicable to very diverse and complex securities market models including discrete-time models as well as continuous-time models with continuous and discontinuous paths.

2. *Appropriateness of the assumptions:* In general, one cannot compute unique prices and optimal portfolios without making strong assumptions concerning the behaviour of the market and the quality of trading strategies. We want these hypotheses and

conditions to be economically intuitive.

3. *Numerical tractability:* Although flipping through the pages of this thesis may not give you this impression, our approach is aimed at the practitioner. Thus we must ensure that the resulting formulas are numerically tractable. This does not mean that one can fall back on existing methods in any given setting. But we want to take care that the results are not too complex to allow for efficient algorithms at all. To avoid this, we simplify the assumptions leading to the extended models.

Black & Scholes (1973) give a very elegant and satisfactory solution to our three-step program in a particular situation (also including the first and the third step). We can summarize their reasoning in an informal manner as follows:

$$\begin{array}{ll} & \text{market regularity conditions} \\ + & \text{hypotheses on the distribution of the underlyings} \\ + & \text{absence of arbitrage} \\ \hline \rightarrow & \text{unique reasonable derivative prices} \\ & \text{perfect hedging strategies} \end{array} \tag{1.1}$$

Here market regularity compromises many assumptions characterizing ideal markets: Securities are traded continuously at any time at a unique market price, traders are price takers and they can buy and sell arbitrary amounts of any asset without any transaction costs, taxes, etc. In the Black-Scholes model the underlyings are stock and a riskless bank account. The interest rate is presumed to be fixed and stock prices are assumed to behave statistically as geometric Brownian motion, which is a reasonable though not entirely satisfactory approximation. The key insight of Black-Scholes is that, under these conditions, the absence of arbitrage (i.e. the impossibility of riskless gains in the market) suffices to derive unique prices for European options on the stock. Their idea is as follows: One constructs a dynamic portfolio consisting of shares of stock and money in the bank account whose value at maturity will certainly equal the payout of the option. The dynamic strategy is self-financing, i.e. after inception of the strategy no further cash infusions (or withdrawels) are needed. The absence of arbitrage implies that investments yielding the same profits must have the same initial costs. Hence in this setting a unique fair option price can be computed in terms of the current stock price. Moreover, this answers the question how we can hedge our risk if we have sold an option and if we can only trade in the stock and the bank account. In order to completely offset the risk, we simply have to buy the duplicating portfolio, which in fact necessitates an uncountable number of very small trades.

The Black-Scholes approach was reformulated in terms of semimartingale theory by Harrison & Pliska (1981). The application of the well-developed *general theory of stochastic processes* to finance led to considerable progress in the field. The paper by Harrison and Pliska was also the main inspiration for this thesis.

The reasoning (1.1) has been applied to many other underlyings (e.g. foreign exchange, zero-coupon bonds, cf. Lamberton & Lapeyre (1996)) and other distributional hypotheses (for an overview see Frey (1997)). However, though the arbitrage-based approach to derivative pricing and hedging is very elegant, it suffers from a severe limitation. The choice of the distribution of the underlyings is quite restricted. An alteration of the probabilistic model not only affects the pricing formulas, it often makes the whole argumentation impossible.

Many papers have addressed derivative pricing and hedging in incomplete models. Since the reasoning (1.1) is not applicable, they usually impose additional conditions. Most approaches are restricted to a certain class of securities price process models (e.g., discrete-time models or continuous-time models with continuous processes driven by Brownian motion). Some of them are based on a general equilibrium framework (cf. Duffie (1992)), some come up with ad-hoc assumptions. The equilibrium framework is appealing from an economic point of view but in complex models the control problems which must be solved in order to derive prices and strategies seem almost intractable. Although our formalism is fundamentally built on maximization of expected utility and on some form of market clearing, we do not place ourselves in a general equilibrium setting. It would be interesting to examine whether our approach could be completely embedded in that framework, but this is beyond our scope here.

As far as hedging is concerned, Schweizer's work (Föllmer & Schweizer (1991), Schweizer (1991)) is related to ours in that he also works in a general semimartingale setting and he also applies a local optimality criterion for trading strategies (minimization of quadratic losses). Contrary to him, we use an increasing utility function since we do not want to penalize strategies that produce gains.

The probabilistic models used to describe the underlyings can be of very different kind. Just consider bivariate diffusions, discrete ARCH time series and hyperbolic jump diffusions (cf. Chapter 4) that are all used to model stock price behaviour. These are not only processes with distinct path properties, they are also expressed in different terms: using stochastic differential equations or infinitesimal generators for diffusions, conditional distributions for time series models and the Lévy jump measure for pure-jump independent increment processes. In order to apply the same formalism in these disparate settings, we have to use a unifying representation that can easily be obtained from the respective notations. The appropriate tool at hand is the notion of *predictable characteristics* for semimartingales, a concept that goes back to Itô, Grigelionis, Jacod & Mémin (cf. Jacod & Shiryaev (1987), p. 573). Although Jacod's comprehensive account (1979) was written almost twenty years ago, this notion seems to be scarcely used in applications. Very loosely speaking, semimartingale characteristics can be compared to the derivative of a time-dependent function. In this respect, *martingale problems* form a stochastic counterpart of ordinary differential equations (ODE's). As with ODE's, the question whether martingale problems have unique solutions is an issue. We give an introduction to predictable characteristics and martingale problems with an emphasis on applications in Chapter 2. No knowledge of finance is needed there.

Although the notions and results from Chapter 2 are necessary to understand our formalism in its full generality, we feel that we should not frighten away the majority of potential readers by confronting them immediately with heavy doses of stochastic calculus. As an appetizer, we present our approach in a lighter fashion in Section 1.2 for the multiperiod model. Although we are applying only moderate portions of probability theory, this exposition contains all the important ideas from an economical point of view. In Chapter 3, we give a mathematically rigorous presentation of our formalism, which is then applied to particular settings in Chapter 4.

Let us mention a peculiarity about our notation that may lead to confusion, but cannot easily be removed. For $x \in \mathbb{R}^n$, x^2 denotes the second component of x, whereas for $x \in \mathbb{R}$, it indicates x squared.

1.2 Intuitive Survey by Means of the Multiperiod Model

1.2.1 The Market Model

In this section we present all economically important ideas at an informal level and without going into mathematical details. We make an effort to be open about the assumptions underlying our results in order to avoid being overinterpreted.

Our object of interest is a securities market with a finite number of traded assets. Like most approaches, we assume some kind of *frictionless market*. In this case, this means that traders can buy and sell arbitrary (including fractional and negative) amounts of any security at a unique market price without any transaction costs, taxes, restrictions or margin requirements. The borrowing and lending interest rate are equal. Any single trader is assumed to be so small that he does not affect market prices. Some of the conditions will be weakened later, but still they form the basis for most of the following. The term *frictionless* is well chosen, since – as in physical models – it means that we make assumptions that are never fulfilled in practice, but allow us to approach the subject by mathematical means. One then hopes that the results form a good approximation of real markets. In general, this will only be true in cases where the "friction" is at least low. In our setting this is to say that we are talking only about heavily traded markets of comparatively large volume and frequency.

The securities at our exchange are termed $0, \dots, n$. The market prices of these assets are described by the $(n+1)$-dimensional stochastic process S, which simply means that S_t^i is the (random) price of security i at time t. Here t takes only the values $0, 1, 2, \dots$, since in this introduction we are working in a discrete-time frame. We assume that the whole market (i.e. the price process S) is governed by some objective probability measure P, on which inference can be made e.g. by statistical means. Security 0 plays a particular role. It serves as a *numeraire* by which all other securities are discounted. The discounted market price of security i at time t is denoted by $Z_t^i := S_t^i / S_t^0$. In the following we consider only the discounted price processes $(Z^0, \dots, Z^n)$ which have to be multiplied by S^0 to return to nominal prices. Usually S^0 is the money market account, i.e. a short-term fixed-income investment with initial value $S_0^0 := 1$. But in principle it could be any traded security. Discounting practically means expressing the value of any asset or portfolio in units of the numeraire S^0. Note that the resulting trading strategies and derivative prices in the following subsections slightly depend on the choice of the numeraire.

We follow the standard approaches in describing trading by another $(n+1)$-dimensional stochastic process φ called *trading strategy*. The random vector $\varphi_t = (\varphi_t^0, \dots, \varphi_t^n)$ is the investor's (hereafter called *you*) portfolio at time t, i.e. at time t you hold φ_t^i shares of security i. The composition of your portfolio can only be based on the information you have, which generally excludes exact knowledge about future price changes. We denote the information that is available to you up to time t as $\mathcal{F}_t$. As is usually done, we assume that you have to order your portfolio for time t strictly before t, i.e. based on the information set $\mathcal{F}_{t-1}$. Mathematically this is to say that φ_t is $\mathcal{F}_{t-1}$-measurable. We call you a *speculator* if you can choose your portfolio freely among all securities $0, \dots, n$. For a *hedger* with fixed positions $\psi^k, \dots, \psi^n$ in assets $k, \dots, n$, the situation is different. He is only free to choose $\varphi_t^0, \dots, \varphi_t^{k-1}$, but the rest of his portfolio is determined by the equalities $\varphi_t^k = \psi^k, \dots, \varphi_t^n = \psi^n$. This is the state of affairs for e.g. a bank that has sold derivatives $k, \dots, n$ and can only trade in the underlyings $0, \dots, k-1$ to hedge the risk.

The *value* of your portfolio (i.e. $\sum_{i=0}^n \varphi^i S^i$ or $\sum_{i=0}^n \varphi^i Z^i$ in discounted terms) changes whenever you gain or lose money due to price changes of the securities or if you invest or withdraw funds. In our approach we are only interested in changes of the first kind. Your *financial gains* in discounted terms at time t are $\Delta G_t(\varphi_t) := \sum_{i=0}^n \varphi_t^i \Delta Z_t^i := \sum_{i=0}^n \varphi_t^i (Z_t^i - Z_{t-1}^i)$, since the discounted securities prices change at time t from Z_{t-1}^i to Z_t^i. We denote your *total gains* up to time t by $G_t(\varphi) := \sum_{s=1}^t \Delta G_s(\varphi_s) = \sum_{s=1}^t \sum_{i=0}^n \varphi_s^i \Delta Z_s^i$.

1.2.2 Optimal Strategies

In this subsection we assume that the probability distribution for future price changes of all assets is known to the investor. We will relax this condition later. It would be great to find an optimal strategy in the sense that it maximizes your financial gains ΔG or G. This will typically not be possible, of course, since you do not know the direction of future price changes in advance. One could now seek to maximize at least the *expected* gain $E(\Delta G_t(\varphi_t))$ or $E(G_t(\varphi))$, but this would contradict economic prudence. Investors usually prefer slightly lower expected returns if they can thereby considerably reduce their risk of losses. One way of taking this into account is by trying to maximize an expected utility instead of the expected gain itself. *Utility* here means a function $u : \mathbb{R} \to \mathbb{R}$ of the gain, i.e. you try to maximize $E(u(\Delta G_t(\varphi_t)))$ or $E(u(G_t(\varphi)))$. If u is appropriately chosen, then optimization of the expected utility takes into account the average return as well as the risk or the degree of uncertainty of the profit. To that end, you want u to be strictly increasing and concave. Strict growth means that you prefer "more" to "less." Concavity is a way of saying that if you earn \$100/month you will be happier about a pay rise of \$50/month than if your salary amounts to \$10,000/month. In particular it means that, when computing expectations, potential losses more than offset potential profits of the same size and likelihood. Utility functions are a common tool in equilibrium theory and they can be backed up in that framework (see e.g. Duffie (1992)). We only use them as a reasonable intuitive concept here. Before we discuss the particular choice of u, we have to decide whether we want to consider ΔG_t or G_t for maximization. If you seek to optimize $E(u(G_T(\varphi)))$ for a given distant time T, you are trading on a long-term basis. You have to maximize only one function, but over a very large set of variables (namely, the set of all strategies between time 0 and T). Alternatively, one can work on a short-term basis by choosing, at each time t, a portfolio φ_{t+1} that maximizes the expected utility $E(u(\Delta G_{t+1}(\varphi_{t+1})))$ for the following period. In economic theory one usually considers *terminal wealth* which is a long-term concept (cf. Korn (1997)). For two reasons we work instead with the one-period gains ΔG_t.

1. A sequence of maximizations in $\mathbb{R}^{n+1}$ is a much simpler mathematical problem than optimizing over the whole set of strategies, which is of a very high dimension. Since numerical tractability is a basic demand for our approach, this alone would be reason enough to consider only local gains.

2. It seems likely to us that many investors really trade on a short-term basis, so that the easier concept may even be as adequate as the other. We also avoid dependencies of the results on the terminal date T.

Now we turn to the shape of the utility function $u : \mathbb{R} \to \mathbb{R}$. We demand the following properties:

1. u is three times continuously differentiable.

2. The derivatives u', u'', u''' are bounded and $\lim_{x\to\infty} u'(x) = 0$.

3. $u(0) = 0$, $u'(0) = 1$.

4. u is strictly increasing (i.e. $u'(x) > 0$ for any $x \in \mathbb{R}$).

5. u is strictly concave (i.e. $u''(x) < 0$ for any $x \in \mathbb{R}$).

$\kappa := -u''(0)$ will be called *risk aversion*. We have already explained that we claim Properties 4 and 5 for economical reasons. The third statement is just a convenient normalisation that does not affect the results. The first two features are set up for mathematical ease and (particularly the boundedness of u') to allow application to a large class of underlying probability distributions. Since we want to give concrete advice to the trader, we propose a one-parametric class of *standard utility functions,* namely

$$u_\kappa : \mathbb{R} \to \mathbb{R},\ x \mapsto \frac{1}{\kappa}(1 + \kappa x - \sqrt{1 + \kappa^2 x^2})$$

for any risk aversion $\kappa > 0$. The functions u_κ are plotted in Figure 1.1 for $\kappa = 0.2$ (dotted line), $\kappa = 1$ (solid line) and $\kappa = 5$ (dashed line). The risk aversion parameter κ must be chosen by the investor according to his tastes. Choosing a very large value means that one tries to minimize the expected losses, almost regardless of the positive gains. As a result, big values of κ may be appropriate for a hedger. On the other hand if κ is small, then $u_\kappa(x)$ behaves like the identity for moderate values of x, so that one is basically maximizing the expected profit without caring about the risk.

We are now ready to define optimal portfolios.

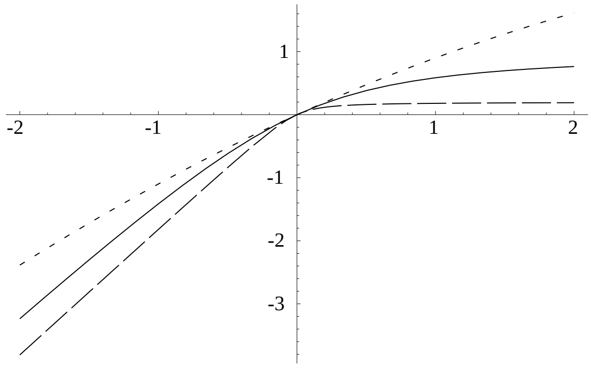

Figure 1.1: Standard utility functions u_κ for $\kappa = 0.2$, $\kappa = 1$, $\kappa = 5$.

Definition 1.1 We call a strategy u-*optimal* if $E(u(\Delta G_t(\varphi_t)))$ is maximal for any $t \in \mathbb{N} \setminus \{0\}$.

Since $E(u(\Delta G_t(\varphi_t))) = E(E(u(\Delta G_t(\varphi_t))|\mathcal{F}_{t-1}))$ (where $E(\cdot|\mathcal{F}_{t-1})$ denotes conditional expectation given $\mathcal{F}_{t-1}$) and since φ_t can be chosen $\mathcal{F}_{t-1}$-measurable, it suffices to maximize the function

$$\psi \mapsto E(u(\Delta G_t(\psi))|\mathcal{F}_{t-1}) = E\Big(u\Big(\sum_{i=1}^n \psi^i \Delta Z_t^i\Big)\Big|\mathcal{F}_{t-1}\Big).$$

Taking partial derivatives yields

Lemma 1.2 *1. A strategy φ is u-optimal for the speculator if and only if for any $t \in \mathbb{N} \setminus \{0\}$*

$$E\Big(u'\Big(\sum_{j=1}^n \varphi_t^j \Delta Z_t^j\Big)\Delta Z_t^i\Big|\mathcal{F}_{t-1}\Big) = 0 \text{ for any } i \in \{1, \dots, n\}.$$

2. A strategy φ is u-optimal for the hedger with fixed positions $\psi^k, \dots, \psi^n$ in the assets $k, \dots, n$ if and only if for any $t \in \mathbb{N} \setminus \{0\}$

(a)

$$E\Big(u'\Big(\sum_{j=1}^n \varphi_t^j \Delta Z_t^j\Big)\Delta Z_t^i\Big|\mathcal{F}_{t-1}\Big) = 0 \text{ for any } i \in \{1, \dots, k-1\},$$

(b)

$$\varphi_t^i = \psi^i \text{ for any } i \in \{k, \dots, n\}.$$

Observe that φ^0 can be arbitrarily chosen because $\Delta Z_t^0 = 0$ for any t. It remains to solve n equations in the n unknowns $\varphi_t^1, \dots, \varphi_t^n$ at any time t. For the rest of this chapter, we assume that the equations in Lemma 1.2 have a unique solution. In Chapter 3 (cf. Theorems 3.28 and 3.26), we show that the existence of optimal strategies is implied by the absence of arbitrage in the following sense.

Definition 1.3 We call a trading strategy φ an *arbitrage* if there is a fixed time $T > 0$ such that $G_T(\varphi) \geq 0$ P-almost surely and $P(G_T(\varphi) > 0) > 0$.

Let us summarize. We have defined a notion of optimality for trading strategies in markets where the distribution of all securities price processes is known. This concept is flexible as to the risk profile of the trader (by adjustment of the risk aversion parameter κ in the standard utility function u_κ) and to his situation (speculator vs. hedger). Since we have chosen a local criterion, optimal strategies can be computed relatively easily by Lemma 1.2.

1.2.3 Trading Corridors

As a real investor you are facing transaction costs. So you are not going to apply a trading strategy necessitating many small adjustments of the portfolio. You have to steer a middle course between too many transactions and positions which are too risky. To assist you, we want to provide you with some sort of alarm that is triggered whenever you are too far off the optimal strategy. More precisely, we define a trading corridor consisting of all portfolios whose expected utility does not fall to more than ε below the optimal value. The utility bandwidth $\varepsilon \in \mathbb{R}_+$ has to be chosen according to the investor's needs. A large parameter ε means accepting a higher risk, whereas a trader who does not want to leave the corridor corresponding to a small ε must reshape his portfolio more often.

Definition 1.4

1. The (u, ε)*-trading corridor* at time t for the speculator is the set of all portfolios $\widehat{\varphi}_t$ such that

$$E\Big(u\Big(\sum_{j=1}^{n} \widehat{\varphi}_t^j \Delta Z_t^j\Big)\Big|\mathcal{F}_{t-1}\Big) \geq E\Big(u\Big(\sum_{j=1}^{n} \varphi_t^j \Delta Z_t^j\Big)\Big|\mathcal{F}_{t-1}\Big) - \varepsilon,$$

   where φ is the u-optimal strategy for the speculator.

2. The (u, ε)*-trading corridor* at time t for the hedger with fixed positions $\psi^k, \dots, \psi^n$ in the assets $k, \dots, n$ is the set of all portfolios $\widehat{\varphi}_t$ such that

$$E\Big(u\Big(\sum_{j=1}^{n} \widehat{\varphi}_t^j \Delta Z_t^j\Big)\Big|\mathcal{F}_{t-1}\Big) \geq E\Big(u\Big(\sum_{j=1}^{n} \varphi_t^j \Delta Z_t^j\Big)\Big|\mathcal{F}_{t-1}\Big) - \varepsilon$$

   and

$$\widehat{\varphi}_t^i = \psi^i \text{ for any } i \in \{k, \dots, n\},$$

   where φ is the u-optimal strategy for the hedger.

It is shown in Chapter 3 that the trading corridors usually form convex subsets of $\mathbb{R}^{n+1}$.

1.2.4 Derivative Pricing

For the computation of optimal strategies we need a probabilistic model for the whole market. Obtaining such a model solely by statistical means has two disadvantages. Firstly, one has to deal with a very large number of stochastic processes which complicates estimation. Secondly, one ignores the fact that some assets are closely linked to others by being derivatives of them. The Black-Scholes model shows that in some settings this connection can be so strong that the derivative price is actually a function of the underlying. In that sense one can interpret the Black-Scholes approach as a model extension from a market with two securities (bank account and stock) to an infinity of assets (bank account, stock and all European options on the stock). In this subsection we will mimic this aspect in a more general situation, albeit on admittedly weaker grounds.

The setting is as follows. We are still considering an exchange using securities $0,\ldots,n$. We assume that the assets, say $l+1,\ldots,n$, are *derivatives* of $0,\ldots,l$ in the sense that, at some future time T, the random vector $(Z_T^{l+1},\ldots,Z_T^n)$ is a deterministic function of the process $(S_t^0, Z_t^1,\ldots,Z_t^l)_{t\in\{0,1,\ldots,T\}}$ (the *underlyings*). As in Subsection 1.2.1, we are given a securities market for the underlyings $0,\ldots,l$, including the probability measure P which governs price changes. However, we do not yet know anything about the derivatives $l+1,\ldots,n$, except their final values $Z_T^{l+1},\ldots,Z_T^n$ in terms of the assets $0,\ldots,l$. Our aim is to build a probabilistic model for the whole market, i.e. to make a reasonable suggestion for the distribution of all securities. The extended model can then be used e.g. to estimate the value of risk of your portfolio or to compute optimal hedging strategies in the sense of Subsection 1.2.2. This extension is only possible under some very strong assumptions which carry a faint equilibrium flavour:

(A 1) We suppose that the vast majority of traders in the derivative market consists of speculators, whereas the influence of other investors (e.g. hedgers) is negligable.

(A 2) Moreover, we assume that the speculators intuitively (by their market experience) know the real distribution of all securities prices including the derivatives and that they trade (maybe unknowingly) by maximizing their expected utility in the sense of Subsection 1.2.2. We suppose that they all work with standard utility functions, but possibly with a differing risk aversion κ.

What is a speculator doing under these assumptions? He is choosing the u_κ-optimal strategy φ according to his risk-aversion κ. By Lemma 1.2 and since $u_\kappa'(x) = u_1'(\kappa x)$ for any $\kappa > 0$, $x \in \mathbb{R}$, this strategy φ satisfies

$$E\Big(u_1'\Big(\sum_{j=1}^n \kappa\varphi_t^j \Delta Z_t^j\Big)\Delta Z_t^i \Big| \mathfrak{F}_{t-1}\Big) = E\Big(u_\kappa'\Big(\sum_{j=1}^n \varphi_t^j \Delta Z_t^j\Big)\Delta Z_t^i \Big| \mathfrak{F}_{t-1}\Big) = 0 \tag{1.2}$$

for $i = 1,\ldots,n$ and any t. It follows that all speculators trade with multiples of the u_1-optimal strategy. In particular, if any speculator has a positive (resp. negative) amount of a certain derivative in his portfolio, then the others do as well. However, according to our first assumption there are only few potential suppliers of these assets compared to a huge crowd of unrestricted traders. Hence, the portfolio of any typical speculator must contain practically no derivative. To phrase it mathematically, we can draw the following

Conclusion 1.5 *If φ is the u_κ-optimal strategy for the speculator, then $\varphi^i = 0$ for $i = l+1,\dots,n$.*

From now on, fix $\kappa > 0$ and let φ denote the u_κ-optimal strategy for the speculator. By Lemma 1.2 it follows that

$$E\Big(u'_\kappa\Big(\sum_{j=1}^{l}\varphi_t^j\Delta Z_t^j\Big)\Delta Z_t^i\Big|\mathcal{F}_{t-1}\Big) = 0 \tag{1.3}$$

for $i = 1,\dots,n$ and any t. In particular (again by Lemma 1.2), $(\varphi^0,\dots,\varphi^l)$ is the u_κ-optimal portfolio in the restricted market consisting only of the underlyings $0,\dots,l$ and can be calculated without knowing the derivative prices. Recall that we have assumed that the optimal portfolios are unique except for φ^0 which can be arbitrarily chosen. In Chapter 3 we see that one can do without this restriction.

Observe that Equation (1.3) allows to compute the derivative prices Z_{T-1}^i, Z_{T-2}^i etc. recursively. Indeed, since $Z^1,\dots,Z^l$ are given and $\varphi^1,\dots,\varphi^l$ do not depend on the derivative prices, Z_{t-1}^i can be obtained from Z_t^i by solving Equation (1.3). Since such a recursive procedure is not applicable in continuous-time models, we will show how to obtain the derivative prices in one step. To that end, we define a new probability measure P^*, equivalent to the objective probability measure P, by its Radon-Nikodým density

$$\frac{dP^*}{dP} := \prod_{t=1}^{T}\frac{u'_\kappa\Big(\sum_{j=1}^{l}\varphi_t^j\Delta Z_t^j\Big)}{E\Big(u'_\kappa\Big(\sum_{j=1}^{l}\varphi_t^j\Delta Z_t^j\Big)\Big|\mathcal{F}_{t-1}\Big)}. \tag{1.4}$$

Proposition 1.6

1. *The expectation of the right-hand side of Equation (1.4) equals 1, so P^* is well-defined.*
2. *The definition of P^* does not depend on κ.*
3. *For $t = 1,\dots,T$ and any $\mathcal{F}_t$-measurable random variable Y we have that*

$$E^*(Y|\mathcal{F}_{t-1}) = \frac{E\Big(Yu'_\kappa\Big(\sum_{j=1}^{l}\varphi_t^j\Delta Z_t^j\Big)\Big|\mathcal{F}_{t-1}\Big)}{E\Big(u'_\kappa\Big(\sum_{j=1}^{l}\varphi_t^j\Delta Z_t^j\Big)\Big|\mathcal{F}_{t-1}\Big)},$$

where E^* denotes expectation with respect to P^* instead of P.

PROOF.

1. Since $E(dP^*/dP) = E(E(dP^*/dP|\mathcal{F}_0))$, it suffices to show that

$$E\left(\frac{dP^*}{dP}\Big|\mathcal{F}_t\right) = \prod_{s=1}^{t}\frac{u'_\kappa\Big(\sum_{j=1}^{l}\varphi_s^j\Delta Z_s^j\Big)}{E\Big(u'_\kappa\Big(\sum_{j=1}^{l}\varphi_s^j\Delta Z_s^j\Big)\Big|\mathcal{F}_{s-1}\Big)} \tag{1.5}$$

for $t = 0, \ldots, T$. (Just take $t = 0$.) By backward induction and the properties of conditional expectation we have that

$$\begin{aligned}
E\left(\frac{dP^*}{dP}\Big|\mathfrak{F}_{t-1}\right) &= E\left(E\left(\frac{dP^*}{dP}\Big|\mathfrak{F}_t\right)\Big|\mathfrak{F}_{t-1}\right) \\
&= \prod_{s=1}^{t-1} \frac{u'_\kappa\left(\sum_{j=1}^l \varphi_s^j \Delta Z_s^j\right)}{E\left(u'_\kappa\left(\sum_{j=1}^l \varphi_s^j \Delta Z_s^j\right)\Big|\mathfrak{F}_{s-1}\right)} \\
&\quad \cdot E\left(\frac{u'_\kappa\left(\sum_{j=1}^l \varphi_t^j \Delta Z_t^j\right)}{E\left(u'_\kappa\left(\sum_{j=1}^l \varphi_t^j \Delta Z_t^j\right)\Big|\mathfrak{F}_{t-1}\right)}\Bigg|\mathfrak{F}_{t-1}\right) \\
&= \prod_{s=1}^{t-1} \frac{u'_\kappa\left(\sum_{j=1}^l \varphi_s^j \Delta Z_s^j\right)}{E\left(u'_\kappa\left(\sum_{j=1}^l \varphi_s^j \Delta Z_s^j\right)\Big|\mathfrak{F}_{s-1}\right)}.
\end{aligned}$$

2. Since $u'_\kappa(x) = u'_1(\kappa x)$ for any x, we have that φ is u_κ-optimal if and only if $\kappa\varphi$ is the u_1-optimal strategy. This implies that P^* does not change if we replace κ with 1 and the u_κ-optimal strategy φ with the u_1-optimal strategy.

3. Let $A \in \mathfrak{F}_{t-1}$. By Equation (1.5) we have

$$\begin{aligned}
E^*(1_A Y) &= E\left(1_A Y \frac{dP^*}{dP}\right) \\
&= E\left(1_A E\left(Y E\left(\frac{dP^*}{dP}\Big|\mathfrak{F}_t\right)\Big|\mathfrak{F}_{t-1}\right)\right) \\
&= E\left(1_A E\left(\frac{dP^*}{dP}\Big|\mathfrak{F}_{t-1}\right) \frac{E\left(Y u'_\kappa\left(\sum_{j=1}^l \varphi_t^j \Delta Z_t^j\right)\Big|\mathfrak{F}_{t-1}\right)}{E\left(u'_\kappa\left(\sum_{j=1}^l \varphi_t^j \Delta Z_t^j\right)\Big|\mathfrak{F}_{t-1}\right)}\right) \\
&= E^*\left(1_A \frac{E\left(Y u'_\kappa\left(\sum_{j=1}^l \varphi_t^j \Delta Z_t^j\right)\Big|\mathfrak{F}_{t-1}\right)}{E\left(u'_\kappa\left(\sum_{j=1}^l \varphi_t^j \Delta Z_t^j\right)\Big|\mathfrak{F}_{t-1}\right)}\right).
\end{aligned}$$

□

By Equation (1.3) and Statement 3 of the previous proposition, it follows that $E^*(\Delta Z_t^i|\mathfrak{F}_{t-1}) = 0$. Since Z_{t-1} is $\mathfrak{F}_{t-1}$-measurable, we obtain $Z_{t-1}^i = E^*(Z_t^i|\mathfrak{F}_{t-1})$ for $t = 1, \ldots, T$ and $i = 0, \ldots, n$. Thus the processes $(Z_t^i)_{t=0,\ldots,T}$ are P^*-martingales and we have shown the following

Lemma 1.7 *The processes $(Z_t^i)_{t=0,\ldots,T}$ are P^*-martingales for $i = 0, \ldots, n$. In particular, the derivative prices are uniquely given by*

$$Z_t^i = E^*(Z_T^i|\mathfrak{F}_t)$$

for $i = l+1, \ldots, n$ and any t.

Note that some regularity conditions are needed to make the previous lemma hold. These can be found in Chapter 3, where we also give more rigorous proofs.

By Lemma 1.7, derivative prices are obtained by calculating conditional expectations under an equivalent martingale measure in the sense of

Definition 1.8 A probability measure $P^* \sim P$ (i.e. P^* and P have the same null sets) is called *equivalent martingale measure (EMM)* for the market with terminal date T if the discounted securities price processes $(Z_t^i)_{t=0,\dots,T}$ are P^*-martingales for $i = 0, \dots, n$.

By a well-known result (cf. Lemma 3.7) the existence of an EMM implies that the extended market admits no arbitrage strategies, which is desirable for a reasonable market model.

The traditional arbitrage-based approach of Black and Scholes is usually applied to complete settings where the cash flow of any derivative can be duplicated by a dynamic portfolio (i.e. a trading strategy) consisting only of underlyings. The only price process consistent with an absence of arbitrage in this case is the value process of the corresponding duplicating portfolio, which can be obtained by calculating conditional expectations under an EMM as in Lemma 1.7. Since there usually exists only one such measure in complete models (cf. Lamberton & Lapeyre (1996), Theorem 1.3.4), both approaches to derivative pricing yield the same result.

Let us mention two alternatives to substitute for the crucial Assumptions (A 1) and (A 2) underlying our pricing approach. We have already observed that the optimal strategies of the speculators differ only by a factor. By Equation (1.2) it is in fact easy to see that the union of the portfolios of p speculators with, say, risk aversions $\kappa_1, \dots, \kappa_p$ is the u_κ-optimal strategy for a speculator with risk aversion $\kappa := 1/(\sum_{i=1}^p \kappa_i^{-1})$. If other investors are virtually absent, then this imaginary trader can be interpreted as a *representative agent* standing in for the whole market. Since any derivative that is bought by some investor has to be sold by another, the union of all portfolios must contain zero derivatives. By loosely applying terms from equilibrium theory one may rephrase Assumptions (A 1) and (A 2) as

(A 1′) Derivative markets *clear*, i.e. the representative agent has a zero position in the assets $l+1, \dots, n$.

(A 2′) The representative agent is a speculator maximizing his expected standard utility for some $\kappa > 0$. In fact, the behaviour of any single trader is irrelevant, as long as the joint strategy of all investors is u_κ-optimal for some $\kappa > 0$.

The third approach leading to Conclusion 1.5 focuses on the issuer and is quite different from the first two. Suppose that a derivative is supplied by a bank for a fixed price. We are interested in the lowest price at which the bank is willing to offer this security. If it uses u_κ-optimal strategies for some $\kappa > 0$, then the threshold is the price at which the optimal portfolio contains zero derivatives. If the price is lower, selling is disadvantageous, if it is higher, it becomes increasingly profitable. Hence, if the bank is a speculator using standard utility functions and if we assume that the market price of the derivative is close to its threshold value, we end up again at Conclusion 1.5.

Let us once again give a brief summary. We have derived derivative prices based only on the probability distribution of the underlyings. This model extension is based on strong economic assumptions. Lemma 1.7 shows that the derivative price processes can be computed by calculating conditional expectations under an equivalent martingale measure. This implies that the extended market model is arbitrage-free and that, in complete models, the derived prices coincide with the unique arbitrage-based values.

What are the limitations of our suggested prices?

1. An extended model can never be better than the underlying probabilistic description of the assets $0, \dots, l$. This is why one should not focus too strongly on complete settings, although the derivative prices are better founded in these models. They often do not fit the distribution of the underlyings very well.

2. The assumption concerning the genesis of derivative prices may be intuitive, but of course it can only be a rough approximation. Except for derivatives that can actually be duplicated, market prices stem from extremely complex, interrelated mechanisms. Therefore, we doubt that any economical model will ever be able to determine derivative prices correctly as a function of the underlyings and some exogenous variables. Still, a lot of investors want reasonable concrete results to base their decisions on. This is exactly the purpose of our pricing approach. In the next two subsections we will present ways to estimate the accuracy of our proposed prices and to improve the market model, although this involves more complicated computations.

1.2.5 Price Regions

In the previous subsection we computed derivative prices under the condition that all investors in the market were speculators. This implied that any of these traders had a zero position in any derivative. In the following two subsections we allow for the existence of other traders who hold a non-zero amount of derivatives in their portfolio. If the positions of these other traders do not offset each other, then the speculators have to assume the counterposition. Hence, the union of the speculators' portfolios does not contain zero derivatives, as was assumed in the previous subsection. We want to examine how this change affects market prices. To this end, we replace the first of the two assumptions in Subsection 1.2.4 with

(A $\hat{1}$) The union of the portfolios of all speculators contains, at any time t and for any $i \in \{l+1, \dots, n\}$, ρ^i shares of Security i,

where $\rho^{l+1}, \dots, \rho^n$ are fixed real numbers (called the *external supply*). Observe that the original Assumption (A 1) is a verbal paraphrase of Condition (A $\hat{1}$) in the case $\rho^{l+1} = 0, \dots, \rho^n = 0$. In the previous subsection we observed that the union of all the speculators' portfolios is again a u_κ-optimal strategy for some $\kappa > 0$. We refer to this κ as the *risk aversion of the representative speculator* (in short: *representative risk aversion*). Given the preceding remark and Condition (A $\hat{1}$), the following definition should be obvious.

Definition 1.9 We call discounted price processes $Z^{l+1}, \dots, Z^n$ *consistent with the representative risk aversion* $\kappa > 0$ *and the external supply* $\rho^{l+1}, \dots, \rho^n$ (in short: $(\kappa, \rho^{l+1}, \dots, \rho^n)$*-consistent*), if the u_κ-optimal strategy φ for the speculator satisfies

$$\varphi_t^i = \rho^i \text{ for } i = l+1, \dots, n \text{ and any } t$$

(i.e. the u_κ-optimal strategy for the hedger with fixed positions $\rho^{l+1}, \dots, \rho^n$ in the assets $l+1, \dots, n$ is u_κ-optimal for the speculator).

In Subsection 1.2.4 we calculate $(\kappa, 0, \dots, 0)$-consistent price processes by computing conditional expectations under an equivalent martingale measure. We will see that this is also possible for non-vanishing external supply. For this purpose, fix $(\kappa, \rho^{l+1}, \dots, \rho^n)$ and let $Z^{l+1}, \dots, Z^n$ and φ be as in Definition 1.9. Define a new probability measure P^*, equivalent to the objective probability measure P, by its Radon-Nikodým density

$$\frac{dP^*}{dP} := \prod_{t=1}^{T} \frac{u'_\kappa\Big(\sum_{j=1}^n \varphi_t^j \Delta Z_t^j\Big)}{E\Big(u'_\kappa\Big(\sum_{j=1}^n \varphi_t^j \Delta Z_t^j\Big)\Big|\mathcal{F}_{t-1}\Big)}. \tag{1.6}$$

With the same proof as in Proposition 1.6, one shows

Proposition 1.10 1. *The expectation of the right-hand side of Equation (1.6) equals* 1*, so* P^* *is well-defined.*

2. *For* $t = 1, \dots, T$ *and any* $\mathcal{F}_t$*-measurable random variable* Y *we have that*

$$E^*(Y|\mathcal{F}_{t-1}) = \frac{E\Big(Y u'_\kappa\Big(\sum_{j=1}^n \varphi_t^j \Delta Z_t^j\Big)\Big|\mathcal{F}_{t-1}\Big)}{E\Big(u'_\kappa\Big(\sum_{j=1}^n \varphi_t^j \Delta Z_t^j\Big)\Big|\mathcal{F}_{t-1}\Big)}.$$

As in Subsection 1.2.4, we conclude from Lemma 1.2 and the second statement of the previous proposition that $E^*(\Delta Z_t^i|\mathcal{F}_{t-1}) = 0$ for $i = 0, \dots, n$ and any t. Hence, the processes $(Z_t^i)_{t=0,\dots,T}$ are again P^*-martingales, but this time for P^* defined by Equation (1.6). Thus we have obtained

Lemma 1.11 *Suppose that the market prices are* $(\kappa, \rho^{l+1}, \dots, \rho^n)$*-consistent. Then the processes* $(Z_t^i)_{t=0,\dots,T}$ *are* P^**-martingales for* $i = 0, \dots, n$*, where the EMM* P^* *is given by Equation (1.6) and* φ *is the* u_κ*-optimal strategy for the speculator in the market* $0, \dots, n$*.*

Let us try to understand what $(\kappa, \rho^{l+1}, \dots, \rho^n)$-consistent prices mean. In complete models derivative prices can be derived solely based on the absence of arbitrage. They are independent of supply and demand, making those models very attractive. In more general settings this is no longer true. Derivative prices are a function not only of the underlyings, but also of the extent to which they are asked for by investors wanting to satisfy their needs. In our pricing approach this is taken into account by specifying the external supply (resp. demand for negative values) $\rho^{l+1}, \dots, \rho^n$. Without considering concrete examples here, one would intuitively expect the current derivative price to be lower (resp. higher) than the $(\kappa, 0, \dots, 0)$-consistent prices from Subsection 1.2.4 if the respective supply is greater (resp. less) than 0, since surplus supply generally tends to lower market prices, whereas excess demand increases them. Note that by Lemma 1.11, $(\kappa, \rho^{l+1}, \dots, \rho^n)$-consistent market prices are always arbitrage-free, since they can be computed by means of an equivalent martingale measure. In particular, they do not depend on the parameter vector $(\kappa, \rho^{l+1}, \dots, \rho^n)$ if there exists but one EMM. This is another way of saying that in complete models (where there is a unique EMM) derivative prices are independent of supply and demand. This property of complete models leads us to measure the degree of incompleteness of the given model or, more precisely, the degree of unattainability of the contingent claims under consideration by the extent to which derivative prices do in fact depend on the supply $\rho^{l+1}, \dots, \rho^n$. To this end, we replace the unique derivative prices from Subsection 1.2.4 with the set of prices corresponding to any external supply that does not exceed a given bound. More specifically, we have the following

Definition 1.12 As before, the underlyings $0, \dots, l$ and the derivatives at maturity $Z_T^{l+1}, \dots, Z_T^n$ are given. Fix a representative risk aversion $\kappa > 0$ and a *supply bound* $r \geq 0$. We say that derivative price processes $(Z_t^{l+1}, \dots, Z_t^n)_{t=0,\dots,T}$ belong to the κr*-price region* if they are $(\kappa, \rho^{l+1}, \dots, \rho^n)$-consistent market price processes for some $\rho^{l+1}, \dots, \rho^n$ satisfying $|\rho^i| \leq r$ for $i = l+1, \dots, n$.

Remark. One easily sees that the price region depends only on the product κr of κ and r. Therefore it makes sense to use the term κr-price region instead of (κ, r)-price region.

Price regions may be compared to confidence regions in statistics, although they have nothing to do with probability. In neither situation we have enough information to uniquely determine a certain quantity (an unknown parameter in statistics, derivative prices in finance). We can now take one of two paths. One option is to choose a particular value (some optimal estimator in statistics, the derivative prices from Subsection 1.2.4 in finance). Alternatively, we may give a set (confidence/price region) consisting of those values that are – according to some criterion – the most reasonable ones. Price regions (as confidence regions) have the advantage that they contain information concerning the precision of the proposed values. Therefore, they are particularly suited for model comparison. If for fixed κr the price region is comparatively small or even zero, then derivative prices are chiefly resp. entirely determined by the underlyings and only weakly dependent on supply and demand. In this case the proposed derivative prices from Subsection 1.2.4 should form a reasonable approximation. On the other hand, in settings where the price region is comparatively large, model extensions solely based on the underlyings might be of limited explanatory power, since the derivative market may follow its own dynamics to some extent.

Although we consider them to be a useful concept, price regions in the sense of Definition 1.12 face two drawbacks because they are defined in terms of $(\kappa, \rho^{l+1}, \dots, \rho^n)$-consistent price processes.

1. We have not shown that $(\kappa, \rho^{l+1}, \ldots, \rho^n)$-consistent derivative prices really exist for any choice of the parameter vector. Especially in the general continuous-time context of Chapter 3 (cf. Section 3.5), no satisfactory sufficient conditions for existence are known so far. This question should be addressed in future research.

2. Except for the simplest case ($\rho^{l+1} = 0, \ldots, \rho^n = 0$), consistent price processes are generally hard to compute explicitly. In order to see this, compare the Lemmas 1.7 and 1.11. The derivative prices are obtained in both cases by computing conditional expectations under an equivalent martingale measure. But whereas the EMM P^* in Subsection 1.2.4 is defined only in terms of the underlyings $Z^0, \ldots, Z^l$, the pricing measure P^* in the current subsection (cf. Equation (1.6)) also depends on the derivative prices $Z^{l+1}, \ldots, Z^n$ that have yet to be calculated. A way out of this vicious circle is to proceed by backward recursion. The derivative prices $Z_t^{l+1}, \ldots, Z_t^n$ for $t = T$ (maturity) are, by assumption, given in terms of the underlyings. If the market is $(\kappa, \rho^{l+1}, \ldots, \rho^n)$-consistent, then there exists, by Lemma 1.2, a strategy φ such that
$$E\left(u'\Big(\sum_{j=1}^n \varphi_t^j \Delta Z_t^j\Big)\Delta Z_t^i \Big| \mathcal{F}_{t-1}\right) = 0 \text{ for any } i \in \{1, \ldots, n\} \tag{1.7}$$
and
$$\varphi_t^i = \rho^i \text{ for any } i \in \{l+1, \ldots, n\}.$$
Since $\varphi_t^{l+1}, \ldots, \varphi_t^n$ are known, Statement (1.7) is a system of n equations in the n unknowns $\varphi_t^1, \ldots, \varphi_t^l, Z_{t-1}^{l+1}, \ldots, Z_{t-1}^n$. Given that a unique solution exists, we may solve for $Z_{t-1}^{l+1}, \ldots, Z_{t-1}^n$ and subsequently in the same manner for Z_{t-2}, Z_{t-3} etc. However, this recursive algorithm has no continuous-time counterpart. Therefore, efficient computation of $(\kappa, \rho^{l+1}, \ldots, \rho^n)$-consistent prices is also an issue for future research.

We now define an alternative notion of price regions that is less satisfactory from a theoretical point of view but avoids the stated problems. To this end, we replace the $(\kappa, \rho^{l+1}, \ldots, \rho^n)$-consistent prices in Definition 1.9 with $(\kappa, \rho^{l+1}, \ldots, \rho^n)$-approximate prices that are obtained as follows:

1. As before, the underlyings $0, \ldots, l$ and the derivatives at maturity $Z_T^{l+1}, \ldots, Z_T^n$ are given as input. Fix $\kappa > 0$ and $\rho^i \in \mathbb{R}$ for $i = l+1, \ldots, n$.

2. Take derivative prices $Z^{l+1}, \ldots, Z^n$ as in Subsection 1.2.4.

3. Let φ be the u_κ-optimal strategy for the hedger with fixed positions $\rho^{l+1}, \ldots, \rho^n$ in the assets $l+1, \ldots, n$, given the derivative prices $Z^{l+1}, \ldots, Z^n$ from step 2.

4. Define a new probability measure P^*, equivalent to the objective probability measure P, by its Radon-Nikodým density
$$\frac{dP^*}{dP} := \prod_{t=1}^T \frac{u'_\kappa\Big(\sum_{j=1}^n \varphi_t^j \Delta Z_t^j\Big)}{E\Big(u'_\kappa\Big(\sum_{j=1}^n \varphi_t^j \Delta Z_t^j\Big)\Big|\mathcal{F}_{t-1}\Big)}.$$

5. Define new derivative price processes $\widehat{Z}^{l+1}, \ldots, \widehat{Z}^n$ by

$$\widehat{Z}_t^i := E^*(Z_T^i|\mathfrak{F}_t) \text{ for } i = l+1, \ldots, n \text{ and any } t.$$

$\widehat{Z}^{l+1}, \ldots, \widehat{Z}^n$ shall be called $(\kappa, \rho^{l+1}, \ldots, \rho^n)$-*approximate* price processes.

In the fourth step of this definition we mimic Equation 1.6, but we replace the $(\kappa, \rho^{l+1}, \ldots, \rho^n)$-consistent prices that we do not know with $(\kappa, 0, \ldots, 0)$-consistent prices as an approximation. Likewise, φ is based on $Z^{l+1}, \ldots, Z^n$ from Subsection 1.2.4 instead of the unknown prices as in Definition 1.9. Approximate prices are then computed as P^*-martingales just as in Lemma 1.11. Observe that if we started with $(\kappa, \rho^{l+1}, \ldots, \rho^n)$-consistent prices instead of the processes from the previous subsection in the second step, $\widehat{Z}^{l+1}, \ldots, \widehat{Z}^n$ would become $(\kappa, \rho^{l+1}, \ldots, \rho^n)$-consistent as well (by Lemma 1.11). Our hope is that for moderate values of $\rho^{l+1}, \ldots, \rho^n$ the approximate prices are close to the corresponding consistent market prices (cf. Subsection 4.1.4), but no rigorous statement has been proved yet. One may also iterate steps 2 to 5 of the above five-step procedure by substituting $\widehat{Z}^{l+1}, \ldots, \widehat{Z}^n$ for $Z^{l+1}, \ldots, Z^n$ in the second step and obtain an improved approximation $\widetilde{Z}^{l+1}, \ldots, \widetilde{Z}^n$ etc. One can perhaps apply this iteration procedure in order to obtain $(\kappa, \rho^{l+1}, \ldots, \rho^n)$-consistent price processes in the limit (cf. Subsection 4.1.4). Be this as it may, we apply approximate market prices here since they are well-defined, can be obtained with sufficient ease, and share the following useful properties.

Lemma 1.13 *1. If $\widehat{Z}^{l+1}, \ldots, \widehat{Z}^n$ are $(\kappa, \rho^{l+1}, \ldots, \rho^n)$-approximate price processes, then P^* from step 4 is an equivalent martingale measure for the market $(Z^0, \ldots, Z^l, \widehat{Z}^{l+1}, \ldots, \widehat{Z}^n)$, which is therefore arbitrage-free.*

2. For $\rho^{l+1} = 0, \ldots, \rho^n = 0$ both $(\kappa, \rho^{l+1}, \ldots, \rho^n)$-approximate and $(\kappa, \rho^{l+1}, \ldots, \rho^n)$-consistent prices coincide with the derivative price processes from Lemma 1.7.

3. If there exists only one EMM, then approximate prices and consistent prices necessarily coincide with the unique arbitrage-free prices.

PROOF.

1. Firstly observe that Proposition 1.10 also holds for P^* from step 4. By Statement 2 in Lemma 1.2 we have that $E(\Delta Z_t^i u_\kappa'(\sum_{j=1}^n \varphi_t^j \Delta Z_t^j)|\mathfrak{F}_{t-1}) = 0$ for $i = 1, \ldots, l$ and any t. By Statement 2 of Proposition 1.10, it follows that $E^*(\Delta Z_t^i|\mathfrak{F}_{t-1}) = 0$ for $i = 1, \ldots, l$ and any t. This implies that $Z^0, \ldots, Z^l$ are P^*-martingales. Moreover, $\widehat{Z}^{l+1}, \ldots, \widehat{Z}^n$ are P^*-martingales by definition.

2. This follows from the definitions, from Lemma 1.7 and from Conclusion 1.5.

3. This follows immediately from Statement 1, Lemma 1.11 and Lemma 1.7. □

Parallel to Definition 1.12 we now define approximate price regions for use in place of κr-price regions.

Definition 1.14 Fix a representative risk aversion $\kappa > 0$ and a supply bound $r \geq 0$. We say that derivative price processes $(Z_t^{l+1}, \ldots, Z_t^n)_{t=0,\ldots,T}$ belong to the *approximate κr-price region* if $(Z_t^{l+1}, \ldots, Z_t^n)_{t=0,\ldots,T}$ are $(\kappa, \rho^{l+1}, \ldots, \rho^n)$-approximate market price processes for some $\rho^{l+1}, \ldots, \rho^n$ satisfying $|\rho^i| \leq r$ for $i = l+1, \ldots, n$.

We want to summarize the notions from this subsection. There are no unique arbitrage-free derivative values in incomplete models. Prices may depend on supply and demand. We take this fact into account by introducing consistent prices that incorporate external supply. By uniting all price processes that correspond to moderate demand, we define price regions that can be used to measure the degree of incompleteness of the market and allow us to assess the accuracy of the prices from Subsection 1.2.4. For computational ease we also introduce approximate prices and approximate price regions as a substitute for consistent prices and price regions. For better justification of the concepts from this subsection, a few questions still must be resolved. Firstly, what are sufficient conditions for the existence of processes that are consistent with given external supply? Secondly, efficient algorithms to compute these prices are desirable. Thirdly, under what conditions and in what sense do approximate prices converge to their consistent counterparts?

1.2.6 Improved Derivative Models

In Subsection 1.2.4 we compute derivative prices mainly for the purpose of model extension. Such an extension should be based on all the available information. Therefore, we should incorporate the initial derivative prices $Z_0^{l+1}, \ldots, Z_0^n$ in our model, since they are observable in the market and have not yet been taken into account. The idea is again to replace the first assumption in Subsection 1.2.4 with Condition (A $\hat{1}$) from Subsection 1.2.5 and hence to work with $(\kappa, \rho^{l+1}, \ldots, \rho^n)$-consistent price processes.

Definition 1.15 In addition to the underlyings $0, \ldots, l$ and the terminal values $Z_T^{l+1}, \ldots, Z_T^n$, fix initial derivative prices $p^{l+1}, \ldots, p^n$. We call discounted price processes $Z^{l+1}, \ldots, Z^n$ *consistent with the initial prices* $p^{l+1}, \ldots, p^n$ (in short: $(p^{l+1}, \ldots, p^n)$*-consistent*) if

1. there exists a $\kappa > 0$ and $\rho^{l+1}, \ldots, \rho^n \in \mathbb{R}$ such that $Z^{l+1}, \ldots, Z^n$ are $(\kappa, \rho^{l+1}, \ldots, \rho^n)$-consistent,
2. $$Z_0^i = p^i \text{ for } i = l+1, \ldots, n.$$

By using $(p^{l+1}, \ldots, p^n)$-consistent processes instead of the prices from Subsection 1.2.4, we are hitting two birds with one stone. Firstly, we avoid contradicting observed and theoretical initial prices. Secondly, we can relax the strong Assumption (A 1) (that speculators hold virtually no derivatives) to the weaker Condition (A $\hat{1}$) (that speculators hold a constant amount of derivatives). However, as noted in the previous subsection, consistent price processes are mathematically intricate. Therefore, we once again introduce a second concept that is less intuitive from an economic point of view but facilitates explicit computations.

Definition 1.16 Fix initial derivative prices $p^{l+1}, \dots, p^n$. Discounted price processes Z^{l+1}, $\dots, Z^n$ shall be called *approximately* $(p^{l+1}, \dots, p^n)$*-consistent* if

1. there exists a $\kappa > 0$ and $\rho^{l+1}, \dots, \rho^n \in \mathbb{R}$ such that $Z^{l+1}, \dots, Z^n$ are $(\kappa, \rho^{l+1}, \dots, \rho^n)$-approximate price processes,

2. $$Z_0^i = p^i \text{ for } i = l+1, \dots, n.$$

Let us make some

Remarks.

1. The risk-aversion κ in Definitions 1.15 and 1.16 can in fact be chosen arbitrarily, since by Lemma 1.2 and $u'_\kappa(x) = u'_1(\kappa x)$ we have that $(\kappa, \rho^{l+1}, \dots, \rho^n)$-consistent processes are $(\widehat{\kappa}, (\kappa/\widehat{\kappa})\rho^{l+1}, \dots, (\kappa/\widehat{\kappa})\rho^n)$-consistent for any $\kappa, \widehat{\kappa} > 0$. Therefore, one may choose $\kappa := 1$ without loss of generality in both definitions.

2. As shown in Lemma 1.11, $(p^{l+1}, \dots, p^n)$-consistent markets always constitute arbitrage-free price systems.

3. It is easy to see that $(p^{l+1}, \dots, p^n)$-consistent prices do not necessarily exist for arbitrary $p^{l+1}, \dots, p^n$. Indeed, the initial derivative values $Z_0^{l+1}, \dots, Z_0^n$ are uniquely determined in complete markets by the absence of arbitrage. Therefore, in this case there is only one price vector $(p^{l+1}, \dots, p^n)$ such that $(p^{l+1}, \dots, p^n)$-consistent prices exist.

4. It is an open question as to whether $(p^{l+1}, \dots, p^n)$-consistent price processes are completely determined by the initial prices $p^{l+1}, \dots, p^n$.

5. Choosing $(p^{l+1}, \dots, p^n)$-consistent price processes is related to the method of *inverting the yield curve* in interest rate theory (cf. Björk (1997), Subection 3.5). In both settings, one considers a parametric family of equivalent martingale measures and one uses initial derivative prices (i.e. bond prices in interest rate theory) to determine the unknown parameters. We will apply our approach to interest rate models in Section 4.9.

6. Remarks 1–5 also hold for approximately $(p^{l+1}, \dots, p^n)$-consistent instead of $(p^{l+1}, \dots, p^n)$-consistent price processes.

7. Similar to the previous subsection, it is desirable to prove convergence results relating $(p^{l+1}, \dots, p^n)$-consistent and approximately $(p^{l+1}, \dots, p^n)$-consistent price processes (for a comparison see Subsection 4.1.4).

Observe that for the construction of $(p^{l+1}, \dots, p^n)$-consistent price processes we assumed that the external supply $\rho^{l+1}, \dots, \rho^n$ stays constant through time. Therefore, the extended model will not keep track of variability that is due to changing demand for derivatives. One can now go one step further and take this variability into account by constructing models with *stochastic external supply*. Since this step towards more flexibility leads to even more demanding computations, we will limit ourselves here to sketching a conceivable procedure as an outlook. *Stochastic supply* stands for a randomly changing vector $(\rho_t^{l+1}, \dots, \rho_t^n)_{t=1,\dots,T}$, where ρ_t^i denotes the supply of Security i at time t which we assume to be known at time $t-1$, i.e. $\mathcal{F}_{t-1}$-measurable. As in Subsection 1.2.5, we may now define $(\kappa, \rho^{l+1}, \dots, \rho^n)$*-approximate price processes* by substituting the random supply $(\rho_t^{l+1}, \dots, \rho_t^n)$ for the fixed supply $\rho^{l+1}, \dots, \rho^n$ in the third step. The statements and proofs of Lemma 1.13 also hold for these generalized $(\kappa, \rho^{l+1}, \dots, \rho^n)$-approximate price processes. For model building one may now proceed as follows.

1. As before, subdivide the market into underlyings $0, \dots, l$ as well as derivatives $l+1, \dots, n$ and take a good probabilistic model for the underlyings. Fix $\kappa := 1$.

2. Take a probabilistic model of Markovian type for the external supply process $(\rho_t^{l+1}, \dots, \rho_t^n)_{t=1,\dots,T}$ (e.g. a Markov chain in discrete time or a diffusion process in continuous time). Do not specify the initial supply $(\rho_1^{l+1}, \dots, \rho_1^n)$ yet. All that is still missing for computation of $(\kappa, \rho^{l+1}, \dots, \rho^n)$-approximate prices is the current value $(\rho_1^{l+1}, \dots, \rho_1^n)$, which is not directly observable.

3. As for approximately $(p^{l+1}, \dots, p^n)$-consistent processes (cf. Definition 1.16), try to evaluate the initial supply $(\rho_1^{l+1}, \dots, \rho_1^n)$ such that theoretical and observed derivative prices in $t = 0$ coincide. The market model for Securities $0, \dots, n$ is now completely determined.

4. The last step is to check whether the model extension and especially the supply dynamic in step 2 fits the real data well. To that end you successively calculate the implied supply $(\rho_t^{l+1}, \dots, \rho_t^n)$ by equating theoretical and observed derivative prices in $t = 0, 1, 2, \dots$. Is it likely that the time series $(\rho_t^{l+1}, \dots, \rho_t^n)_{t=1,\dots,T}$ is generated by your model in step 2? If yes, that is fine. If not, then you should change it.

Through this procedure, you obtain a model that can keep track of a dynamic that originates in the derivatives market but which is still definitely arbitrage-free and conforms to the initial market prices.

In summary, we have obtained models that can be made consistent with the initially observed derivative prices. Firstly, this was done by assuming constant instead of vanishing external supply as in Subsection 1.2.4. Secondly, we approximated this approach in order to avoid the computational problems which already appeared in the previous subsection. Thirdly, we briefly sketched how one may construct models that can incorporate an independent dynamic of the derivative market without producing arbitrage.

1.2.7 American Options

American options are more complicated derivatives than those we have considered so far. They allow early exercise in the following sense: At any time t before expiration, you can return the option and in exchange receive the amount of a payment Y_t that depends on the underlyings up to time t. One may regard the derivatives from the previous subsections (at least the non-negative ones) as particular American options with $Y_T := Z_T^i$ and $Y_t := 0$ for $t = 0, \dots, T-1$. There is a well-established theory for pricing American options in complete models (cf. Lamberton & Lapeyre (1996)). We will see that the basic arguments and results carry over to our more general setting. To that end, we place ourselves in the setting of Subsection 1.2.4 with a small exception. We assume that Security n is an American option on the random payout $(Y_t)_{t=0,\dots,T}$. Fix a time $t \in \{1, \dots, T\}$ and suppose for the moment that either no trader exercises the option at time $t-1$ or that it is even forbidden. Then the derivative n behaves at time $t-1$ as an ordinary security in the sense that it does not suddenly vanish from the market by early exercise. Fix $\kappa > 0$. By the same argumentation as in Subsection 1.2.4 (cf. Equation (1.3)) we obtain

$$E\left(u_\kappa'\left(\sum_{j=1}^{l} \varphi_t^j \Delta Z_t^j\right) \Delta Z_t^i \middle| \mathcal{F}_{t-1}\right) = 0 \tag{1.8}$$

for $i = 1, \dots, n$, where $(\varphi^0, \dots, \varphi^l)$ is the u_κ-optimal strategy for the speculator in the restricted market consisting only of the underlyings $0, \dots, l$. By Statement 3 in Proposition 1.6 we conclude that $E^*(\Delta Z_t^n | \mathcal{F}_{t-1}) = 0$, where P^* is defined as in Equation (1.4). Hence,

$$Z_{t-1}^n = E^*(Z_t^n | \mathcal{F}_{t-1}). \tag{1.9}$$

Now we make three weak assumptions in addition to those in Subsection 1.2.4.

1. No trader exercises the option if the market price is higher than the exercise price. He would rather sell it on the market than exercise it.

2. The market price cannot fall below the exercise price. This is evident if we assume the absence of arbitrage.

3. The market price Z_{t-1}^n at time $t-1$ is $\geq E^*(Z_t^n | \mathcal{F}_{t-1})$. Above we have shown that the market price would equal $E^*(Z_t^n | \mathcal{F}_{t-1})$ if exercise at time $t-1$ were not allowed. Therefore, our assumption means that the additional right to exercise the option at time $t-1$ may increase but not decrease the price.

By Assumption 1 we have that Equation 1.9 holds in the case $Z_{t-1}^n > Y_{t-1}$. If on the other hand, Z_{t-1}^n is not strictly greater than Y_{t-1}, then the option may suddenly vanish from the market if everybody returns it. Speculators may thus face the short sale restriction $\varphi_{t-1}^n \geq 0$, since potential buyers can use their right to immediately exercise the option. Hence, Equation (1.8) may no longer hold, because its derivation by Lemma 1.2 is based on the assumption that speculators can freely choose their portfolio in $\mathbb{R}^{n+1}$. So Equation (1.9) cannot be derived in this case. On the other hand, we know by Assumption 2 that the option price cannot fall below Y_{t-1}. Thus we have

$$Z^n_{t-1} = Y_{t-1} \tag{1.10}$$

in the case $Z^n_{t-1} \leq Y_{t-1}$. Putting together Equations (1.9) and (1.10) as well as Assumption 3 yields

Lemma 1.17 *Under Assumptions 1–3 we have*

$$Z^n_{t-1} = \max\{Y_{t-1}, E^*(Z^n_t|\mathfrak{F}_{t-1})\} \tag{1.11}$$

for $t = 1, \ldots, T$.

Since $Z^n_T = Y_T$ by definition, the derivative price process is uniquely determined by the recursive Equation (1.11). The process $(Z^n_t)_{t=1,\ldots,T}$ is called the *Snell envelope* of $(Y_t)_{t=1,\ldots,T}$ (cf. Lamberton & Lapeyre (1996), Section 2.2). Using well-known results on Snell envelopes, we immediately obtain

Corollary 1.18 *Under the assumptions of Lemma 1.17 we have that*

1. Z^n *is the smallest* P^**-supermartingale such that* $Z^n_t \geq Y_t$ *for* $t = 0, \ldots, T$.
2. *For any* $t \in \{0, \ldots, T\}$, *we have that*

$$\begin{aligned} Z^n_t &= \operatorname{ess\,sup}\{E^*(Z^n_\tau|\mathfrak{F}_t) : \tau \text{ stopping time assuming values in } \{0, \ldots, T\}\} \\ &= E^*(Z^n_{\tau_t}|\mathfrak{F}_t), \end{aligned}$$

where the stopping time τ_t *is defined by* $\tau_t := \inf\{s \geq t : Z^n_s = Y_s\}$.

PROOF. Gihman & Skorohod (1979), Section 1.5 and Lamberton & Lapeyre (1996), Section 2.2. □

Let us give a short summary. In complete models it is well-known that American options are obtained as a Snell envelope of the exercise price process. Under essentially the same assumptions as in Subsection 1.2.4, the same is true in our general setting, where the pricing measure is obtained as before by Equation 1.4.

1.2.8 Foreign Exchange and Stochastic Interest Rates

If the stochastic process $(Y_t)_{t=0,1,\ldots}$ denotes the price of a security in terms of a foreign currency with a random exchange rate $(F_t)_{t=0,1,\ldots}$, then the price of this asset relative to your underlying currency is obviously given by $S^1_t := Y_t F_t$, hence $Z^1_t = Y_t F_t / S^0_t$ in discounted terms. Investments in foreign exchange are thus covered by our approach. But observe that when working with more than one currency, there is more than one natural choice of the numeraire. One may choose the fixed income investments in any of the currencies involved.

In incomplete models the obtained optimal strategies and derivative prices depend on this choice.

The modelling of markets with stochastic interest rates also poses no conceptual problems, since the numeraire S^0 was not assumed to be deterministic. However, the dynamic of the discounted securities $Z^i := S^i/S^0$ may be very complex for involved interest rate models. As in the case of foreign exchange, there is more than one natural choice of the numeraire. Beside the money market account $S_t^0 = \prod_{s=1}^t (1 + r_s)$ (or $S_t^0 = \exp(\int_0^t r_s\, ds)$ in continuous-time), where r_s is the instantaneous interest rate, one may also take long-term fixed-income investments (i.e. bonds). Again, the resulting strategies and prices may differ.

It is an open question whether this dependence plays an important role in practice from a numerical point of view (for a concrete example see Section 4.1). For both cases we propose the following guideline for the choice of S^0. For computing optimal strategies you can choose the numeraire according to your own needs, whereas for derivative pricing one should take S^0 to be the investment that the leading market powers consider riskless. For example, if you are an investor from Reykjavik trading in the US equities market, you may consider fixed-income investments in Icelandic crowns riskless, whereas US-Dollars contain a currency risk for you. Since optimal trading in the sense of Subsection 1.2.2 is risk-averse trading and the numeraire is by definition the benchmark of risklessness, you should base the calculation of your optimal portfolio on Icelandic crowns. The computation of derivative prices, however, is not based on your interests, but instead on assumptions about how the market behaves as a whole, i.e. how influential investors trade. They are more likely to consider US-Dollars as riskless. As a result you should take US-Dollar fixed-income investments as a numeraire for model extension.

Chapter 2

Martingale Problems as a Means to Model Dynamical Phenomena

Since the early days of analysis, time-dependent deterministic phenomena have been modelled using *derivatives* and *ordinary differential equations*. *Predictable semimartingale characteristics* and *martingale problems* can be viewed as stochastic counterparts of these notions, but they seem to be rarely used in the same spirit for modelling purposes. Details of these concepts can be found in Jacod (1979), Jacod & Shiryaev (1987), Métivier (1982), and Liptser & Shiryaev (1989), (1998). Here we want to present the basic ideas underlying predictable characteristics and martingale problems starting from real analysis and applications. Most of the statements in this chapter are reformulations or consequences of well-known results that can be found in Jacod (1979) and Jacod & Shiryaev (1987). In Section 2.8 we present an existence and uniqueness theorem for martingale problems under local Lipschitz conditions. Its statement and proof are closely related to similar classical results for stochastic differential equations (SDE's), but it is new in the sense that it is directly applicable to martingale problems. To understand everything in this chapter excepting the proofs, semimartingale theory and stochastic calculus, as found in Jacod & Shiryaev (1979), Chapter I, or Protter (1992), complemented by notions from Appendix A, should form a sufficient background. For easier readability, we relegate all proofs to the end of the respective sections.

2.1 Real Analysis as a Motivation

If you want to model a quantitative, time-dependent deterministic phenomenon mathematically, you may do this in terms of a function $X : \mathbb{R}_+ \to \mathbb{R}^d$, i.e. $X_t \in \mathbb{R}^d$ describes the state of your system at any time $t \geq 0$. The set of all mappings X of that kind is usually too large to work with in practice. In order to derive concrete results, one restricts the attention to classes of relatively simple functions. One such class consists of all *linear* mappings, i.e. functions of the type $X : \mathbb{R}_+ \to \mathbb{R}^d$, $X_t = bt$, where $b \in \mathbb{R}^d$ is a constant. Linear functions can be used to describe systems that grow steadily through time. They are uniquely determined by the constant growth rate b. Although linear mappings take you quite far in light of their simplicity, they are of limited use when you are dealing with a system that is not growing constantly.

A way out of this fix is to consider instead the larger class of differentiable functions. Intuitively, a *differentiable* function $X : \mathbb{R}_+ \to \mathbb{R}^d$ with *derivative* $b : \mathbb{R}_+ \to \mathbb{R}^d$ can be viewed as a mapping that, locally around any $t \in \mathbb{R}_+$, behaves as a linear function with growth rate b_t (more exactely: $X_{t+h} \approx X_t + b_t h$ for small h). These functions can be used to describe systems of approximately constant increase in small time intervals. The great success of analysis may be due to the fact that differentiable mappings are sufficiently regular to derive a large number of useful results, but still flexible enough to model many real-world phenomena.

In the following sections, we use a slightly more general notion. A function $X : \mathbb{R}_+ \to \mathbb{R}^d$ is called *absolutely continuous* if there is a Lebesgue-integrable function $b : \mathbb{R}_+ \to \mathbb{R}^d$ (more precisely: $\int_0^t |b_s|\, ds < \infty$ for any $t \geq 0$) such that $X_t = X_0 + \int_0^t b_s\, ds$ for any $t \geq 0$. Since absolutly continuous functions are differentiable in λ-almost all $t \geq 0$ (with derivative b_t) (cf. Elstrodt (1996), VII.4.12, VII.4.14), it makes sense to take absolute continuity as a slight generalization of differentiability.

When applied e.g. in the natural sciences, dynamical phenomena are often modelled by *ordinary differential equations*. The state of the system is described by a differentiable (or absolutely continuous) function $X : \mathbb{R}_+ \to \mathbb{R}^d$. The derivative b_t of X, which characterizes the local change of the system, is given as a function of the current state X_t (or, more generally, of the past $(X_s)_{s\in[0,t]}$), e.g. by the ODE

$$b_t = f(X_t, t) \text{ (or, equivalently, } X_t = X_0 + \int_0^t f(X_s, s)\, ds\text{)},$$

where $f : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$ is a given continuous function. Under Lipschitz and growth conditions, any ODE has a unique solution on $\mathbb{R}_+$ (given a fixed $X_0 \in \mathbb{R}^d$). Since explicit computation is often impossible, one has to fall back on numerical methods to obtain the solution function X.

In the following sections we discuss stochastic analogues of the above notions.

2.2 Lévy Processes

If deterministic, time-dependent phenomena are described by a deterministic function $X : \mathbb{R}_+ \to \mathbb{R}^d$ for fixed $d \in \mathbb{N}^*$, then it seems natural to model a stochastic system by a stochastic function X, i.e. a $\mathbb{R}^d$-valued stochastic process.

General setting for Chapter 2 (unless otherwise stated): Our terminology is chosen as in Jacod & Shiryaev (1987), which is abbreviated JS in the following. We fix a stochastic basis (filtered probability space) $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ in the sense of JS, Definition I.1.2, i.e. the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is right-continuous but not necessarily complete. By X we denote a

$\mathbb{R}^d$-valued stochastic process on Ω for some $d \in \mathbb{N}^*$. For what follows, we usually consider only adapted processes with càdlàg paths. We do not distinguish between the process X as a family of random variables $(X_t)_{t\in\mathbb{R}_+}$ with $X_t : \Omega \to \mathbb{R}^d$ for any $t \in \mathbb{R}_+$, as a mapping $X : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ and as a mapping $X : \Omega \to (\mathbb{R}^d)^{\mathbb{R}_+}$ (or $X : \Omega \to \mathbb{D}(\mathbb{R}^d)$ if X is càdlàg). Moreover, equalities etc. are usually only meant up to indistinguishability.

What is the stochastic analogue of a linear function? We are looking for processes that keep growing steadily, but *steadily* here is to be understood in a stochastic sense. Constant growth for linear functions means that $X_t - X_s$ depends only on $t - s$. Our stochastic translation is that the *distribution* of $X_t - X_s$ depends only on $t - s$, and to avoid feedback between successive parts of the process, that $X_t - X_s$ is independent of the σ-field $\mathcal{F}_s$. Hence, we consider processes with stationary, independent increments in the sense of the following definition to be a natural stochastic counterpart of linear functions.

Definition 2.1 A càdlàg, adapted process X with $X_0 = 0$ is called *Lévy process* (or *process with stationary, independent increments (PIIS)*) if the distribution of $X_t - X_s$ depends only on $t - s$ and if $X_t - X_s$ is independent of $\mathcal{F}_s$ for any $s, t \in \mathbb{R}_+$ with $s \le t$.

Since we want to work in a semimartingale framework, the following statement is useful.

Lemma 2.2

1. *Lévy processes are semimartingales.*
2. *A Lévy process X is a special semimartingale if and only if it is* integrable *(in the sense that $E(|X_t|) < \infty$ for any $t \in \mathbb{R}_+$, or, equivalently, $E(|X_1|) < \infty$).*

Although the general theory of stochastic processes is usually formulated in terms of semimartingales, we want to restrict our attention to special semimartingales, since these are a little easier to understand from an intuitive point of view. This allows us to replace the truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$ appearing in the Lévy-Khintchine formula as well as in the semimartingale characteristics with the identity mapping $h : \mathbb{R}^d \to \mathbb{R}^d$, $x \mapsto x$. The difference between semimartingales and special semimartingales can be interpreted as an integrability condition on the jumps (cf. JS, Proposition II.2.29a). In the case of Lévy processes, our restriction means that we consider only those with existing first moments, i.e. we exclude e.g. α-stable Lévy motions with $\alpha \le 1$ (cf. Samorodnitsky & Taqqu (1994)). Still, most statements can be generalized by reintroducing the truncation function.

In Section 2.1 we observe that linear functions are characterized by a constant $b \in \mathbb{R}^d$. By the Lévy-Khintchine formula a similar statement is true for Lévy processes. Their distribution is completely determined by a constant characteristic triplet $(b, c, F)^L$. This is another reason why one may consider them a rightful stochastic counterpart of linear functions.

Theorem 2.3 *Let X be an integrable Lévy process.*

1. *There is a unique triplet $(b, c, F)^L$, consisting of $b \in \mathbb{R}^d$, a symmetric, non-negative definite matrix $c \in \mathbb{R}^{d\times d}$ and a measure F on $\mathbb{R}^d$ satisfying $\int(|x^2| \wedge |x|)\, F(dx) < \infty$ and $F(\{0\}) = 0$, such that for any $t \in \mathbb{R}_+$ we have*

(a)

$$B_t = bt,$$

where $B \in \mathscr{V}^d$ *is the predictable part of finite variation in the canonical decomposition of the special semimartingale* X*,*

(b)

$$\langle X^{i,c}, X^{j,c}\rangle_t = c^{ij}t \text{ for any } i,j \in \{1,\dots,d\},$$

(c)

$$\nu([0,t] \times G) = F(G)t \text{ for any } G \in \mathcal{B}^d, \tag{2.1}$$

where ν *denotes the compensator of the random measure of jumps* μ^X *of* X *(cf. Definition A.3 in Appendix A).*

2. *The triplet* $(b,c,F)^L$ *uniquely determines the distribution of* X*.*

3. *We have*

$$E(e^{iu\cdot X_t}) = \exp\left(t\left(iu\cdot b - \frac{1}{2}u^\top cu + \int (e^{iu\cdot x} - 1 - iu\cdot x)\,F(dx)\right)\right) \tag{2.2}$$

for any $t \in \mathbb{R}_+$ *and any* $u \in \mathbb{R}^d$*.*

Definition 2.4 We call $(b,c,F)^L$ from the previous theorem the *characteristic triplet* of the Lévy process X.

Remarks.

1. A deterministic process X (i.e. $X_t(\omega)$ does not depend on ω) is a Lévy process if and only if $X(\omega) : \mathbb{R}_+ \to \mathbb{R}^d$ is a linear function, i.e. $X_t = bt$ for some $b \in \mathbb{R}^d$. Its characteristic triplet is $(b,0,0)^L$.

2. A continuous, adapted, $\mathbb{R}$-valued process is a Lévy process if and only if it is a Wiener process with drift (more precisely, if it is of the form $X_t = bt + \sigma W_t$ for some $b \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$ and (if $\sigma \neq 0$) some standard Wiener process W). Its characteristic triplet is $(b,\sigma^2,0)^L$.

3. Any càdlàg Poisson process X with arrival rate $\lambda \in \mathbb{R}_+$ (in the sense of Protter (1992), Section I.3) is a Lévy process with characteristic triplet $(\lambda, 0, \lambda\varepsilon_1)^L$.

4. Intuitively speaking, an arbitrary integrable Lévy process can be interpreted as an independent sum of a linear function with derivative b, a d-dimensional Wiener process with covariance matrix c and a (possibly uncountable) number of rescaled, *compensated* Poisson processes, where $F(G)\,dt$ is the probability of a jump of size $\Delta X_t \in G$ in an infinitesimal interval of length dt (cf. Figure 2.1). Compensation basically means subtracting a predictable drift (which is even deterministic and linear for Poisson processes) to transform the process into a local martingale. If the jump measure F is

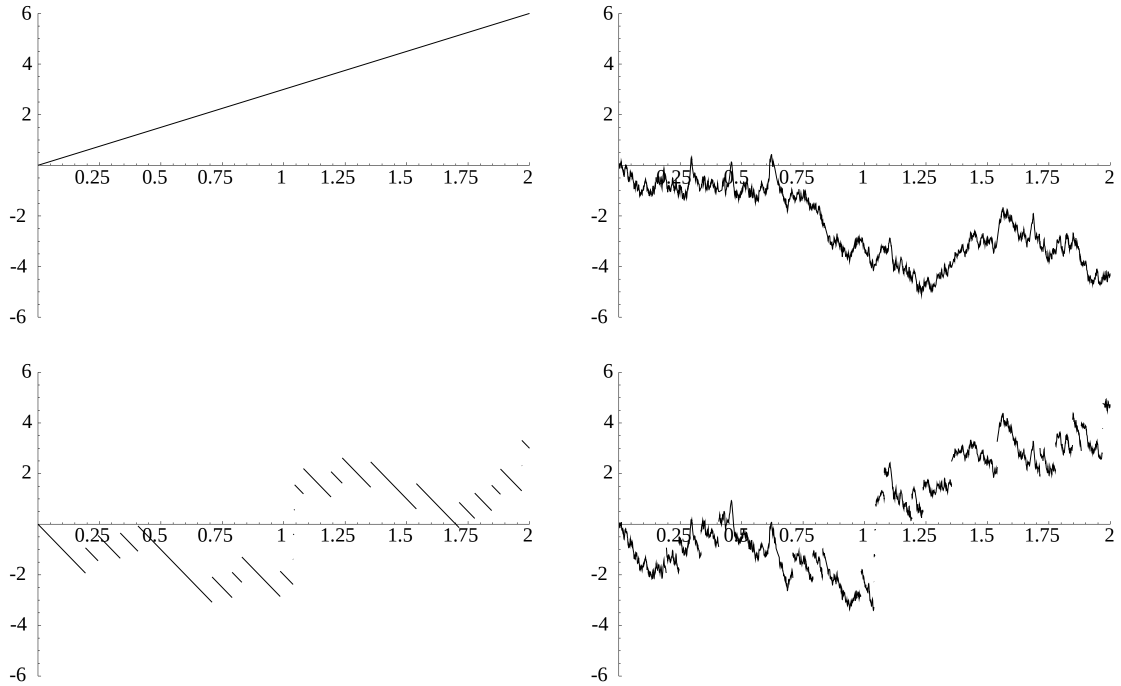

Figure 2.1: Sample paths of Lévy processes with characteristic triplets $(3,0,0)^L$, $(0,10,0)^L$, $(0,0,10\varepsilon_1)^L$ and $(3,10,10\varepsilon_1)^L$, respectively

finite, one may view the Lévy process alternatively as an additive superposition of a linear drift (generally with derivative $\neq b$), a Wiener process and rescaled, *uncompensated* Poisson processes. But for unbounded F, this interpretation is not appropriate.

5. An integrable Lévy process is a martingale if and only if $b = 0$.

There is another way in which Lévy processes are "linear," as is shown in Lemma 2.6.

Definition 2.5 Let X be a special semimartingale. We call $(B, C, \nu)^I$ the *integral characteristics* of X, where

1. $B \in \mathscr{V}^d$ is the predictable part of finite variation in the canonical decomposition of X,

2. $C \in \mathscr{V}^{d\times d}$ is the continuous process defined by $C^{ij} := \langle X^{i,c}, X^{j,c}\rangle$ for any $i, j \in \{1,\ldots,d\}$,

3. ν is the compensator of the random measure of jumps μ^X of X.

Remarks.

1. The integral characteristics $(B, C, \nu)^I$ are not the characteristics $(B(h), C, \nu)$ of X in JS, Definition II.2.6 (or Jacod (1979), Definition 3.46, from now on denoted as

$(B(h), C, \nu)^{JS}$), but they are closely related. By JS, Proposition II.2.29a the only difference is that

$$B(h) = B - (x - h(x)) * \nu, \tag{2.3}$$

where $h : \mathbb{R}^d \to \mathbb{R}^d$ is the truncation function in this definition. We see that if we chose $h(x) = x$ (which is not allowed for arbitrary semimartingales, cf. JS, Definition II.2.3), then both notions would coincide.

2. In the above definition one observes that the first part of the integral characteristics tells us about the drift of X (since $X - B$ is a local martingale), the second part about the continuous part of unbounded variation and the third about the jumps.

Lemma 2.6 *Let X be a special semimartingale, $b \in \mathbb{R}^d$, $c \in \mathbb{R}^{d\times d}$ symmetric and non-negative definite and F a measure on $\mathbb{R}^d$ such that $\int (|x^2| \wedge |x|)\, F(dx) < \infty$ and $F(\{0\}) = 0$. Then we have equivalence between*

1. *X is an integrable Lévy process with characteristic triplet $(b, c, F)^L$.*

2. *The integral characteristics $(B, C, \nu)^I$ of X are linear in the sense that $B_t = bt$, $C_t = ct$, $\nu([0,t] \times G) = F(G)t$ for any $t \in \mathbb{R}_+$, $G \in \mathcal{B}^d$.*

Proofs

Proposition 2.7 *Let X be a semimartingale and ν the compensator of the measure of jumps μ^X. Then we have equivalence between*

1. *X is a special semimartingale.*

2. *$(|x|^2 \wedge |x|) * \nu \in \mathscr{A}_{\text{loc}}$*

3. *For any $t \in \mathbb{R}_+$, we have $(|x|^2 \wedge |x|) * \nu_t < \infty$ P-almost surely.*

PROOF. $1 \Leftrightarrow 2$: cf. JS, II.2.29a

$2 \Rightarrow 3$: This is obvious.

$3 \Rightarrow 2$: The last statement implies that $(|x|^2 \wedge |x|) * \nu \in \mathscr{V}$ which, by JS, I.3.10, means that it is also in $\mathscr{A}_{\text{loc}}$. □

Proposition 2.8 *Let μ be an integer-valued random measure with compensator ν and let $W : \Omega \times \mathbb{R}_+ \times E \to \mathbb{R}$ be predictable, where $(E, \mathcal{E})$ denotes a Blackwell space. Assume that ν is of the form $\nu(dt, dx) = F_t(dx)\, dA_t$ for some transition kernel F from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ into $(E, \mathcal{E})$ and an increasing function A as in Definition 2.15. If $E((|W|^2 \wedge |W|) * \nu_T) < \infty$ for $T \in \mathbb{R}_+$, then $W * (\mu - \nu)$ is a uniformly integrable martingale on $[0, T]$.*

PROOF. Since $W = W1_{\{|W|\le 1\}} + W1_{\{|W|>1\}}$, we may assume that either $|W| \le 1$ or $|W| > 1$. If $|W| > 1$, then $E(|\widehat{W_t}|) < \infty$ for $t \in \Theta \cap [0,T]$ and $\widehat{W}_t = 0$ for $t \notin \Theta$. By $|\Theta \cap [0,T]| < \infty$, we have $E(|\widehat{W}| * \nu_T) < \infty$. Since also $E(|W| * \nu_T) < \infty$, we have

$\overline{C}(W1_{[0,T]}) \in \mathscr{A}^+$ and hence, by JS, II.1.33b, $(W * (\mu - \nu))^T \in \mathscr{A}$. Thus, $(W * (\mu - \nu))^T$ is a local martingale of class (D), hence a uniformly integrable martingale (JS, I.1.47c). The proof for $|W| \leq 1$ is similar, but using JS, II.1.33a and the fact that a square-integrable martingale is uniformly integrable. □

PROOF OF LEMMA 2.2. *1.* cf. JS, II.4.19

2. $\Rightarrow$: By JS, II.4.19 the characteristics $(B(h), C, \nu)^{JS}$ of X are of the form $B(h)_t = \widetilde{b}t$, $C_t = ct$, $\nu(dt, dx) = F(dx)dt$ for deterministic $\widetilde{b}, c, F$. Since X is a special semimartingale we have, by Proposition 2.7, $t \int (|x|^2 \wedge |x|) \, F(dx) = (|x|^2 \wedge |x|) * \nu_t < \infty$ P-almost surely for any $t \in \mathbb{R}_+$, and hence $\int (|x|^2 \wedge |x|) \, F(dx) < \infty$. Therefore, we have $E((|x|^2 \wedge |x|) * \nu_t) < \infty$ for any $t \in \mathbb{R}_+$. By Proposition 2.8 we conclude that $x * (\mu^X - \nu)$ is a martingale. Moreover, X^c is a martingale, because $E(\langle X^{i,c}, X^{i,c} \rangle_t) = c^{ii} t < \infty$ for any $i \in \{1, \ldots, d\}$, $t \in \mathbb{R}_+$ (cf. JS, I.4.50). Finally, the last part in the canonical decomposition $X = X_0 + X^c + x * (\mu^X - \nu) + A$ of the special semimartingale X (cf. JS, II.2.38), is, by JS, II.2.29a, deterministic (and linear in time). Hence A_t is integrable for any $t \in \mathbb{R}_+$.

$\Leftarrow$: W.l.o.g. $d = 1$. If X_1 is integrable, we have $\int (|x|^2 \wedge |x|) \, F(dx) < \infty$ (cf. Wolfe (1971), Theorem 2) and hence $(|x|^2 \wedge |x|) * \nu_t = t \int (|x|^2 \wedge |x|) \, F(dx) < \infty$ P-almost surely for any $t \in \mathbb{R}_+$. By Proposition 2.7, X is a special semimartingale. □

PROOF OF THEOREM 2.3. *1.* The existence of c, F is stated in JS, II.4.19. For the integrability condition on F, cf. the previous proof. By JS, II.4.19 and II.2.29a the process B is also linear and deterministic. The uniqueness of b, c, F follows at once.

2. This follows from Statement 3, since P^{X_1} uniquely determines the distribution of a Lévy process X.

3. From Statement 1 (and JS, II.2.29a) we know that the characteristics $(B(h), C, \nu)^{JS}$ are of the form $B(h)_t = bt - \int (x - h(x)) \, F(dx)t$, $C_t = ct$, $\nu(dt, dx) = F(dx)dt$. By JS, II.4.19, we have

$$\begin{aligned} E(e^{iu \cdot X_t}) &= \exp\Big(t\Big(iu \cdot \Big(b - \int (x - h(x)) \, F(dx)\Big) - \frac{1}{2} u^\top cu \\ &\qquad + \int (e^{iu \cdot x} - 1 - iu \cdot h(x)) \, F(dx)\Big)\Big) \\ &= \exp\Big(t\Big(iu \cdot b - \frac{1}{2} u^\top cu + \int (e^{iu \cdot x} - 1 - iu \cdot x) \, F(dx)\Big)\Big), \end{aligned}$$

where the F-integrability of $(x - h(x))$ e.g. for $h(x) = x1_{\{|x| \leq 1\}}$ has been shown in the proof of Lemma 2.2. □

PROOF OF THE REMARKS. *1.* If X is deterministic, then its characteristic triplet is $(b, 0, 0)^L$ for some $b \in \mathbb{R}$, since a deterministic local martingale starting in 0 is 0.

2. If X is a continuous Lévy-process with $c \neq 0$, then $W := (\frac{1}{\sqrt{c}}(X_t - bt))_{t \in \mathbb{R}_+}$ is a continuous local martingale with $\langle W, W \rangle_t = t$, hence a standard Wiener process (cf. JS, II.4.4).

3. This follows from the definition and from Equation (2.2).

5. The "if"-part follows from the proof of Lemma 2.2, where we have shown the local martingale part of the special martingale X to be a martingale. □

PROOF OF LEMMA 2.6. By Theorem 2.3 we have $1 \Rightarrow 2$. The converse follows from JS, II.4.19 and Statement 2 of Lemma 2.2. □

We will occasionally need the following statement relating the moments of a Lévy process to those of its Lévy measure.

Proposition 2.9 *Let X be a real-valued integrable Lévy process with characteristic triplet $(b, c, F)^L$.*

1. *For any $p \in [1, \infty)$, we have equivalence between*
   - *(a)* $E(|X_1|^p) < \infty$.
   - *(b)* $E(|X_t|^p) < \infty$ *for any* $t \in \mathbb{R}_+$.
   - *(c)* $\int |x|^p 1_{\{|x| \geq 1\}} \, F(dx) < \infty$.
2. *For any $p \in \mathbb{R}_+$, we have equivalence between*
   - *(a)* $E(\exp(p|X_1|)) < \infty$.
   - *(b)* $E(\exp(p|X_t|)) < \infty$ *for any* $t \in \mathbb{R}_+$.
   - *(c)* $\int \exp(p|x|) 1_{\{|x| \geq 1\}} \, F(dx) < \infty$.

   If any of these conditions holds, then $E(\exp(pX_t)) = (E(\exp(pX_1))^t$ for any $t \in \mathbb{R}_+$.

PROOF. We will only prove the second statement. The first one follows along the same lines. Fix $p \in \mathbb{R}_+^*$.

(b)⇒(a): This is obvious.

(a)⇒(c): Since the distribution of X_1 is infinitely divisible with Lévy measure F, it follows from Wolfe (1971), Theorem 2 that $E(\exp(p|X_1|)) < \infty$ if and only if $\int \exp(p|x|)$ $1_{\{|x| \geq 1\}} \, F(dx) < \infty$.

(c)⇒(b): Let $t \in \mathbb{R}_+^*$. By Theorem 2.3, we have that X_t has an infinitely divisible distribution with Lévy measure tF. From Wolfe (1971), Theorem 2, we conclude that $E(\exp(p|X_t|)) < \infty$.

It remains to show the equality $E(\exp(pX_t)) = (E(\exp(X_1)))^t$ for any $t \in \mathbb{R}_+$. Firstly, suppose that $t \in \mathbb{Q}$, say $t = n/m$ for $n, m \in \mathbb{N}^*$. Since X has independent and stationary increments, we have that $E(\exp(pX_1)) = (E(\exp(pX_{\frac{1}{m}})))^m$ and $E(\exp(pX_{\frac{n}{m}})) = (E(\exp(pX_{\frac{1}{m}})))^n$, which yields the claim for t. For arbitrary $t \in \mathbb{R}_+$, consider a sequence $t_n \downarrow t$ in $\mathbb{Q}$. Since $t \mapsto \exp(pX_t)$ is right-continuous, Fatou's lemma implies that

$E(\exp(pX_t)) \leq \lim_{n\to\infty} E(\exp(pX_{t_n})) = e^{tC}$, where $C := \log(E(\exp(pX_1)))$. Again using the independent, stationary increments, we have

$$\begin{aligned} e^{([t]+1)C} &= E(\exp(pX_{[t]+1})) \\ &= E(\exp(pX_t))E(\exp(pX_{[t]+1-t})) \\ &\leq e^{tC}e^{([t]+1-t)C} = e^{([t]+1)C} \end{aligned}$$

for any $t \in \mathbb{R}_+$. Hence, the inequality is actually an equality. This is only possible if $E(\exp(pX_t)) = e^{tC}$, which proves the claim. □

2.3 Grigelionis Processes and their Derivative

In the previous section, we found Lévy processes to be in some sense "stochastic linear functions." Now we want to define a reasonable stochastic counterpart of *locally* linear (i.e. differentiable or absolutely continuous) functions and their derivative. To that end, we focus on Lemma 2.6. Since Lévy processes are the semimartingales whose integral characteristics are linear in time, we consider those processes whose integral characteristics are pathwise absolutly continuous in time to be a stochastic analogue of absolutly continuous functions. For want of a shorter name we call them *Grigelionis processes*, since they are studied by Grigelionis in a series of papers (cf. Grigelionis (1973)). Let us begin with a

Lemma 2.10 *Let X be a special semimartingale with integral characteristics $(B, C, \nu)^I$. Then there exists a predictable, real-valued process $A \in \mathscr{A}^+_{\mathrm{loc}}$, a predictable $\mathbb{R}^d$-valued process $(b_t)_{t\in\mathbb{R}_+}$, a predictable $\mathbb{R}^{d\times d}$-valued process $(c_t)_{t\in\mathbb{R}_+}$ whose values are symmetric, non-negative definite matrices and a transition kernel F from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ into $(\mathbb{R}^d, \mathcal{B}^d)$ such that for any $t \in \mathbb{R}_+$ we have*

$$B_t = \int_0^t b_s \, dA_s,$$

$$C_t = \int_0^t c_s \, dA_s,$$

$$\nu([0,t] \times G) = \int_0^t F_s(G) \, dA_s \quad \textit{for any } G \in \mathcal{B}^d.$$

Remark. We usually drop the argument ω in the notation of transition kernels from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ into $(\mathbb{R}^d, \mathcal{B}^d)$, as is done for stochastic processes.

Definition 2.11 We call a special semimartingale X as in the previous lemma *Grigelionis process* or *locally infinitely divisble process* if A can be chosen such that its paths $A(\omega) : \mathbb{R}_+ \to \mathbb{R}$ are absolutely continuous in time.

The following lemma shows that a special semimartingale is a Grigelionis process if and only if its integral characteristics are absolutely continuous.

Lemma 2.12 *Let X be a special semimartingale with integral characteristics $(B, C, \nu)^I$. Then we have equivalence between*

1. *X is a Grigelionis process.*

2. *There exist a predictable $\mathbb{R}^d$-valued process $(b_t)_{t\in\mathbb{R}_+}$, a predictable $\mathbb{R}^{d\times d}$-valued process $(c_t)_{t\in\mathbb{R}_+}$ whose values are symmetric, non-negative definite matrices and a transition kernel F from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ into $(\mathbb{R}^d, \mathcal{B}^d)$ such that for any $t \in \mathbb{R}_+$ we have*

$$B_t = \int_0^t b_s \, ds,$$

$$C_t = \int_0^t c_s \, ds,$$

$$\nu([0, t] \times G) = \int_0^t F_s(G) \, ds \text{ for any } G \in \mathcal{B}^d.$$

Definition 2.13 Let X be a Grigelionis process. We call any triplet $(b, c, F)^D$ with b, c, F as in Lemma 2.12 *differential characteristics* or a *derivative* of X.

Remark. Grigelionis (1973) calls a similar object (the difference being that it corresponds to the truncation function $h(x) = x1_{[0,1]}(|x|)$ instead of $h(x) = x$) *local characteristics* of the process. We avoid this term here, since it is used by Jacod (1979) and Métivier (1982) to denote the integral characteristics $(B(h), C, \nu)^{JS}$ (called *characteristics* in JS) or $(B, C, \nu)^I$, respectively.

Lemma 2.14 *Any two derivatives of a Grigelionis process coincide outside some $(P \otimes \lambda)$-null set $N \in \mathcal{P}$.*

By Lemma 2.6 the derivative of any integrable Lévy process can be chosen deterministic and constant. Moreover, it coincides with its characteristic triplet. For general Grigelionis processes one may interpret the derivative $(b, c, F)^D$ so that, locally around $t \in \mathbb{R}_+$, the process statistically resembles a Lévy process with drift $b_t(\omega)$, Brownian part with covariance matrix $c_t(\omega)$ and local jump intensity $F((\omega, t), \cdot)$. It would be nice to support this way of talking with an appropriate limit theorem.

Remark. A Grigelionis process X with derivative $(b, c, F)^D$ has (P-almost surely) only continuous paths if and only if $F = 0$ $(P \otimes \lambda)$-almost surely.

Examples.

1. Let X be a deterministic process with absolutely continuous, càdlàg paths (e.g. $X : \mathbb{R}_+ \to \mathbb{R}^d$ differentiable). Then X is a Grigelionis process with derivative $((X'_t)_{t\in\mathbb{R}_+}, 0, 0)^D$.

2. Let X be a $\mathbb{R}$-valued *Itô process*, i.e. $X_t = X_0 + \int_0^t \mu_s\,ds + \int_0^t \sigma_s\,dW_s$, where W is a standard Wiener process and μ, σ are predictable, locally bounded $\mathbb{R}$-valued processes with $\sigma \geq 0$. Then X is a Grigelionis process with derivative $(\mu, \sigma^2, 0)^D$.

One should be aware that the analogy between derivatives for deterministic functions on the one hand and differential characteristics for Grigelionis processes on the other hand is limited. A differentiable function $X : \mathbb{R}_+ \to \mathbb{R}^d$ is uniquely determined by X_0 and its derivative $(X'_t)_{t\in\mathbb{R}_+}$. But it is generally not true that, given some stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$, as well as the starting value X_0 and the derivative $(b, c, F)^D$ of a Grigelionis process $X : \Omega \to \mathbb{D}(\mathbb{R}^d)$, the whole process X as a mapping could be recovered. Usually, not even the distribution P^X of X (on $\mathcal{D}(\mathbb{R}^d)$) is uniquely determined by the given information (unless the derivative is deterministic). As an example, consider some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ and two real-valued, independent standard Wiener processes U, V on that space. Moreover, we define a stopping time $T := \inf\{t \geq 0 : U_t = 1\}$. It is easy to see that both U^T and V^T are Grigelionis processes with derivative $(0, (1_{[0,T]}(t))_{t\in\mathbb{R}_+}, 0)^D$. But since $P(\lim_{n\to\infty} U^T_n = 1) = P(U_T = 1) = 1 \neq P(V_T = 1) = P(\lim_{n\to\infty} V^T_n = 1)$, the laws of U^T and V^T obviously differ.

Nevertheless, this non-uniqueness is not important from a practical point of view. For use in applications we are rather interested in an analogue of ordinary differential equations. In these the derivative is not given explicitly, but in terms of the unknown solution process itself. The stochastic translation would be that we are given the derivative b_t, c_t, F_t of a Grigelionis process X at time t as a deterministic function of the current value $X_t(\omega)$ (or, more generally, the past $(X_s)_{s\in[0,t]}(\omega)$). As in real analysis, the question of existence and uniqueness of solutions arises. It is immediately clear that we can only hope for uniqueness in the sense of distributions, since even for a deterministic derivative $(0, 1, 0)^D$ (which corresponds to any standard Wiener process), only the distribution is determined, but not the process itself as a mapping $X : \Omega \to \mathbb{D}(\mathbb{R}^d)$. We formally introduce the stochastic counterpart of an ODE under the notion *martingale problem* in Section 2.7. The term is approximately in line with Jacod (1979) and Jacod & Shiryaev (1987), where the integral characteristics $(B(h), C, \nu)^{JS}$ are considered instead of $(b, c, F)^D$ (making their approach more general).

Proofs

PROOF OF LEMMA 2.10. This follows from JS, II.2.9 and Equation (2.3). □

PROOF OF LEMMA 2.12. $1 \Rightarrow 2$: Since A is predictable, one can find a non-negative, predictable process $(a_t)_{t\in\mathbb{R}_+}$ such that $A = \int_0^\cdot a_t\,dt$ (cf. JS, I.3.13). Now let $\widetilde{b}_t := b_t a_t$, $\widetilde{c}_t := c_t a_t$, $\widetilde{F}_t(dx) := F_t(dx) a_t$ for any $t \in \mathbb{R}_+$. □

PROOF OF LEMMA 2.14. This follows from Lemma 2.19 and 2.18 in the following section. □

PROOF OF THE REMARK. If its compensator ν is 0, then $\mu^X = 0$ by JS, II.1.8(i). Thus X is continuous. $\square$

2.4 Extended Grigelionis Processes

Unlike linear functions, Lévy processes may have jumps, but the probability of a jump at any fixed time is still 0. The same is true for Grigelionis processes. This fact makes them useless for discrete-time models, where any process changes its value only at fixed (e.g. integer) times. Therefore, we want to extend the class of semimartingales under consideration slightly to be able to apply the results to discrete-time and mixed settings as well.

Definition 2.15 Let X be a special semimartingale. If in Lemma 2.10 the process A can be chosen as

$$A_t = t + \sum_{s \leq t} 1_\Theta(s) \text{ for any } t \in \mathbb{R}_+,$$

where $\Theta \subset \mathbb{R}_+^*$ is a discrete (and hence at most countable) set of times, then we call X an *extended Grigelionis process*.

Lemma 2.16 *Let X be a special semimartingale with integral characteristics $(B, C, \nu)^I$. Then we have equivalence between*

1. *X is an extended Grigelionis process.*

2. *There exists a discrete set $\Theta \subset \mathbb{R}_+^*$, a predictable $\mathbb{R}^d$-valued process $(b_t)_{t\in\mathbb{R}_+}$, a predictable $\mathbb{R}^{d\times d}$-valued process $(c_t)_{t\in\mathbb{R}_+}$ whose values are symmetric, non-negative definite matrices and a transition kernel F from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ into $(\mathbb{R}^d, \mathcal{B}^d)$ such that for any $t \in \mathbb{R}_+$ we have*

$$B_t = \int_0^t b_s \, ds + \sum_{s\in\Theta\cap[0,t]} \Delta B_s,$$

$$C_t = \int_0^t c_s \, ds,$$

$$\nu([0,t] \times G) = \int_0^t F_s(G) \, ds + \sum_{s\in\Theta\cap[0,t]} \nu(\{s\} \times G) \text{ for any } G \in \mathcal{B}^d.$$

3. *There exists a discrete set $\Theta \subset \mathbb{R}_+^*$, a $(\mathcal{F} \otimes \mathcal{B}_+)$-measurable $\mathbb{R}^d$-valued process $(b_t)_{t\in\mathbb{R}_+}$, a $(\mathcal{F}\otimes\mathcal{B}_+)$-measurable $\mathbb{R}^{d\times d}$-valued process $(c_t)_{t\in\mathbb{R}_+}$ whose values are symmetric, non-negative definite matrices and a transition kernel F from $(\Omega\times\mathbb{R}_+, \mathcal{F}\otimes\mathcal{B}_+)$ into $(\mathbb{R}^d, \mathcal{B}^d)$ such that for any $t \in \mathbb{R}_+$ the equations in Statement 2 hold.*

Definition 2.17 For any choice as in the previous lemma, we call $(\Theta, P^{X_0}, b, c, F, K)^E$ *extended (differential) characteristics* of the extended Grigelionis process X, where the transition kernel K from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ into $(\mathbb{R}^d, \mathcal{B}^d)$ is defined by

$$K_t(G) := \begin{cases} \nu(\{t\} \times G) + \varepsilon_0(G)(1 - \nu(\{t\} \times \mathbb{R}^d)) & \text{if } t \in \Theta \\ 0 & \text{else.} \end{cases}$$

Remarks.

1. For P-almost all $\omega \in \Omega$ and any $t \in \Theta$ we have that $K((\omega, t), \cdot)$ is a probability measure.
2. For any $t \in \Theta$, we have $\Delta B_t = \int x \, K_t(dx)$ P-almost surely.
3. For any $t \in \Theta$ and any $G \in \mathcal{B}^d$ (or any $t \in \mathbb{R}_+$ and any $G \in \mathcal{B}^d$ with $0 \notin G$) we have
$$P^{\Delta X_t | \mathcal{F}_{t-}}(G) = K_t(G) \ \ P\text{-almost surely}.$$
4. Intuitively, an extended Grigelionis process with extended differential characteristics $(\Theta, P^{X_0}, b, c, F, K)^E$ is a locally infinitely divisible process with derivative $(b, c, F)^D$ plus some jumps at fixed times $t \in \Theta$. These are characterised by the conditional jump distributions K_t for $t \in \Theta$. The initial distribution P^{X_0} has been added to the characteristics for later use.

The extended characteristics are unique in the following sense.

Lemma 2.18 *Let X be an extended Grigelionis process with extended characteristics $(\Theta, P^{X_0}, b, c, F, K)^E$. Then*

1. *For any $t \in \mathbb{R}_+$ we have*
$$\int_0^t \int (|x|^2 \wedge |x|) \, F_s(dx) \, ds < \infty \ \ P\text{-almost surely}.$$
There is a $(P \otimes \lambda)$-null set $N \in \mathcal{P}$ such that for any $(\omega, t) \in N^C$ we have
$$\int (|x|^2 \wedge |x|) F((\omega, t), dx) < \infty.$$
There is an evanescent set $N \in \mathcal{P}$ such that for any $(\omega, t) \in N^C$ we have
$$\int |x| K((\omega, t), dx) < \infty.$$
2. *Let $(\widetilde{\Theta}, P^{X_0}, \widetilde{b}, \widetilde{c}, \widetilde{F}, \widetilde{K})^E$ be other extended characteristics of X. Then there is a $(P \otimes \lambda)$-null set $N \in \mathcal{P}$ such that b, c, F and $\widetilde{b}, \widetilde{c}, \widetilde{F}$ coincide outside N. Moreover, we have $K((\omega, t), \cdot \cap (\mathbb{R}^d \setminus \{0\})) = \widetilde{K}((\omega, t), \cdot \cap (\mathbb{R}^d \setminus \{0\}))$ up to indistinguishability.*

Lemma 2.19 *Let X be a special semimartingale. For any discrete set $\Theta \subset \mathbb{R}_+^*$ there is equivalence between*

1. *X is a Grigelionis process with derivative $(b, c, F)^D$.*
2. *X is an extended Grigelionis process with extended characteristics $(\Theta, P^{X_0}, b, c, F, (\varepsilon_0)_{t\in\Theta})^E$.*

In a discrete-time setting the extended characteristics are particularly easy.

Lemma 2.20 *Assume that the filtration as well as the adapted process X are discrete (in the sense of Definition A.4 in Appendix A). Then there is equivalence between*

1. *X is an extended Grigelionis process.*
2. *X is a special semimartingale.*
3. *For any $t \in \mathbb{N}^*$ we have $\int |x| P^{\Delta X_t|\mathcal{F}_{t-1}}(dx) < \infty$ P-almost surely.*

In this case $(\mathbb{N}^*, P^{X_0}, 0, 0, 0, K)^E$ are extended characteristics for X, where

$$K_t := \begin{cases} P^{\Delta X_t|\mathcal{F}_{t-1}} & \text{for } t \in \mathbb{N}^* \\ 0 & \text{for } t \in \mathbb{R}_+ \setminus \mathbb{N}^*. \end{cases}$$

Remark. It should be obvious how to transfer Lemma 2.20 to an arbitrary discrete set $\Theta \subset \mathbb{R}_+^*$, i.e. if we consider a Θ-discrete filtration and a Θ-discrete process in the following sense. We call the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ (or the process X) Θ-*discrete* if the mapping $t \mapsto \mathcal{F}_t$ is constant (resp. the mapping $t \mapsto X_t(\omega)$ is constant for P-almost all $\omega \in \Omega$) on the open intervals between neighbouring points of $\Theta \cup \{0, \infty\}$.

For later proofs we now relate extended differential characteristics and semimartingale characteristics as in Jacod (1979) and JS.

Lemma 2.21 *Let X be a special semimartingale and h a truncation function as in JS, Definition II.2.3.*

1. *If X is an extended Grigelionis process with extended characteristics $(\Theta, P^{X_0}, b, c, F, K)^E$, then its characteristics $(B(h), C, \nu)^{JS}$ in the sense of JS, Definition II.2.6 are given by*

$$\nu([0,t] \times G) := \int_0^t F_s(G)\, ds + \sum_{s\in\Theta\cap[0,t]} K_s(G \setminus \{0\}) \tag{2.4}$$

$$B(h)_t := \int_0^t b_s\, ds + \sum_{s\in\Theta\cap[0,t]} x\, K_s(dx) + \int_{[0,t]\times\mathbb{R}^d} (h(x) - x)\, \nu(ds, dx) \tag{2.5}$$

$$C_t := \int_0^t c_s\, ds \tag{2.6}$$

for any $t \in \mathbb{R}_+, G \in \mathcal{B}^d$.

2. *If the semimartingale characteristics $(B(h), C, \nu)^{JS}$ of X can be written as in Equations (2.4) – (2.6) for some discrete set $\Theta \subset \mathbb{R}_+^*$, some predictable $\mathbb{R}^d$-valued process $(b_t)_{t\in\mathbb{R}_+}$, some predictable $\mathbb{R}^{d\times d}$-valued process $(c_t)_{t\in\mathbb{R}_+}$ whose values are symmetric, non-negative definite matrices and some transition kernels F, K from $(\Omega\times\mathbb{R}_+, \mathcal{P})$ into $(\mathbb{R}^d, \mathcal{B}^d)$, where K is a probability measure for $(\omega, t) \in \Omega\times\Theta$ and 0 (as a measure) for $(\omega, t) \in \Omega\times(\mathbb{R}_+ \setminus \Theta)$, then X is an extended Grigelionis process with extended characteristics $(\Theta, P^{X_0}, b, c, F, K)^E$.*

The following result concerns stochastic integrals with respect to extended Grigelionis processes.

Lemma 2.22 *Let X be an extended Grigelionis process with extended characteristics $(\Theta, \eta, b, c, F, K)^E$, and let $H^{ij} = (H_t^{ij})_{t\in\mathbb{R}_+}$ be predictable, locally bounded processes for $i \in \{1, \dots, d'\}$, $j \in \{1, \dots, d\}$. Then Y, defined by $Y^i := \sum_{j=1}^d \int_0^{\cdot} H_s^{ij}\, dX_s^j$ for $i = 1, \dots, d'$, is an extended Grigelionis process with extended characteristics $(\Theta, \varepsilon_0, \widetilde{b}, \widetilde{c}, \widetilde{F}, \widetilde{K})^E$, where*

$$\widetilde{b}_t^i = \sum_{\beta=1}^d H_t^{i\beta} b_t^\beta,$$

$$\widetilde{c}_t^{ik} = \sum_{\beta,\gamma=1}^d H_t^{i\beta} c_t^{\beta\gamma} H_t^{k\gamma},$$

$$\widetilde{F}_t(G) = \int 1_{G\setminus\{0\}}\Big(\sum_{j=1}^d H_t^{\cdot j} x^j\Big)\, F_t(dx),$$

$$\widetilde{K}_t(G) = \int 1_G\Big(\sum_{j=1}^d H_t^{\cdot j} x^j\Big)\, K_t(dx)$$

for any $t \in \mathbb{R}_+$, any $i, k \in \{1, \dots, d'\}$ and any $G \in \mathcal{B}^{d'}$.

We need the following technical result for Chapter 4.

Lemma 2.23 *Let X be an extended Grigelionis process with extended characteristics $(\Theta, P^{X_0}, b, c, F, K)^E$. Moreover, let $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$ be another filtration on $\mathcal{F}$ such that for any $t \in \mathbb{R}_+$ we have $\mathcal{F}_t \subset \mathcal{G}_t \subset \sigma(\mathcal{F}_t \cup \mathcal{C})$, where $\mathcal{C}$ is a sub-σ-field of $\mathcal{F}$ that is independent of $\mathcal{F}_{\infty-} := \sigma(\bigcup_{t\in\mathbb{R}_+}\mathcal{F}_t)$. Then on the space $(\Omega, \mathcal{F}, (\mathcal{G}_t)_{t\in\mathbb{R}_+}, P)$, X is still an extended Grigelionis process with the same extended characteristics.*

Proofs

PROOF OF LEMMA 2.16. $1 \Rightarrow 2$: We have $B_t = \int_0^t b_s\, dA_s = \int_0^t b_s\, ds + \sum_{s\in\Theta\cap[0,t]} b_s = \int_0^t b_s\, ds + \sum_{s\in\Theta\cap[0,t]} \Delta B_s$, and similarly for C and ν. Note that C is, by definition, continuous.

$3 \Rightarrow 1$: From the proof of JS, Proposition II.2.9 and Equation 2.3, it follows that A in Lemma 2.10 can be chosen as

$$A = \sum_{i=1}^{d} \mathrm{Va}(B^i - (x^i - h^i(x)) * \nu) + \sum_{i,j=1}^{d} \mathrm{Va}(C^{ij}) + (|x|^2 \wedge 1) * \nu.$$

Thus, A is absolutely continuous with respect to $(t + \sum_{s \leq t} 1_\Theta(s))_{t \in \mathbb{R}_+}$. Statement 1 now follows as in the proof of Lemma 2.12, 1⇒2. □

PROOF OF THE REMARKS. *1.* This follows from JS, II.1.17b.

2. We have $\Delta X_t = \Delta B_t + \Delta(x * (\mu^X - \nu))_t = \Delta B_t + \Delta X_t - \Delta(x * \nu)_t$. Hence, $\Delta B_t = \Delta(x * \nu)_t = \int x\, K_t(dx)$ P-almost surely for any $t \in \Theta$.

3. $P(\Delta X_t \in G \setminus \{0\} | \mathcal{F}_{t-}) = E(\mu^X(\{t\} \times (G \setminus \{0\})) | \mathcal{F}_{t-}) = E(\nu(\{t\} \times (G \setminus \{0\})) + W * (\mu^X - \nu)_t | \mathcal{F}_{t-})$, where $W(s,x) = 1_{\{t\} \times (G \setminus \{0\})}(s,x)$. Since ν is predictable and by JS, I.2.27, the right-hand side equals $\nu(\{t\} \times (G \setminus \{0\})) = K_t(G \setminus \{0\})$. □

PROOF OF LEMMA 2.18. *1.* Since X is a special semimartingale, we have, by Proposition 2.7,

$$\int_0^t \int (|x|^2 \wedge |x|)\, F_s(dx)\, ds + \sum_{s \in \Theta \cap [0,t]} \int (|x|^2 \wedge |x|)\, K_s(dx) = (|x|^2 \wedge |x|) * \nu_t < \infty$$

P-almost surely for any $t \in \mathbb{R}_+$. This implies the first statement.

2. Let $\mathcal{E}$ denote a countable $\cap$-stable generator of $\mathcal{B}^d$ and define $N := \{(\omega,t) : b(\omega,t) \neq \widetilde{b}(\omega,t)$ or $c(\omega,t) \neq \widetilde{c}(\omega,t)$ or $F((\omega,t),G) \neq \widetilde{F}((\omega,t),G)$ for some $G \in \mathcal{E}\}$. One easily sees that $N \in \mathcal{P}$ and $(P \otimes \lambda)(N) = 0$. Similarly for K. □

PROOF OF LEMMA 2.19. $1 \Rightarrow 2$: Take $\Theta = \varnothing$.

$2 \Rightarrow 1$: By Remark 2 we have $\Delta B_t = \int x\, \varepsilon_0(dx) = 0$ P-almost surely for any $t \in \Theta$. Hence, X is a Grigelionis process. □

PROOF OF LEMMA 2.20. A discrete process is a semimartingale if and only if it is adapted (cf. JS, Subsection I.4g). Moreover, we have $\nu((\mathbb{R}_+ \setminus \mathbb{N}^*) \times \mathbb{R}^d) = 0$ P-almost surely, since $\mu^X((\mathbb{R}_+ \setminus \mathbb{N}^*) \times \mathbb{R}^d) = 0$.

$2 \Rightarrow 3$: By Remark 3 we have $P^{\Delta X_t | \mathcal{F}_{t-}}(G \setminus \{0\}) = \nu(\{t\} \times (G \setminus \{0\}))$ for any $G \in \mathcal{B}^d$ P-almost surely. Since $\mathcal{F}_{t-} = \mathcal{F}_{t-1}$ and by Proposition 2.7, we have $\int |x|\, P^{\Delta X_t | \mathcal{F}_{t-1}}(dx) \leq 1 + \int (|x|^2 \wedge |x|)\, \nu(\{t\} \times dx) < \infty$ P-almost surely for any $t \in \mathbb{N}$.

$3 \Rightarrow 1$: It suffices to prove that X is a special semimartingale. The form of the integral characteristics in Lemma 2.16 then follows from the fact that X is discrete. By $\nu((\mathbb{R}_+ \setminus \mathbb{N}^*) \times \mathbb{R}^d) = 0$, by $P^{\Delta X_t | \mathcal{F}_{t-1}}(\cdot \setminus \{0\}) = \nu(\{t\} \times (\cdot \setminus \{0\}))$ for $t \in \mathbb{N}^*$ and by assumption, it follows that $(|x|^2 \wedge |x|) * \nu_t \leq 0 + \sum_{s \in \mathbb{N}^* \cap [0,t]} \int |x|\, P^{\Delta X_s | \mathcal{F}_{s-1}} < \infty$ P-almost surely for any $t \in \mathbb{R}_+$. By Proposition 2.7, X is a special semimartingale.

The shape of the characteristics follows once more from Remark 3. □

Proof of Lemma 2.21. *1.* This follows immediately from Remark 2 and Equation (2.3).

2. By Equation (2.4) we have that $K_t(G \setminus \{0\}) = \nu(\{t\} \times G \setminus \{0\})$ for any $t \in \Theta$, $G \in \mathcal{B}^d$. Statement 2 then follows from Equation (2.3). □

Proposition 2.24 *Let X be an extended Grigelionis process with extended characteristics $(\Theta, \eta, b, c, F, K)^E$ and $W : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d'}$ a predictable mapping with $W^i \in G_{\mathrm{loc}}(\mu^X)$ for $i = 1, \dots, d$. Then the local martingale $Y := W * (\mu^X - \nu)$ is an extended Grigelionis process in $\mathbb{R}^{d'}$ with extended characteristics $(\Theta, \varepsilon_0, 0, 0, \widetilde{F}, \widetilde{K})^E$, where*

$$\widetilde{F}_t(G) = \int 1_{G\setminus\{0\}}(W(t,x))\, F_t(dx)$$

for any $t \in \mathbb{R}_+$ and any $G \in \mathcal{B}^{d'}$. (We do not say anything about $\widetilde{K}$ here.)

Proof. Since Y is a special semimartingale without drift and continuous local martingale part, we only have to prove that the compensator ν^Y of the jump measure μ^Y is absolutely continuous with respect to A from Definition 2.15. Note that, by definition of $W * (\mu^X - \nu)$, we have, up to an evanescent set, $\Delta Y_t 1_{\Theta^C}(t) = W(t, \Delta X_t) 1_{\Theta^C}(t)$. Hence, $\mu^Y([0,t] \times G) = \int_{[0,t]\times\mathbb{R}^d} 1_{G\setminus\{0\}}(W(s,x)) 1_{\Theta^C}(s)\, \mu^X(ds,dx) + \mu^Y((\Theta \cap [0,t]) \times G)$ for any $t \in \mathbb{R}_+$, $G \in \mathcal{B}^{d'}$. By the form of ν in Lemma 2.16, we have $\nu^Y([0,t] \times G) = \int_0^t \int 1_{G\setminus\{0\}}(W(s,x))\, F_s(dx)\, ds + \nu^Y((\Theta \cap [0,t]) \times G)$ for any $t \in \mathbb{R}_+$, $G \in \mathcal{B}^{d'}$, where ν^Y denotes the compensator of μ^Y. By Lemma 2.16 we are done. □

Remark. The previous proposition still holds if we replace μ^X with any integer-valued random measure μ whose compensator is absolutely continuous with respect to the process A in Definition 2.15.

Proof of Lemma 2.22. If $X = X_0 + B + X^c + x * (\mu^X - \nu)$ denotes the decomposition of the special semimartingale X in the sense of JS, II.2.38, we obviously have $Y^i = \int_0^\cdot H_s^{i\cdot} \cdot dB_s + \int_0^\cdot H_s^{i\cdot} \cdot dX_s^c + (\sum_{j=1}^d H_s^{ij} x^j) * (\mu^X - \nu)$, where the terms are a predictable one of finite variation, a continuous local martingale, and a discontinuous local martingale, respectively. Moreover, from $B_t = \int_0^t b_s\, ds + \sum_{s\in\Theta\cap[0,t]} \Delta B_s$ and $C_t = \int_0^t c_s\, ds$ for any $t \in \mathbb{R}_+$, we immediately obtain that the first two of the integral characteristics $(\widetilde{B}, \widetilde{C}, \widetilde{\nu})^I$ of Y are as in Lemma 2.16, but with $\widetilde{b}, \widetilde{c}$ instead of b, c. By Proposition 2.24 it follows that also $(\sum_{j=1}^d H_s^{\cdot j} x^j) * (\mu^X - \nu)$ and hence Y altogether is an extended Grigelionis process. Moreover, we have that the extended characteristics of Y are $(\Theta, \varepsilon_0, \widetilde{b}, \widetilde{c}, \widetilde{F}, \widetilde{K})^E$, where $\widetilde{F}$ is, by Proposition 2.24, as in Lemma 2.22. It remains to show that $\widetilde{K}$ is of the claimed form. By Remark 3 in this section we have that $\widetilde{K}_t(G) = P^{\Delta Y_t | \mathcal{F}_{t-}}(G)$ for any $t \in \Theta$ and any $G \in \mathcal{B}^{d'}$. Since $\Delta Y_t = \sum_{j=1}^d H^{\cdot j} \Delta X_t^j$, it follows that $\widetilde{K}_t(G) = \int 1_G(\sum_{j=1}^d H^{\cdot j} x^j)\, P^{\Delta X_t | \mathcal{F}_{t-}}(dx)$, which, by $P^{\Delta X_t | \mathcal{F}_{t-}} = K_t$, yields the claim. □

PROOF OF LEMMA 2.23. Firstly, we prove that any $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-local martingale is also a $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$-local martingale. Since any $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$- stopping time is also a $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$- stopping time, it suffices to prove the claim for martingales instead of local martingales. By Jacod (1979), (9.29) we have to show that $E(E(X|\mathcal{F}_{\infty-})|\mathcal{G}_t) = E(X|\mathcal{F}_t)$ P-almost surely for any bounded random variable and any $t \in \mathbb{R}_+$. By Bauer (1978), Satz 54.4 we have $E(E(X|\mathcal{F}_\infty)|\sigma(\mathcal{F}_t \cup \mathcal{C})) = E(E(X|\mathcal{F}_\infty)|\mathcal{F}_t)$ P-almost surely. Taking $E(\cdot|\mathcal{G}_t)$ yields the claim.

By Lemma 2.21 we have that the characteristics $(B(h), C, \nu)^{JS}$ of X are given by Equations (2.4) – (2.6). By JS, II.2.42 this is equivalent to the fact that for any bounded C^2-function $f : \mathbb{R}^d \to \mathbb{R}$, the process

$$\begin{aligned} Y \quad := \quad & f(X) - f(X_0) - \sum_{i=1}^d \int_0^\cdot D_i f(X_{s-})\, dB(h)^i_s - \frac{1}{2}\sum_{i,j=1}^d \int_0^\cdot D_{ij} f(X_{s-})\, dC^{ij}_s \\ & - \Big(f(X_- + x) - f(X_-) - \sum_{i=1}^d D_i f(X_-) h^i(x)\Big) * \nu \end{aligned}$$

is a local martingale. By the reasoning above we have that Y is also a $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$-local martingale. Again applying JS, II.2.42, the characteristics of X relative to the space $(\Omega, \mathcal{F}, (\mathcal{G}_t)_{t\in\mathbb{R}_+}, P)$ are still $(B(h), C, \nu)^{JS}$. Statement 2 of Lemma 2.22 now yields Lemma 2.23. □

2.5 Itô's Formula for Extended Characteristics

Itô's formula can also be expressed in terms of extended characteristics. If we look upon these as a derivative (as in Section 2.3), then we may call it a stochastic chain rule.

Theorem 2.25 *Let X be an extended Grigelionis process with extended characteristics $(\Theta, P^{X_0}, b, c, F, K)^E$, and $f : \mathbb{R}^d \to \mathbb{R}^{d'}$ a function such that $f(X)$ is a special semimartingale and one of the two following conditions is fulfilled.*

1. $f \in C^2$.

2. $b = 0, c = 0, F = 0$.

Then $f(X)$ is an extended Grigelionis process in $\mathbb{R}^{d'}$ with extended characteristics $(\Theta, P^{f(X_0)}, \widetilde{b}, \widetilde{c}, \widetilde{F}, \widetilde{K})^E$, where

$$\begin{aligned} \widetilde{b}^i_t \quad = \quad & \sum_{\alpha=1}^d D_\alpha f^i(X_{t-}) b^\alpha_t + \frac{1}{2}\sum_{\alpha,\beta=1}^d D_{\alpha\beta} f^i(X_{t-}) c^{\alpha\beta}_t \\ & + \int \Big(f^i(X_{t-} + x) - f^i(X_{t-}) - \sum_{\alpha=1}^d D_\alpha f^i(X_{t-}) x\Big)\, F_t(dx), \end{aligned}$$

$$\widetilde{c}_t^{ij} = \sum_{\alpha,\beta=1}^{d} D_\alpha f^i(X_{t-}) c_t^{\alpha\beta} D_\beta f^j(X_{t-}),$$

$$\widetilde{F}_t(G) = \int 1_{G\setminus\{0\}}(f(X_{t-}+x) - f(X_{t-}))\, F_t(dx),$$

$$\widetilde{K}_t(G) = \int 1_G(f(X_{t-}+x) - f(X_{t-}))\, K_t(dx),$$

for any $t \in \mathbb{R}_+$, $i, j \in \{1, \dots, d'\}$, $G \in \mathcal{B}^{d'}$.

Remark. Sufficient conditions for $f(X)$ to be a special semimartingale are each of the following:

1. For any $t \in \mathbb{R}_+$ one has $\int_0^t \int (|x|^2 \wedge |x|)\, \widetilde{F}_s(dx)\, ds < \infty$ and $\int |x|\, \widetilde{K}_s(dx) < \infty$ P-almost surely, where $\widetilde{F}, \widetilde{K}$ are defined as in the previous theorem.

2. There is some $M \in \mathbb{R}_+$ such that for any $x \in \mathbb{R}^d$ one has $\|Df(x)\| \leq M$.

Proofs

PROOF OF THEOREM 2.25. Regardless of whether X is a special semimartingale or only a semimartingale, we have

$$\mu^{f(X)}([0,t] \times G) = \int_{[0,t]\times\mathbb{R}^d} 1_G(f(X_{s-}+x) - f(X_{s-}))\, \mu^X(ds, dx)$$

and hence for its compensator $\nu^{f(X)}$

$$\begin{aligned}\nu^{f(X)}([0,t] \times G) &= \int_{[0,t]\times\mathbb{R}^d} 1_G(f(X_{s-}+x) - f(X_{s-}))\, \nu(ds, dx) \\ &= \int_0^t \int 1_G(f(X_{s-}+x) - f(X_{s-}))\, F_s(dx)\, ds \\ &\quad + \sum_{s\in\Theta\cap[0,t]} \int 1_G(f(X_{s-}+x) - f(X_{s-}))\, K_s(dx)\end{aligned}$$

for any $t \in \mathbb{R}_+$ and any $G \in \mathcal{B}^{d'}$ with $0 \notin G$. Thus in the situation of Theorem 2.25, the third part of the integral characteristics $(\widetilde{B}, \widetilde{C}, \widetilde{\nu})^I$ of $f(X)$ is as in Lemma 2.16 resp. Definition 2.17, but with $\widetilde{F}$, $\widetilde{K}$ instead of F, K.

Assume that the second condition holds. Then X and hence $f(X)$ are Θ-discrete. Therefore, we have $\widetilde{B}_t = \sum_{s\in\Theta\cap[0,t]} \Delta\widetilde{B}_s$ and $\widetilde{C}_t = 0$ for any $t \in \mathbb{R}_+$, where $(\widetilde{B}, \widetilde{C}, \widetilde{\nu})^I$ denotes the integral characteristics of $f(X)$. Hence, $f(X)$ is an extended Grigelionis process and the statement is proved.

Now let $f \in C^2$. By Jacod (1979), (3.89) we have that

$$\begin{aligned} f^i(X_t) = f^i(X_0) \quad &+ \sum_{\alpha=1}^d \int_0^t D_\alpha f^i(X_{s-})\, dX_s^{\alpha,c} \\ &+ \int_{[0,t]\times\mathbb{R}^d} (f^i(X_{s-}+x) - f^i(X_{s-}))\, (\mu^X - \nu)(ds,dx) \\ &+ \sum_{\alpha=1}^d \int_0^t D_\alpha f^i(X_{s-})\, dB_s^\alpha \\ &+ \sum_{\alpha,\beta=1}^d \frac{1}{2} \int_0^t D_{\alpha\beta} f^i(X_{s-})\, d\langle X^{\alpha,c}, X^{\beta,c}\rangle_s \\ &+ \int_{[0,t]\times\mathbb{R}^d} \Big(f^i(X_{s-}+x) - f^i(X_{s-}) - \sum_{\alpha=1}^d D_\alpha f^i(X_{s-})x \Big)\, \nu(ds,dx) \end{aligned}$$

for any $i = 1, \ldots, d'$ and any $t \in \mathbb{R}_+$, where B denotes the predictable part of finite variation of X. One immediately sees that the second term is the continuous martingale part, the third term the discontinuous martingale part and the last three terms the predictable part of finite variation of the special semimartingale $f^i(X)$. Elementary calculations yield that the first two of the integral characteristics $(\widetilde{B}, \widetilde{C}, \widetilde{\nu})^I$ are as in Lemma 2.16, but with $\widetilde{b}$, $\widetilde{c}$ instead of b, c. □

PROOF OF THE REMARKS. *1.* Firstly, observe that $f(X)$ is a semimartingale, either by JS, I.4.57 (if f is a C^2-function), or by the fact that X and hence $f(X)$ has only finitely many jumps in any interval $[0,t]$ (in the case $b = 0$, $c = 0$, $F = 0$). By the first part of the proof of Theorem 2.25 and the assumption, we have that

$$(|x|^2 \wedge |x|) * \nu_t^{f(X)} \leq \int_0^t \int (|x|^2 \wedge |x|)\, \widetilde{F}_s(dx)\, ds + \sum_{s\in\Theta\cap[0,t]} \int |x|\, \widetilde{K}_s(dx) < \infty$$

P-almost surely for any $t \in \mathbb{R}_+$. By Proposition 2.7, $f(X)$ is a special semimartingale.

2. By the mean value theorem, we have $|f(X_{s-}+x) - f(X_{s-})| \leq M|x|$. Hence, by the first part of the proof of Theorem 2.25,

$$\begin{aligned} &(|x|^2 \wedge |x|) * \nu_t^{f(X)} \\ &= \int_{[0,t]\times\mathbb{R}^d} \Big(|f(X_{s-}+x) - f(X_{s-})|^2 \wedge |f(X_{s-}+x) - f(X_{s-})| \Big)\, \nu(ds,dx) \\ &\leq (M^2 \vee M)(|x|^2 \wedge |x|) * \nu_t \end{aligned}$$

P-almost surely for any $t \in \mathbb{R}_+$. Since X is a special semimartingale, Proposition 2.7 yields that $f(X)$ is a special semimartingale as well. □

2.6 Girsanov's Theorem

Girsanov's theorem tells us how the extended characteristics behave under an absolutely continuous change of the underlying probability measure P.

Theorem 2.26 *Let $\widetilde{P}$ be a probability measure on $(\Omega, \mathcal{F}, P)$ with $\widetilde{P} \overset{\text{loc}}{\ll} P$ and denote by Z the density process of $\widetilde{P}$ relative to P (cf. Definition A.5 in Appendix A). Moreover, let X be an extended Grigelionis process with extended characteristics $(\Theta, P^{X_0}, b, c, F, K)^E$ that is a $\widetilde{P}$-special semimartingale. Then X is an extended Grigelionis process relative to $\widetilde{P}$ and its extended $\widetilde{P}$-characteristics $(\Theta, \widetilde{P}^{X_0}, \widetilde{b}, c, \widetilde{F}, \widetilde{K})^E$ are given as follows. There exist a $(\mathcal{P} \otimes \mathcal{B}^d)$-measurable mapping $Y : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ and a predictable process $\beta : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ such that for any $t \in \mathbb{R}_+$, $i \in \{1, \ldots, d\}$ we have $\widetilde{P}$-almost surely*

$$\int_0^t \int |x(Y(s,x) - 1)|\, F_s(dx)\, ds + \sum_{s \in \Theta \cap [0,t]} \int |x(Y(s,x) - 1)|\, K_s(dx) < \infty, \tag{2.7}$$

$$\int_0^t \Big| \sum_{\alpha=1}^d c_s^{i\alpha} \beta_s^\alpha \Big|\, ds < \infty,$$

$$\int_0^t \Big(\sum_{\alpha,\gamma=1}^d \beta_s^\alpha c_s^{\alpha\gamma} \beta_s^\gamma \Big)\, ds < \infty,$$

and such that for any $t \in \mathbb{R}_+$, $i \in \{1, \ldots, d\}$, $G \in \mathcal{B}^d$ we have

$$\widetilde{P}^{X_0}(G) = E(1_G(X_0) Z_0),$$

$$\widetilde{b}_t^i = b_t^i + \sum_{\alpha=1}^d c_s^{i\alpha} \beta_s^\alpha + \int x^i (Y(t,x) - 1)\, F_t(dx), \tag{2.8}$$

$$\widetilde{F}_t(G) = \int 1_G(x) Y(t,x)\, F_t(dx), \tag{2.9}$$

$$\widetilde{K}_t(G \setminus \{0\}) = \int 1_{G \setminus \{0\}}(x) Y(t,x)\, K_t(dx). \tag{2.10}$$

Moreover, a $(\mathcal{P} \otimes \mathcal{B}^d)$-measurable mapping $Y : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ and a predictable process $\beta : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ meet all the above conditions if and only if

1. *$E(\int_{[0,\infty) \times \mathbb{R}^d} Z_t U(t,x)\, \mu^X(dt,dx)) = E(\int_{[0,\infty) \times \mathbb{R}^d} Y(t,x) Z_{t-} U(t,x)\, \mu^X(dt,dx))$ for any $(\mathcal{P} \otimes \mathcal{B}^d)$-measurable mapping $U : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ (where μ^X denotes as before the random measure of jumps of X),*
2. *$\langle Z^c, X^{i,c} \rangle = \int_0^\cdot (\sum_{\alpha=1}^d c_t^{i\alpha} \beta_t^\alpha Z_{t-})\, dt$ for any $i \in \{1, \ldots, d\}$.*

Remark. For X to be a $\widetilde{P}$-special semimartingale, any of the following conditions suffices.

1. $[Z, M^i] \in \mathscr{A}_{\text{loc}}(P)$ for any $i \in \{1, \ldots, d\}$, where M is the P-local martingale part of the P-canonical decomposition of the P-special semimartingale X.

2. $[Z, M^{d,i}] \in \mathscr{A}_{\text{loc}}(P)$ for any $i \in \{1,\ldots,d\}$, where M^d is the purely discontinuous part of the P-local martingale M in Condition 1, i.e. $M^d = x * (\mu^X - \nu)$.

3. For some $(\mathcal{P} \otimes \mathcal{B}^d)$-measurable mapping $Y : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ meeting Condition 1 in the previous theorem, we have that for any $t \in \mathbb{R}_+$

$$\int_0^t \int (|x|^2 \wedge |x|) Y(s,x)\, F_s(dx)\, ds < \infty \quad P\text{-almost surely,}$$

$$\int |x| Y(t,x)\, K_t(dx) < \infty \quad P\text{-almost surely.}$$

The following lemma shows how to obtain β and Y if the density process is of exponential form, which is often the case.

Lemma 2.27 *Assume that in Theorem 2.26, Z is of the form $Z = \mathscr{E}(\int_0^\cdot \beta_s \cdot dX_s^c + (Y-1) * (\widehat{\mu} - \widehat{\nu}))$, where the integer-valued random measure $\widehat{\mu}$ on $\mathbb{R}_+ \times \mathbb{R}^d$ is defined by*

$$\widehat{\mu}([0,t] \times G) := \mu^X([0,t] \times G) + \varepsilon_0(G) \sum_{s \in \Theta \cap [0,t]} (1 - \mu^X(\{s\} \times \mathbb{R}^d))$$

(for any $t \in \mathbb{R}_+$, $G \in \mathcal{B}^d$), $\widehat{\nu}$ denotes the compensator of $\widehat{\mu}$, $\beta : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ is a predictable process and $Y : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ a predictable mapping such that $\int Y(t,x)\, K_t(dx) = 1$ P-almost surely for any $t \in \Theta$. Then β and Y meet the Conditions 1 and 2 in Theorem 2.26 (even if we do not assume that X is also a $\widetilde{P}$-special semimartingale).

Proofs

PROOF OF THEOREM 2.26. Observe that, by the definition of the extended characteristics and by Remark 2 in Section 2.4, Equations (2.8) – (2.10) and the claim concerning c can be rephrased as

$$\widetilde{\nu}([0,t] \times G) = \int_0^t \int 1_G(s) Y(s,x)\, F_s(dx) + \sum_{s \in \Theta \cap [0,t]} \int 1_{G \setminus \{0\}}(s) Y(s,x)\, K_s(dx), \quad (2.11)$$

$$\begin{aligned} \widetilde{B}_t^i &= B_t^i + \sum_{\alpha=1}^d \int_0^t c_s^{i\alpha} \beta_s^\alpha\, ds + \int_0^t \int x^i (Y(s,x) - 1)\, F_s(dx) \\ &\quad + \sum_{s \in \Theta \cap [0,t]} \int x^i (Y(s,x) - 1)\, K_s(dx), \end{aligned}$$

$$\widetilde{C}_t = C_t$$

for any $t \in \mathbb{R}_+$, $i \in \{1,\ldots,d\}$, $G \in \mathcal{B}^d$, where $(B,C,\nu)^I$ and $(\widetilde{B}, \widetilde{C}, \widetilde{\nu})^I$ denote the integral characteristics of X relative to P and $\widetilde{P}$, respectively. Using Equation (2.3), we rephrase this once more by the statement that $(\widetilde{B}(h), \widetilde{C}, \widetilde{\nu})^{JS}$ are the characteristics of X relative to $\widetilde{P}$, where

$$\widetilde{B}^i_t(h) = B^i_t(h) + \sum_{\alpha=1}^{d} \int_0^t c^{i\alpha}_s \beta^\alpha_s \, ds + h^i(x)(Y-1) * \nu_t$$

for any $t \in \mathbb{R}_+$. Observe that Condition 1 is a rephrase of $YZ_- = M^P_{\mu^X}(Z|\widetilde{\mathcal{P}})$ in the notation of JS. Theorem 2.26 now follows from JS, Theorem III.3.24 if we replace (2.7) with $|h(x)(Y-1)| * \nu_t < \infty$ $\widetilde{P}$-almost surely for any $t \in \mathbb{R}_+$. But note that $|x - h(x)| * \nu_t < \infty$ and $|(x - h(x))Y| * \nu_t = |x - h(x)| * \widetilde{\nu}_t < \infty$ $\widetilde{P}$-almost surely for any $t \in \mathbb{R}_+$ holds anyway, since X is a P- and $\widetilde{P}$-special semimartingale (cf. Proposition 2.7). Hence, the stronger inequality $|x(Y-1)| * \nu_t < \infty$ $\widetilde{P}$- almost surely for any $t \in \mathbb{R}_+$ is also met. $\square$

Proof of the remark. *1.* It suffices to show that M is a $\widetilde{P}$-special semimartingale. By Jacod (1979), (7.29) or JS, III.3.11, this follows from $[Z, M^i] \in \mathscr{A}_{\mathrm{loc}}(P)$ for $i \in \{1, \dots, d\}$.

2. It suffices to show that M^d is a $\widetilde{P}$-special semimartingale. This follows again from Jacod (1979), (7.29) or JS, III.3.11.

3. By JS, III.3.24, we have that (2.11) holds for the $\widetilde{P}$-compensator of μ^X even if X is not special. The claim now follows from Proposition 2.7. $\square$

Proof of Lemma 2.27. Since Z is by definition a solution to the SDE $dZ_t = Z_{t-}\beta_t \cdot dX^c_t + Z_{t-}(Y(t,x) - 1)\,(\widehat{\mu} - \widehat{\nu})(dt, dx)$, one easily verifies that Condition 2 holds for β. Moreover observe that, by definition of the integral with respect to $(\widehat{\mu} - \widehat{\nu})$, the jumps of Z are, up to an evanescent set, given by

$$\Delta Z_t = \begin{cases} Z_{t-}(Y(t, \Delta X_t) - 1) 1_{\mathbb{R}^d \setminus \{0\}}(\Delta X_t) - 0 & \text{if } t \notin \Theta \\[2ex] Z_{t-}(Y(t, \Delta X_t) - 1) - \int Z_{t-}(Y(t,x) - 1)\,\widehat{\nu}(\{t\} \times dx) & \text{if } t \in \Theta. \end{cases}$$

Since $\widehat{\nu}(\{t\} \times dx) = K_t(dx)$ for any $t \in \Theta$, we have that $\int (Y(t,x) - 1)\,\widehat{\nu}(\{t\} \times dx) = 0$ and hence $\Delta Z_t = Z_{t-}(Y(t, \Delta X_t) - 1)$ for any $t \in \mathbb{R}_+$. This in turn implies $Z_t = Z_{t-} + \Delta Z_t = Z_{t-}Y(t, \Delta X_t) = Z_{t-}Y(t,x)$ for μ^X-almost all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Thus, Condition 1 in Theorem 2.26 also holds. $\square$

2.7 Martingale Problems

Now we are ready to define the promised stochastic analogue to ODE's. For the reasons mentioned in the previous section we will not do this in terms of Grigelionis processes and their derivative, but consider instead the respective extended notions. In this section the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P)$ is no longer given. Let us fix some

Notation. By $(\mathbb{D}^d, \mathcal{D}^d, (\mathcal{D}^d_t)_{t \in \mathbb{R}_+})$ we denote the *Skorohod space* $\mathbb{D}^d := \mathbb{D}(\mathbb{R}^d)$ of càdlàg functions $\mathbb{R}_+ \to \mathbb{R}^d$ with its Borel-σ-field $\mathcal{D}^d := \mathcal{D}(\mathbb{R}^d)$ and the canonical filtration

$(\mathcal{D}^d_t)_{t\in\mathbb{R}_+} := (\mathcal{D}(\mathbb{R}^d)_t)_{t\in\mathbb{R}_+}$ (cf. Appendix A, Definition A.7). $\mathcal{P}^d$ stands for the predictable σ-field on $\mathbb{D}^d \times \mathbb{R}_+$.

Definition 2.28 1. A *martingale problem* $(\Theta, \eta, b, c, F, K)^M$ in $\mathbb{R}^d$ is given by

- a discrete set $\Theta \subset \mathbb{R}^*_+$,
- a probability measure η on $(\mathbb{R}^d, \mathcal{B}^d)$,
- a $\mathcal{P}^d$-measurable mapping $b : \mathbb{D}^d \times \mathbb{R}_+ \to \mathbb{R}^d$ such that $\int_0^t |b_s(\bar\omega)|\, ds < \infty$ for any $(\bar\omega, t) \in \mathbb{D}^d \times \mathbb{R}_+$,
- a $\mathcal{P}^d$-measurable mapping $c : \mathbb{D}^d \times \mathbb{R}_+ \to \mathbb{R}^{d\times d}$ whose values are symmetric, non-negative matrices such that $\sum_{i,j=1}^d \int_0^t |c^{ij}_s(\bar\omega)|\, ds < \infty$ for any $(\bar\omega, t) \in \mathbb{D}^d \times \mathbb{R}_+$,
- a transition kernel F from $(\mathbb{D}^d \times \mathbb{R}_+, \mathcal{P}^d)$ into $(\mathbb{R}^d, \mathcal{B}^d)$ such that $F((\bar\omega, t), \{0\}) = 0$ and $\int_0^t \int (|x^2| \wedge |x|)\, F((\bar\omega, s), dx)\, ds < \infty$ for any $(\bar\omega, t) \in \mathbb{D}^d \times \mathbb{R}_+$,
- a transition kernel K from $(\mathbb{D}^d \times \mathbb{R}_+, \mathcal{P}^d)$ into $(\mathbb{R}^d, \mathcal{B}^d)$ such that for any $(\bar\omega, t) \in \mathbb{D}^d \times \mathbb{R}_+$, we have $\int |x|\, K((\bar\omega, t), dx) < \infty$ and

$$K((\bar\omega, t), \mathbb{R}^d) = \begin{cases} 1 & \text{if } t \in \Theta \\ 0 & \text{else.} \end{cases}$$

2. We call a $\mathbb{R}^d$-valued extended Grigelionis process X on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ (or, more exactly, we call the tupel $((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P), X)$) *solution-process* to the martingale problem $(\Theta, \eta, b, c, F, K)^M$ if $(\Theta, \eta, \widetilde{b}, \widetilde{c}, \widetilde{F}, \widetilde{K})^E$ is a version of its extended characteristics, where we define $\widetilde{b}_t(\omega) := b_t(X(\omega))$, $\widetilde{c}_t(\omega) := c_t(X(\omega))$, $\widetilde{F}((\omega, t), \cdot) := F((X(\omega), t), \cdot)$ and $\widetilde{K}((\omega, t), \cdot) := K((X(\omega), t), \cdot)$ for any $(\omega, t) \in \Omega \times \mathbb{R}_+$.

3. For any solution-process X the law P^X on $(\mathbb{D}^d, \mathcal{D}^d)$ is called a *solution-measure* to the martingale problem.

Remark. Strictly speaking, a semimartingale X on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ is càdlàg only up to an evanescent set, so that $b_t(X(\omega))$ etc. may only be defined for P-almost all $\omega \in \Omega$. However, by Jacod (1979), (1.1) we can define $b(X)$ (up to indistinguishability uniquely) by taking a $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-predictable version of $b(\widetilde{X})$, where $\widetilde{X}$ is a $(\mathcal{F}^P_t)_{t\in\mathbb{R}_+}$-semimartingale with càdlàg paths (for *all* $\omega \in \Omega$) and $X = \widetilde{X}$ up to indistinguishability.

If two processes solve the same or similar martingale problems, we say that they share the same (resp. a similar) *dynamic*. This term will also be loosely applied if we talk about the extended characteristics of an extended Grigelionis process. Before we prove a number of results about martingale problems for later use, we give some examples showing that various probabilistic models can be obtained as particular martingale problems.

Examples.

1. Let η be a starting distribution on $(\mathbb{R}^d, \mathcal{B}^d)$ and Q a Markov transition kernel from $(\mathbb{R}^d, \mathcal{B}^d)$ into $(\mathbb{R}^d, \mathcal{B}^d)$ such that $\int |x|\, Q(y, dx) < \infty$ for any $y \in \mathbb{R}^d$. Moreover, let $((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P), X)$ be a solution-process to the martingale problem $(\mathbb{N}^*, \eta, 0, 0, 0, K)^M$, where the transition kernal K from $(\mathbb{D}^d \times \mathbb{R}_+, \mathcal{P}^d)$ into $(\mathbb{R}^d, \mathcal{B}^d)$ is defined by
$$K((\bar{\omega}, t), G) = \begin{cases} Q(\bar{\omega}_{t-1}, G + \bar{\omega}_{t-1}) & \text{if } t \in \mathbb{N}^* \\ 0 & \text{else} \end{cases}$$
for any $(\bar{\omega}, t) \in \mathbb{D}^d \times \mathbb{R}_+$, $G \in \mathcal{B}^d$. Then $(X_t)_{t\in\mathbb{N}}$ is a *Markov chain* corresponding to the initial distribution η and the transition kernel K.

2. Consider now an ordinary differential equation or, more specifically, the initial value problem
$$x(0) = x_0, \quad x'(t) = f(x(t), t) \tag{2.12}$$
for some $x_0 \in \mathbb{R}^d$ and some continuous function $f : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$. Then any solution to this initial value problem is a deterministic solution-process to the martingale problem $(\varnothing, \varepsilon_{x_0}, b, 0, 0, 0)^M$, where $b : \mathbb{D}^d \times \mathbb{R}_+ \to \mathbb{R}^d$ is defined by $b_t(\bar{\omega}) := f(\bar{\omega}_t, t)$ for any $(\bar{\omega}, t) \in \mathbb{D}^d \times \mathbb{R}_+$. Moreover, if $((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P), X)$ is a solution-process to this martingale problem, then P-almost all paths of X are solutions to the initial value problem (2.12). Hence, there are only deterministic solution-processes to the martingale problem if uniqueness holds for the initial value problem (2.12).

3. Let $((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P), (Z, X))$ be a solution-process to a martingale problem $(\mathbb{N}^*, \eta, 0, 0, 0, K)^M$ in $\mathbb{R}^2$ such that for any $((\bar{\omega}^1, \bar{\omega}^2), t) \in \mathbb{R}^2 \times (\mathbb{N}^* \setminus \{1, 2, \ldots, (p \vee q) - 1\})$ we have that $K(((\bar{\omega}^1, \bar{\omega}^2), t), \cdot)$ is the image of $N(0, 1)$ under the mapping $\mathbb{R} \to \mathbb{R}^2$, $z \mapsto (z - \bar{\omega}^1_{t-1}, z - \bar{\omega}^2_{t-1} + \sum_{i=1}^p \varphi_i \bar{\omega}^2_{t-i} + \sum_{i=1}^q \vartheta_i \bar{\omega}^1_{t-i})$, where $p, q \in \mathbb{N}$ and $\varphi_1, \ldots, \varphi_p, \vartheta_1, \ldots, \vartheta_q \in \mathbb{R}$ are given. Then $(Z_t)_{t\in\mathbb{N}^*}$ is a sequence of independent, $N(0, 1)$-distributed random variables. Moreover, we have that
$$X_t - \varphi_1 X_{t-1} - \ldots - \varphi_p X_{t-p} = Z_t + \vartheta_1 Z_{t-1} + \ldots + \vartheta_q Z_{t-q}$$
P-almost surely for any $t \in \mathbb{N} \setminus \{0, 1, \ldots, p \vee q\}$. Therefore, $(X_t)_{t\in\mathbb{N}\setminus\{0,1,\ldots,p\vee q\}}$ is a *ARMA*(p, q) *time series*.

4. Consider a one-dimensional *diffusion* satisfying the SDE
$$dX_t = \mu(X_t, t)\, dt + \sigma(X_t, t)\, dW_t, \quad X_0 = x_0$$
where $x_0 \in \mathbb{R}$, $\mu, \sigma : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ are given continuous functions and W is a standard Wiener process. Then X is a solution-process to the martingale problem $(\varnothing, \varepsilon_{x_0}, b, c, 0, 0)^M$, where $b(\bar{\omega}, t) = \mu(\bar{\omega}_t, t)$, $c(\bar{\omega}, t) = (\sigma(\bar{\omega}_t, t))^2$ for any $(\bar{\omega}, t) \in \mathbb{R} \times \mathbb{R}_+$.

The following lemma relates our martingale problems with those from JS, Definition III.2.4 and Jacod (1979), Problème 12.9. These two use a slightly different notation for the same object: $S^{II}(\mathcal{H}; X|P_H; B, C, \nu)$ in Jacod (1979) corresponds to $\mathscr{s}(\mathcal{H}, X|P_H; B, C, \nu)$ in JS.

Remark. If η denotes a probability measure on $\mathbb{R}^d$ and Y is the canonical process on $\mathbb{D}^d$, then $Y_0^{-1}(A) \mapsto \eta(A)$ for any $A \in \mathcal{B}^d$ uniquely defines a probability measure on $(\mathbb{D}^d, \sigma(Y_0))$, which we denote again by η (*l'image réciproque* in Jacod (1979), p.395).

Lemma 2.29 *Let $(\Theta, \eta, b, c, F, K)^M$ be a martingale problem in $\mathbb{R}^d$ as in Definition 2.28.*

1. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ be a filtered probability space and X a $\mathbb{R}^d$-valued semimartingale on that space. Then we have equivalence between*

   (a) X is a solution-process to the martingale problem $(\Theta, \eta, b, c, F, K)^M$.

   (b) P is a solution to the martingale problem $\mathscr{s}(\sigma(X_0), X|(P|_{\sigma(X_0)}); B(h), C, \nu)$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+})$ in the sense of JS, Definition III.2.4, where the mappings $B(h) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$, $C : \Omega \times \mathbb{R}_+ \to \mathbb{R}^{d\times d}$ and the random measure ν on $\mathbb{R}_+ \times \mathbb{R}^d$ are defined by

$$B(h)_t := \int_0^t b_s(X)\, ds + \int_0^t \int (h(x) - x)\, F((X, s), dx)\, ds + \sum_{s\in\Theta\cap[0,t]} \int h(x)\, K((X, s), dx),$$

$$C_t := \int_0^t c_s(X)\, ds,$$

$$\nu([0, t] \times G) := \int_0^t F((X, s), G)\, ds + \sum_{s\in\Theta\cap[0,t]} K((X, s), G \setminus \{0\})$$

   for any $t \in \mathbb{R}_+$, $G \in \mathcal{B}^d$. Moreover, $P^{X_0} = \eta$.

2. *If $((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P), X)$ is a solution-process to the martingale problem, then $((\mathbb{D}^d, \mathcal{D}^d, (\mathcal{D}^d_t)_{t\in\mathbb{R}_+}, P^X), Y)$ is a solution-process as well, where Y here denotes the* canonical process *on $\mathbb{D}^d$ (i.e. $Y_t(\bar{\omega}) = \bar{\omega}_t$ for any $(\bar{\omega}, t) \in \mathbb{D}^d \times \mathbb{R}_+$).*

3. *On the space $(\mathbb{D}^d, \mathcal{D}^d, (\mathcal{D}^d_t)_{t\in\mathbb{R}_+})$ with canonical process Y, consider the (JS-sense) martingale problem $\mathscr{s}(\sigma(Y_0), Y|\eta; B(h), C, \nu)$, where $B(h), C, \nu$ are defined as in Statement 2, but on $\mathbb{D}^d$ instead of Ω. Then we have equivalence between*

   (a) P is a solution-measure to $(\Theta, \eta, b, c, F, K)^M$.

   (b) $P \in \mathscr{s}(\sigma(Y_0), Y|\eta; B(h), C, \nu)$.

   In particular, $(\Theta, \eta, b, c, F, K)^M$ has a unique solution-measure if and only if $\mathscr{s}(\sigma(Y_0), Y|\eta; B(h), C, \nu)$ has a unique solution.

Now we turn to the connection of martingale problems and stochastic differential equations (SDE's).

Theorem 2.30 *Let $x \in \mathbb{R}^d$, let $b : \mathbb{D}^d \times \mathbb{R}_+ \to \mathbb{R}^{d'}$ and $u : \mathbb{D}^d \times \mathbb{R}_+ \to \mathbb{R}^{d\times d'}$ be predictable. Then the following statements are equivalent.*

1. *P is a weak solution (or solution-measure) to the SDE*

$$dX_t = b_t(X)\,dt + u_t(X)\,dW_t, \;\; X_0 = x$$

(in the sense of Jacod (1979), Definition 14.79), where W denotes a $\mathbb{R}^{d'}$-valued standard Wiener process.

2. *P is a solution-measure to the martingale problem $(\varnothing, \varepsilon_x, b, c, 0, 0)^M$, where $c := uu^\top$.*

The preceding theorem deserves a short reflection. In applications stochastic phenomena are often modelled by SDE's with respect to Wiener processes. This common choice is, by Theorem 2.30, also natural from the point of view of martingale problems as long as one considers only models with continuous paths (i.e. $F = 0$, $K = 0$). The situation is less obvious in the discontinuous case. Although formally martingale problems (with $\Theta = \varnothing$) can be transformed into a weak sense SDE with respect to a Wiener process and a Poisson random measure (cf. Jacod (1979), Théorèmes 14.80, 14.45, 14.53), the choice and the meaning of the coefficients is not evident. Therefore we think that, especially in the discontinuous case, martingale problems may be the more intuitive concept from the point of view of modelling.

The following theorem states that the existence of a unique solution-measure to a martingale problem carries over to related problems with different drift coefficients in the continuous case. Its proof is based on a Girsanov transformation.

Theorem 2.31 *Let $(\varnothing, \eta, b, c, 0, 0)^M$ be a martingale problem in $\mathbb{R}^d$ having a unique solution-measure P. Moreover, let $h : \mathbb{D}^d \times \mathbb{R}_+ \to \mathbb{R}^d$ be a $\mathcal{P}^d$-measurable mapping such that $\int_0^t |h_s(\bar\omega)^\top c_s(\bar\omega) h_s(\bar\omega)|\,ds < \infty$ for any $(\bar\omega, t) \in \mathbb{D}^d \times \mathbb{R}_+$. Then the martingale problem $(\varnothing, \eta, b + \sum_{\alpha=1}^d h^\alpha c^{\alpha\cdot}, c, 0, 0)^M$ has a unique solution-measure P', which is, in addition, locally equivalent to P (cf. Appendix A, Definition A.5). The density process of P' relative to P is $Z := \mathscr{E}(\int_0^\cdot h_s \cdot dX_s^c)$, where X denotes the canonical process on $\mathbb{D}^d$.*

The next two technical lemmas are for later use.

Lemma 2.32 *Let $((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P), X)$ be a solution-process to a martingale problem $(\Theta, \eta, b, c, F, K)^M$ in $\mathbb{R}^d$. Then $((\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in\mathbb{R}_+}, P), X)$ is also a solution-process to the problem, where $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$ denotes any sub-filtration of $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ to which X is adapted or the P-completion of such a filtration and $\mathcal{G} = \mathcal{F}$ resp. $\mathcal{F}^P$.*

Lemma 2.33 *Let $((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P), X)$ be a solution-process to a martingale problem $(\Theta, \eta, b, c, F, K)^M$ in $\mathbb{R}^d$. Assume that $(\Omega, \mathcal{F})$ is a Blackwell space (cf. Remark 2 below). Fix $t \in \mathbb{R}_+$ and let $\widetilde{\Theta} := \{s - t : s \in \Theta \cap (t, \infty)\}$. For fixed $\omega \in \Omega$ we define $x \in \mathbb{R}_+$, the mappings $\widetilde{b} : \mathbb{D}^d \times \mathbb{R}_+ \to \mathbb{R}^d$, $\widetilde{c} : \mathbb{D}^d \times \mathbb{R}_+ \to \mathbb{R}^{d\times d}$ and the transition kernels F, K*

from $(\mathbb{D}^d \times \mathbb{R}_+, \mathcal{P}^d)$ *into* $(\mathbb{R}^d, \mathcal{B}^d)$ *by* $x := X_t(\omega)$, $\widetilde{b}_s(\bar{\omega}) := b_{t+s}(\iota(X(\omega), \bar{\omega}))$, $\widetilde{c}_s(\bar{\omega}) := c_{t+s}(\iota(X(\omega), \bar{\omega}))$, $\widetilde{F}((\bar{\omega}, s), \cdot) := F((\iota(X(\omega), \bar{\omega}), t+s), \cdot)$, $\widetilde{K}((\bar{\omega}, s), \cdot) := K((\iota(X(\omega), \bar{\omega}), t+s), \cdot)$ *for any* $(\bar{\omega}, s) \in \mathbb{D}^d \times \mathbb{R}_+$*, where*

$$\iota(X(\omega), \bar{\omega})_s := \begin{cases} X_s(\omega) & \text{for } s \in [0,t) \\ \bar{\omega}_{s-t} & \text{for } s \in [t, \infty). \end{cases}$$

Then both $P^{(X_{t+s})_{s\in\mathbb{R}_+}|\mathcal{F}_t}(\omega)$ *and* $P^{(X_{t+s})_{s\in\mathbb{R}_+}|\sigma(X_u:u\in[0,t])}(\omega)$ *are for* P*-almost all* $\omega \in \Omega$ *solution-measures to the (random) martingale problem* $(\widetilde{\Theta}, \varepsilon_x, \widetilde{b}, \widetilde{c}, \widetilde{F}, \widetilde{K})^M$.

Remarks.

1. Lemma 2.33 still holds if we replace $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ with its P-completion $(\Omega, \mathcal{F}^P, (\mathcal{F}_t^P)_{t\in\mathbb{R}_+}, P)$.

2. Blackwell spaces are defined in Dellacherie & Meyer (1978), III.24. Any Polish space with its Borel-σ-field as e.g. $(\mathbb{D}^d, \mathcal{D}^d)$ is a Blackwell space (cf. JS, p. 65). Moreover, if $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is the canonical filtration of a càdlàg process X and $\mathcal{F} = \mathcal{F}_{\infty-}$, then $(\Omega, \mathcal{F})$ is also a Blackwell space.

Proofs

PROOF OF THE EXAMPLES. *1.* Since $b = 0$, $c = 0$, $F = 0$, we have that $X_t = X_0 + \sum_{s\in\Theta\cap[0,t]} \Delta X_s$ for any $t \in \mathbb{R}_+$. By Remark 3 in Section 2.4 we obtain $P(X_t \in G|\mathcal{F}_{t-}) = P^{\Delta X_t|\mathcal{F}_{t-}}(G - X_{t-1}) = K((X,t), G - X_{t-1}) = Q(X_{t-1}, G)$ P-almost surely for any $t \in \mathbb{N}^*$, $G \in \mathcal{B}^d$. Therefore, also $P(X_t \in \cdot|\sigma(X_0, \ldots, X_{t-1})) = Q(X_{t-1}, \cdot)$ P-almost surely for any $t \in \mathbb{N}^*$, and the claim follows.

3. Similarly to Example 1, it follows that

$$\begin{aligned} P(Z_t \in G, X_t \in H|\mathcal{F}_{t-}) &= K(((Z,X),t), (G - Z_{t-1}) \times (H - X_{t-1})) \\ &= \int 1_G(z) 1_H\Big(z + \sum_{i=1}^p \varphi_i X_{t-i} + \sum_{i=1}^q \vartheta_i Z_{t-i}\Big) \, N(0,1)(dz) \end{aligned}$$

P-almost surely for $t \in \mathbb{N}^* \setminus \{1, 2, \ldots, (p \vee q)\}$ and $G, H \in \mathcal{B}$. This shows the claim.

4. This will be shown in Theorem 2.30. □

Proposition 2.34 *Let* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ *be a filtered probability space,* X *a* $\mathbb{R}^d$*-valued semimartingale on that space and* η *a probability measure on* $\sigma(X_0)$*. By* $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$ *we denote a sub-filtration of* $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$*, to which* X *is adapted or the* P*-completion of such a filtration; moreover,* $\mathcal{G} := \mathcal{F}$ *resp.* $\mathcal{F}^P$*. Assume that* P *is a solution to the martingale problem* $\mathscr{s}(\sigma(X_0), X|\eta; B(h), C, \nu)$*, where* $B(h), C, \nu$ *are as in JS, III.2.3, but with predictability also relative to the filtration* $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$*. Then* P *is a solution to the martingale problem* $\mathscr{s}(\sigma(X_0), X|\eta; B(h), C, \nu)$ *on* $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in\mathbb{R}_+}, P)$*.*

PROOF. By JS, II.2.42, we have that

$$\begin{aligned} Y &:= f(X) - f(X_0) - \sum_{i=1}^{d} \int_0^{\cdot} D_i f(X_{s-})\, dB(h)^i_s - \frac{1}{2} \sum_{i,j=1}^{d} \int_0^{\cdot} D_{ij} f(X_{s-})\, dC^{ij}_s \\ &\quad - \Big(f(X_- + x) - f(X_-) - \sum_{i=1}^{d} D_i f(X_-) h^i(x) \Big) * \nu \end{aligned}$$

is a $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-local martingale for any bounded C^2-function $f : \mathbb{R}^d \to \mathbb{R}$. Since the last three terms are $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$-predictable and, moreover, of finite variation, they are, by JS, I.3.10, $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$-locally bounded. Hence, Y is a $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$-locally bounded $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-local martingale. By Jacod (1979), (9.18), (iii)⇒(i), one has that Y is also a $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$-local martingale. Again by JS, II.2.42, the statement follows. □

PROOF OF LEMMA 2.29. *1.* (a)⇒(b): By Lemma 2.21 the characteristics $(B(h), C, \nu)^{JS}$ are given by the equations in (b), which yields the claim.

(b)⇒(a): By the integrability conditions on F, K in Definition 2.28, we have that $(|x|^2 \wedge |x|) * \nu_t < \infty$ P-almost surely for any $t \in \mathbb{R}_+$. Therefore, X is a special semimartingale (cf. Proposition 2.7). The claim now follows from Statement 2 in Lemma 2.21.

2. By Statement 1, P is a solution to the martingale problem $\mathscr{I}(\sigma(X_0), X|(P|_{\sigma(X_0)}); B(h), C, \nu)$ on $((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}), X)$. By Proposition 2.34 one may replace $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ with the canonical filtration $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$ of X. The corresponding martingale problem on the space $(\mathbb{D}^d, \mathcal{D}^d, (\mathcal{D}^d_t)_{t\in\mathbb{R}_+})$ (which is called $\mathscr{I}(\sigma(Y_0), Y|\eta; B(h), C, \nu)$ in Statement 3 of Lemma 2.29) is the image of that problem in the sense of Jacod (1979), (12.65). By Jacod (1979), (12.66), we have that $P^X \in \mathscr{I}(\sigma(Y_0), Y|\eta; B(h), C, \nu)$. The claim now follows from the inclusion (b)⇒(a) in Statement 1.

3. By Statement 2, we have that (a) is equivalent to

(c) $((\mathbb{D}^d, \mathcal{D}^d, (\mathcal{D}^d_t)_{t\in\mathbb{R}_+}, P), Y)$ is a solution-process to $(\Theta, \eta, b, c, F, K)^M$.

Statement 1 implies that (c) is equivalent to (b). □

PROOF OF THEOREM 2.30. By Jacod (1979), (14.80), the first statement is equivalent to the assertion that P is a solution to the martingale problem $\mathscr{I}(\sigma(X_0), X|\varepsilon_x; B, C, 0)$ on the space $(\mathbb{D}^d, \mathcal{D}^d, (\mathcal{D}^d_t)_{t\in\mathbb{R}_+})$ with canonical process X, where $B_t := \int_0^t b_s(X)\, ds$, $C_t = \int_0^t u_s(X) u_s(X)^\top\, ds$ for any $t \in \mathbb{R}_+$. The claim now follows from Statement 3 of the previous lemma. □

PROOF OF THEOREM 2.31. Related versions of this theorem can be found in Revuz & Yor (1994), Theorems IX.1.10 and IX.1.11.

Firstly, we will show the existence part. For any $r \in \mathbb{R}_+$ define a stopping time T_r on $(\mathbb{D}^d, \mathcal{D}^d, (\mathcal{D}^d_t)_{t\in\mathbb{R}_+}, P)$ by $T_r := \inf\{t \in \mathbb{R}_+ : \exp(\frac{1}{2}\int_0^t |h_s^\top c_s h_s|\, ds) > r\}$. By assumption, we have $T_r \uparrow \infty$ for $r \uparrow \infty$ and hence $\mathcal{D}^d = \mathcal{D}^d_{\infty-} = \sigma(\cup_{r\in\mathbb{N}} \mathcal{D}^d_{T_r})$ (cf. Jacod (1979), (1.9a)).

We define a new filtration $(\mathcal{G}_r)_{r\in\mathbb{R}_+}$ on $(\mathbb{D}^d, \mathcal{D}^d)$ by $\mathcal{G}_r := \mathcal{D}^d_{T_r}$. Observe that $(\mathcal{G}_r)_{r\in\mathbb{R}_+}$ is right-continuous, since $T_{\widetilde{r}} \downarrow T_r$ for $\widetilde{r} \downarrow r$ (cf. JS, I.1.18). By Lemma 2.29, Statement 2 we have that X is a solution-process to the martingale problem $(\varnothing, \eta, b, c, 0, 0)^M$. According to the Novikov condition (cf. Revuz & Yor (1994), Proposition VIII.1.15), Z^{T_r} is a uniformly integrable martingale starting in 1. Therefore, we may define for any $r \in \mathbb{R}_+$ a probability measure $P^r \sim P$ on $\mathcal{D}^d$ by $dP^r/dP := Z^{T_r}_\infty$. By use of the stopping theorem (cf. JS, I.1.39b) one easily shows that $P^{r'}|_{\mathcal{G}_r} = P^r|_{\mathcal{G}_r}$ for $r \leq r'$. Since $(\mathbb{D}^d, \mathcal{D}^d)$ is a Polish space, there exists, by Ikeda & Watanabe (1989), IV.4.1, a probability measure P' on $(\mathbb{D}^d, \mathcal{D}^d)$ such that $P'|_{\mathcal{G}_r} = P^r|_{\mathcal{G}_r}$ for any $r \in \mathbb{R}_+$. Since $Z^{T_r} = \mathscr{E}(\int_0^\cdot 1_{[0,T_r]} h_s \cdot dX^c_s)$, it follows from Lemma 2.27 and Theorem 2.26 that $((\mathbb{D}^d, \mathcal{D}^d, (\mathcal{D}^d_t)_{t\in\mathbb{R}_+}, P^r), X)$ is for any $r \in \mathbb{R}_+$ a solution-process to the martingale problem $(\varnothing, \eta, b^{(r)}, c, 0, 0)^M$, where $b^{(r)} := b + \sum_{\alpha=1}^d h^\alpha c^{\alpha\cdot} 1_{[0,T_r]}$. By Statement 3 of Lemma 2.29, P^r is a solution to the martingale problem $\mathscr{J}(\sigma(X_0), X|\eta; B^{(r)}, C, 0)$ on $((\mathbb{D}^d, \mathcal{D}^d, (\mathcal{D}^d_t)_{t\in\mathbb{R}_+}), X)$, where $B^{(r)}_t := \int_0^t b^{(r)}_s(X)\, ds$, $C_t := \int_0^t c_s(X)\, ds$ for any $t \in \mathbb{R}_+$. By JS, II.2.21 the processes $M^{(r)} := X - X_0 - B^{(r)}$ and $N^{(r),ij} := M^{(r),i} M^{(r),j} - C^{ij}$ are P^r-local martingales for $i, j \in \{1, \ldots, d\}$. The stopping theorem implies that this also holds for the stopped processes $(M^{(r)})^{T_r}, (N^{(r),ij})^{T_r}$. Now define the processes B, M, N^{ij} by $B_t := \int_0^t (b_s + \sum_{\alpha=1}^d h^\alpha_s c^{\alpha\cdot}_s)\, ds$, $M_t := X_t - X_0 - B_t$, $N^{ij} := M^i M^j - C^{ij}$ for any $t \in \mathbb{R}_+$. Since $(M^{(r)})^{T_r} = M^{T_r}$, $(N^{(r),ij})^{T_r} = (N^{ij})^{T_r}$ and since P^r, P' coincide on $\mathcal{D}^d_{T_r}$, we have that $M^{T_r}, (N^{ij})^{T_r}$ are P'-local martingales for any $r \in \mathbb{N}$. Hence M, N^{ij} are also P'-local martingales for $i, j \in \{1, \ldots, d\}$. By JS, II.2.21 we have that P' is a solution to the martingale problem $\mathscr{J}(\sigma(X_0), X|\eta; B, C, 0)$ on $((\mathbb{D}^d, \mathcal{D}^d, (\mathcal{D}^d_t)_{t\in\mathbb{R}_+}), X)$. Statement 3 of Lemma 2.29 yields that P' is a solution-measure to the martingale problem $(\varnothing, \eta, b + \sum_{\alpha=1}^d h^\alpha c^{\alpha\cdot}, c, 0, 0)^M$ as well. By JS, III.3.3, P and P' are locally equivalent. The fact that Z is the density process of P' relative to P follows from an easy calculation using the martingale property of the processes Z^{T_r}, $r \in \mathbb{N}$.

In order to show uniqueness, assume that there are two solution-measures P', P'' as in Theorem 2.31. By applying the existence part of the theorem to $-h$ instead of h, we have that there are probability measures $\widetilde{P}' \overset{\text{loc}}{\sim} P'$, $\widetilde{P}'' \overset{\text{loc}}{\sim} P''$, defined by $d\widetilde{P}''|_{\mathcal{D}^d_t}/dP''|_{\mathcal{D}^d_t} := d\widetilde{P}'|_{\mathcal{D}^d_t}/dP'|_{\mathcal{D}^d_t} := \mathscr{E}(-\int_0^\cdot h_s \cdot d\overline{X}^c_s)_t$ for any $t \in \mathbb{R}_+$, which are solution-measures to the original martingale problem $(\varnothing, \eta, b, c, 0, 0)^M$, where $\overline{X}^c := X - \int_0^\cdot (b_s(X) + \sum_{\alpha=1}^d h^\alpha_s(X)$ $c^{\alpha\cdot}_s(X))\, ds$ denotes the continuous local martingale part of X relative to P' as well as P''. Since the martingale problem $(\varnothing, \eta, b, c, 0, 0)^M$ has a unique solution-measure, we have $\widetilde{P}' = \widetilde{P}''$ and, by the positivity of $\mathscr{E}(-\int_0^\cdot h_s \cdot d\overline{X}^c_s)$, also $P' = P''$, which yields the claim. $\square$

PROOF OF LEMMA 2.32. By Lemma 2.29, P is a solution to the martingale problem $\mathscr{J}(\sigma(X_0), X|(P|_{\sigma(X_0)}); B(h), C, \nu)$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+})$ where $B(h), C, \nu$ are defined as in Statement 1 of this lemma. Proposition 2.34 implies that we may substitute $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in\mathbb{R}_+})$ for the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+})$. Again by Lemma 2.29 the statement follows. $\square$

Proposition 2.35 *If $(X_t)_{t\in\mathbb{R}_+}$ is a càdlàg, adapted process such that $(X_t)_{t\in\mathbb{Q}_+}$ is a martingale, then $(X_t)_{t\in\mathbb{R}_+}$ is a martingale as well.*

PROOF. Fix $T \in \mathbb{Q}_+$. It suffices to show that $(X_t)_{t\in[0,T]}$ is a martingale. By JS, I.1.42 there exists a martingale $(\widetilde{X}_t)_{t\in\mathbb{R}_+}$ with $\widetilde{X}_t = E(X^T|\mathcal{F}_t)$ P-almost surely for any $t \in \mathbb{R}_+$. Since X and $\widetilde{X}$ are càdlàg and coincide on $\mathbb{Q} \cap [0,T]$, we have that $X^T = \widetilde{X}^T$ up to indistinguishability, which implies that $(X_t)_{t\in[0,T]}$ is a martingale. □

Proposition 2.36 *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ be a filtered probability space such that $(\Omega, \mathcal{F})$ is a Blackwell space. Fix $t \in \mathbb{R}_+$ and define the filtration $(\mathcal{G}_s)_{s\in\mathbb{R}_+}$ on $(\Omega, \mathcal{F})$ by $\mathcal{G}_s := \mathcal{F}_{s+t}$ for any $s \in \mathbb{R}_+$. Let X be a local martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$. Then the process $\widetilde{X}$, defined by $\widetilde{X}_s := X_{s+t} - X_t$ for any $s \in \mathbb{R}_+$, is a local martingale on $(\Omega, \mathcal{F}, (\mathcal{G}_s)_{s\in\mathbb{R}_+}, P^{|\mathcal{F}_t}(\omega))$ for P-almost all $\omega \in \Omega$, where $P^{|\mathcal{F}_t}$ denotes the regular conditional distribution of the identity given the σ-field $\mathcal{F}_t$.*

PROOF. Let $(T_n)_{n\in\mathbb{N}}$ be a localizing sequence for X. One easily shows that the sequence $(\widetilde{T}_n)_{n\in\mathbb{N}}$, defined by $\widetilde{T}_n := (T_n - t) \vee 0$, is a sequence of stopping times on $(\Omega, \mathcal{F}, (\mathcal{G}_s)_{s\in\mathbb{R}_+})$ with $\widetilde{T}_n \uparrow \infty$ $P^{|\mathcal{F}_t}(\omega)$-almost surely for P-almost all $\omega \in \Omega$. We will now show that $\widetilde{X}^{\widetilde{T}_n}$ is a $(\Omega, \mathcal{F}, (\mathcal{G}_s)_{s\in\mathbb{R}_+}, P^{|\mathcal{F}_t}(\omega))$-martingale for any $n \in \mathbb{N}$ and P-almost all $\omega \in \Omega$. Fix $n \in \mathbb{N}$. Since $\widetilde{X}^{\widetilde{T}_n}$ is $(\mathcal{G}_s)_{s\in\mathbb{R}_+}$-adapted and càdlàg, it suffices to prove that $(\widetilde{X}^{\widetilde{T}_n}_s)_{s\in\mathbb{Q}^+}$ is a $P^{|\mathcal{F}_t}(\omega)$-martingale for P-almost all $\omega \in \Omega$ (cf. Proposition 2.35). We will only prove the martingale equality; the integrability follows along the same lines. Fix $r, s \in \mathbb{Q}^+$ with $r \leq s$. It remains to be shown that there is a P-null set $N \in \mathcal{F}$ such that, for any $G \in \mathcal{F}$ and any $\omega \in N^C$, we have $\int(\widetilde{X}^{\widetilde{T}_n}_s(\bar{\omega}) - \widetilde{X}^{\widetilde{T}_n}_r(\bar{\omega}))E(1_G|\mathcal{G}_r)(\bar{\omega})\, P^{|\mathcal{F}_t}(\omega, d\bar{\omega}) = 0$. By a Dynkin argument it suffices to consider a countable generating algebra (which always exists in a Blackwell space) instead of all $G \in \mathcal{F}$. Therefore, we may let the P-null set N depend on the chosen set $G \in \mathcal{F}$. So, we are left to prove that for any $G \in \mathcal{F}$ and any $F \in \mathcal{F}_t$, we have

$$\iint \Big(\widetilde{X}^{\widetilde{T}_n}_s(\bar{\omega}) - \widetilde{X}^{\widetilde{T}_n}_r(\bar{\omega})\Big)E(1_G|\mathcal{G}_r)(\bar{\omega})\, P^{|\mathcal{F}_t}(\omega, d\bar{\omega})1_F(\omega)\, P(d\omega) = 0.$$

But, by definition of conditional probabilities and by $\widetilde{X}^{\widetilde{T}_n}_s(\bar{\omega}) = X^{T_n\vee t}_{s+t}(\bar{\omega}) - X_t(\bar{\omega})$, the left-hand side equals $\int(X^{T_n}_{t+s}(\omega) - X^{T_n}_{t+r}(\omega))E(1_G|\mathcal{G}_r)(\omega)1_F(\omega)\, P(d\omega)$, which is 0, since $\mathcal{F}_t \subset \mathcal{G}_r = \mathcal{F}_{t+r}$ and since X^{T_n} is a P-martingale. □

PROOF OF LEMMA 2.33. By Lemma 2.29, P is a solution to the martingale problem $\mathscr{I}(\sigma(X_0), X|(P|_{\sigma(X_0)}); B(h), C, \nu)$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+})$, where $B(h)$, C, ν are defined in Statement 1 of that lemma. By JS, II.2.21 this implies that the processes $M(h) := X(h) - B(h) - X_0$, $M(h)^i M(h)^j - \widetilde{C}^{ij}(h)$ for any $i, j \in \{1, \ldots, d\}$, and $g * \mu^X - g * \nu$ for any $g \in \mathscr{C}^+(\mathbb{R}^d)$ are local martingales, where we refer to JS for notation to avoid lengthy definitions here. But note that $\mathscr{C}^+(\mathbb{R}^d)$ can be chosen countable (cf. JS, II.2.20). Let now $t \in \mathbb{R}_+$ be fixed as in Lemma 2.33. Define a new filtration $(\mathcal{G}_s)_{s\in\mathbb{R}_+}$ on $(\Omega, \mathcal{F})$ by $\mathcal{G}_s := \mathcal{F}_{s+t}$ for any $s \in \mathbb{R}_+$. Moreover, define $(\mathcal{G}_s)_{s\in\mathbb{R}_+}$-local martingales $\widetilde{M}(h)$ etc. by $\widetilde{M}(h)_s := M(h)_{t+s} - M(h)_t$ for any $s \in \mathbb{R}_+$ (and accordingly for the processes $M(h)^i M(h)^j - \widetilde{C}^{ij}(h)$ and $g * \mu^X - g * \nu$). Proposition 2.36 yields that $\widetilde{M}(h)$ etc. are $P^{|\mathcal{F}_t}(\omega)$-local martingales for P-almost all $\omega \in \Omega$. Again by JS, II.2.21, this implies that for P-almost all $\omega \in \Omega$ we have $P^{|\mathcal{F}_t}(\omega) \in \mathscr{I}(\sigma(Y_0), Y|(P^{|\mathcal{F}_t}(\omega)|_{\sigma(Y_0)}); \widetilde{B}(h), \widetilde{C}, \widetilde{\nu})$ on $(\Omega, \mathcal{F}, (\mathcal{G}_s)_{s\in\mathbb{R}_+})$ with fundamental process $(Y_s)_{s\in\mathbb{R}_+} := (X_{t+s})_{s\in\mathbb{R}_+}$, where $\widetilde{B}(h)_s := B(h)_{t+s} - B(h)_t$, $\widetilde{C}_s := C_{t+s} - C_t$, $\widetilde{\nu}([0,s]\times G) := \nu([t,t+s]\times G)$ for $s\in\mathbb{R}_+$, $G\in\mathcal{B}^d$. Fix $\omega\in\Omega$. Observe that we have $b_{t+s}(X(\omega)) = b_{t+s}(\iota(X(\omega),(X_{t+s})_{s\in\mathbb{R}_+}(\omega)) = \widetilde{b}_s(Y(\omega))$ (and likewise for c,F,K) for any $s\in\mathbb{R}_+$. Therefore, we have by definition of $B(h),C,\nu$ that

$$\widetilde{B}(h)_s = \int_0^s \widetilde{b}_u(Y)\,du + \int_0^s\!\!\int (h(x)-x)\widetilde{F}((Y,u),dx)du + \sum_{u\in\widetilde{\Theta}\cap[0,s]}\int h(x)\widetilde{K}((Y,u),dx),$$

$$\widetilde{C}_s = \int_0^s \widetilde{c}_u(Y)\,du,$$

$$\widetilde{\nu}([0,s]\times G) = \int_0^s \widetilde{F}((Y,u),G)\,du + \sum_{u\in\widetilde{\Theta}\cap[0,s]}\widetilde{K}((Y,u),G\setminus\{0\}).$$

By Lemma 2.29, Statement 1 we can therefore conclude that $((\Omega,\mathcal{F},(\mathcal{G}_s)_{s\in\mathbb{R}_+},P^{|\mathcal{F}_t}(\omega)),Y)$ is a solution-process to the martingale problem $(\widetilde{\Theta},\varepsilon_{X_t(\omega)},\widetilde{b},\widetilde{c},\widetilde{F},\widetilde{K})^M$. Therefore $(P^{|\mathcal{F}_t}(\omega))^Y = P^{Y|\mathcal{F}_t}(\omega) = P^{(X_{t+s})_{s\in\mathbb{R}_+}|\mathcal{F}_t}(\omega)$ is a solution-measure to this random martingale problem for P-almost any $\omega\in\Omega$. The proof works basically unchanged for $\sigma(X_u : u\in[0,t])$ instead of $\mathcal{F}_t$. □

PROOF OF THE REMARKS.

1. By Jacod (1979), (1.1), X is indistinguishable from an $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted process $\widetilde{X}$. It follows from Lemma 2.32 that $((\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{R}_+},P),\widetilde{X})$ is also a solution-process to the martingale problem $(\Theta,\eta,b,c,F,K)^M$. Moreover, it is easy to see that for any Blackwell space-valued, measurable mapping $Z:\Omega\to(E,\mathcal{E})$, any version of the regular conditional distribution $P^{Z|\mathcal{F}_t}$ is a version of $P^{Z|\mathcal{F}_t^P}$ as well. This shows the claim.

2. If the filtration is the canonical filtration of a càdlàg process X and $\mathcal{F}=\mathcal{F}_{\infty-}$, then $\mathcal{F}=\sigma(X)$, where X here means the mapping $\Omega\to(\mathbb{D}^d,\mathcal{D}^d)$. Using Theorem III.25 in Dellacherie & Meyer (1978), one easily verifies that if the σ-field $\mathcal{F}$ on Ω is generated by a Blackwell space-valued mapping X, then $(\Omega,\mathcal{F})$ is a Blackwell space as well. □

2.8 Existence and Uniqueness Theorems

We have motivated martingale problems as stochastic analogues of ordinary differential equations. In order for them to be useful in practice, we need some existence and uniqueness results. Statements of this kind are proved in this section. We have seen in Example 2 in Section 2.7 that ODE's can be interpreted as specific cases of martingale problems. Since existence and uniqueness results for ODE's usually rely on some kind of Lipschitz and growth conditions, we cannot hope for more for arbitrary martingale problems. The assumptions in the following theorem, which is explained in greater detail below, are exactly of this kind.

Theorem 2.37 *Let $(\Theta, \eta, b, c, F, K)^M$ be a martingale problem in $\mathbb{R}^d$. We make the following assumptions.*

Existence: *Suppose that $c = uu^\top$ for some $\mathcal{P}^d$-measurable mapping $u : \mathbb{D}^d \times \mathbb{R}_+ \to \mathbb{R}^{d\times d}$. Moreover, assume that F can be written as a sum $F = \kappa_1 + \kappa_2 + \kappa_3$ of transition kernels $\kappa_1, \kappa_2, \kappa_3$ from $(\mathbb{D}^d \times \mathbb{R}_+, \mathcal{P}^d)$ into $(\mathbb{R}^d, \mathcal{B}^d)$ with the following properties.*

1. *There is some $M_1 \in \mathbb{R}_+$ such that for any $(\bar{\omega}, t) \in \mathbb{D}^d \times \mathbb{R}_+$ we have $\kappa_1((\bar{\omega}, t), \mathbb{R}^d) \le M_1$.*
2. *For any $\beta \in \mathbb{R}_+$ there is an increasing mapping $M_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $(\bar{\omega}, t) \in \mathbb{D}^d \times \mathbb{R}_+$ with $\|\bar{\omega}\|_t^* := \sup\{|\bar{\omega}_s| : s \in [0, t]\} \le \beta$ we have $\kappa_2((\bar{\omega}, t), \mathbb{R}^d) \le M_2(t)$.*
3. *There exist a finite measure Γ on the p-dimensional sphere $S^p := \{x \in \mathbb{R}^{p+1} : |x| = 1\}$ (for some $p \in \mathbb{N}$), a $(\mathcal{P}^d \otimes \mathcal{B}(S^p) \otimes \mathcal{B}_+)$-measurable mapping $g : \mathbb{D}^d \times \mathbb{R}_+ \times S^p \times \mathbb{R}_+ \to \mathbb{R}^d$ with $g(\cdot, \cdot, \cdot, 0) = 0$ and a $(\mathcal{P}^d \otimes \mathcal{B}(S^p) \otimes \mathcal{B}_+)$-measurable mapping $\rho : \mathbb{D}^d \times \mathbb{R}_+ \times S^p \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $(\bar{\omega}, t) \in \mathbb{D}^d \times \mathbb{R}_+$ we have*
$$\kappa_3((\bar{\omega}, t), \bullet) = \mu((\bar{\omega}, t), \cdot)^{g(\bar{\omega}, t, \cdot, \cdot)}(\bullet)$$
(i.e. the measure $\kappa_3((\bar{\omega}, t), \cdot)$ is the image of the measure $\mu((\bar{\omega}, t), \cdot)$ under the mapping $(n, r) \mapsto g(\bar{\omega}, t, n, r)$), where the transition kernel μ from $(\mathbb{D}^d \times \mathbb{R}^d, \mathcal{P}^d)$ into $(S^p \times \mathbb{R}_+, \mathcal{B}(S^p) \otimes \mathcal{B}^d)$ is defined by
$$\mu((\bar{\omega}, t), d(n, r)) := \rho(\bar{\omega}, t, n, r)\, dr\, \Gamma(dn).$$

Linear growth conditions: *There are measurable mappings $M_3, M_4, M_5, M_6 : \mathbb{R}_+ \to \mathbb{R}_+$ with $\int_0^t M_i(s)\, ds < \infty$ for $i = 3, 4, 5, 6$ and any $t \in \mathbb{R}_+$ such that for any $(\bar{\omega}, t) \in \mathbb{D}^d \times \mathbb{R}_+$ we have*
$$\left| b_t(\bar{\omega}) - \int x\, (\kappa_1 + \kappa_2)((\bar{\omega}, t), dx) \right| \le M_3(t)(1 + \|\bar{\omega}\|_t^*),$$
$$\sum_{i,j=1}^{d} |u_t^{ij}(\bar{\omega})|^2 \le M_4(t)(1 + \|\bar{\omega}\|_t^*)^2,$$
$$\int |x|\, \kappa_2((\bar{\omega}, t), dx) \le M_5(t)(1 + \|\bar{\omega}\|_t^*),$$
$$\int |x|^2\, \kappa_3((\bar{\omega}, t), dx) \le M_6(t)(1 + \|\bar{\omega}\|_t^*)^2,$$
where $\|\bar{\omega}\|_t^* := \sup\{|\bar{\omega}_s| : s \in [0, t]\}$.

Regularity conditions on ρ: *There is a $(\mathcal{B}_+ \otimes \mathcal{B}(S^p))$-measurable mapping $R : \mathbb{R}_+ \times S^p \to \overline{\mathbb{R}}_+$ such that for any $(t, n) \in \mathbb{R}_+ \times S^p$ we have*

1. *$\rho(\bar{\omega}, t, n, r) \ne 0$ if and only if $r \in (0, R(t, n)] \setminus \{\infty\}$,*

2. *$\int_r^{R(t,n)} \rho(\bar{\omega}, t, n, \widetilde{r})\, d\widetilde{r} < \infty$ for any $\bar{\omega} \in \mathbb{D}^d$, $r > 0$,*
3. *The mapping $\rho(\cdot, t, n, \cdot) : \mathbb{D}_t^d \times ((0, R(t,n)] \setminus \{\infty\}) \to \mathbb{R}$ is continuous and in its first argument Fréchet-differentiable with continuous derivative $D_1\rho(\cdot, t, n, \cdot) : \mathbb{D}_t^d \times ((0, R(t,n)] \setminus \{\infty\}) \to \mathscr{L}(\mathbb{D}_t^d, \mathbb{R})$.*
4. *For any $\beta \in \mathbb{R}_+$, there exists a measurable mapping $M_7 : \mathbb{R}_+ \to \mathbb{R}_+$ with $\int_r^\infty M_7(\widetilde{r})\, d\widetilde{r} < \infty$ for any $r > 0$ and such that*
$$\|D_1\rho(\bar{\omega}, t, n, r)\| \le M_7(r) \tag{2.13}$$
for any $\bar{\omega} \in \mathbb{D}_t^d$ with $\|\bar{\omega}\|_t^* \le \beta$ and any $r \in (0, R(t,n))$.

(Observe that by predictability the mapping $\bar{\omega} \mapsto \rho(\bar{\omega}, t, n, r)$ is $\mathcal{D}_{t-}^d$-measurable and thus depends only on $(\bar{\omega}_s)_{s\in[0,t]}$. Therefore, we may identify the mapping $\rho(\cdot, t, n, \cdot) : \mathbb{D}^d \times \mathbb{R}_+ \to \mathbb{R}$ with a mapping $\mathbb{D}_t^d \times \mathbb{R}_+ \to \mathbb{R}$ (which we call again ρ), where $\mathbb{D}_t^d := \{\alpha : [0,t] \to \mathbb{R}^d : \alpha \text{ càdlàg}\}$. By endowing $\mathbb{D}_t^d$ with the sup-norm $\|\cdot\|_t^*$ (i.e. $\|\alpha\|_t^* := \sup\{|\alpha_s| : s \in [0,t]\}$), we obtain a Banach space and hence continuity, Fréchet-differentiability etc. as above make sense. If we write $D_1\rho(\bar{\omega}, t, n, r)$ for $\bar{\omega} \in \mathbb{D}^d$ in the sequel, then this is to be understood as $D_1\rho((\bar{\omega}_s)_{s\in[0,t]}, t, n, r)$.)

Local Lipschitz conditions: *For any $\beta \in \mathbb{R}_+$ there exist measurable mappings $M_8 : \mathbb{R}_+ \times S^p \times \mathbb{R}_+ \to \overline{\mathbb{R}}_+$, $M_9 : \mathbb{R}_+ \times S^p \times \mathbb{R}_+ \to \overline{\mathbb{R}}_+$, $L_1 : \mathbb{R}_+ \to \mathbb{R}_+$, $L_2 : \mathbb{R}_+ \to \mathbb{R}_+$, $L_3 : \mathbb{R}_+ \times S^p \to \mathbb{R}_+$, $L_4 : \mathbb{R}_+ \times S^p \times \mathbb{R}_+ \to \mathbb{R}_+$ with $L_4(\cdot,\cdot,0) = 0$ such that for any $t \in \mathbb{R}_+$ the following conditions hold.*

1. *For any $n \in S^p$, $r \in \mathbb{R}_+$ and any $\bar{\omega} \in \mathbb{D}^d$ with $\|\bar{\omega}\|_t^* \le \beta$ we have*
$$\frac{\left(\int_r^{R(t,n)} \|D_1\rho(\bar{\omega}, t, n, \widetilde{r})\|\, d\widetilde{r}\right)^2}{\rho(\bar{\omega}, t, n, r)} \le M_8(t,n,r),$$
$$\rho(\bar{\omega}, t, n, r) \le M_9(t,n,r),$$
where we set $0/0 := 0$.
2.
$$\int_0^t L_i(s)\, ds < \infty \ \text{ for } i = 1, 2,$$
$$\int_0^t \int L_3^2(s,n) \int_{\mathbb{R}_+} M_8(s,n,r)\, dr\, \Gamma(dn)\, ds < \infty,$$
$$\int_0^t \int \int_{\mathbb{R}_+} L_4^2(s,n,r) M_9(s,n,r)\, dr\, \Gamma(dn)\, ds < \infty.$$
3. *For any $\omega, \bar{\omega} \in \mathbb{D}^d$ with $\|\omega\|_t^* \le \beta$, $\|\bar{\omega}\|_t^* \le \beta$, any $n \in S^p$ and any $r, \bar{r} \in \mathbb{R}_+$ we have*
$$\begin{aligned}\Big| b_t(\omega) - \int x\,(\kappa_1+\kappa_2)((\omega,t),dx) - b_t(\bar{\omega}) + \int x\,(\kappa_1+\kappa_2)((\bar{\omega},t),dx) \Big| \\ \le\ L_1(t)\|\omega - \bar{\omega}\|_t^*,\end{aligned}$$

$$\sum_{i,j=1}^{d} |u_t^{ij}(\omega) - u_t^{ij}(\bar{\omega})|^2 \leq L_2(t)(\|\omega - \bar{\omega}\|_t^*)^2,$$
$$|g(\bar{\omega}, t, n, r) - g(\bar{\omega}, t, n, \bar{r})| \leq L_3(t, n)|r - \bar{r}|,$$
$$|g(\omega, t, n, r) - g(\bar{\omega}, t, n, r)| \leq L_4(t, n, r)\|\omega - \bar{\omega}\|_t^*.$$

Under all these conditions there exists a unique solution-measure to the given martingale problem.

Before we turn to some corollaries we want to explain the assumptions of Theorem 2.37. For the drift and diffusion coefficients b, c (or b, u) we assume local Lipschitz and linear growth conditions, as is well-known from results on SDE's. There is no condition on K. The jump transition kernel F is decomposed into three parts. The first one (κ_1) has to be of bounded jump intensity M_1, but faces no additional condition. Another part (κ_2) is of locally bounded jump intensity M_2 and must fulfill a growth, but no Lipschitz condition. Finally, we have a third part κ_3, which is of more complicated structure and only comes into play if the local jump intensity is infinite. The kernel κ_3 is the image of another kernel μ under some mapping g that has been introduced to add some flexibility, but which can often be chosen very simple (e.g. $g(\bar{\omega}, t, \pm 1, r) := \pm r$ in the case $d = 1$, $p = 0$). The radial part of the measure μ is assumed to have a density ρ that is in some sense continuously differentiable, which is a hidden local Lipschitz condition. The mapping g also has to fulfill Lipschitz conditions in the first and the fourth argument. The role of g and ρ may become clearer in the two examples below.

The proof of Theorem 2.37 basically works by transforming the martingale problem into a stochastic differential equation with respect to a Wiener process and a fixed Poisson random measure, so that existence and uniqueness results for SDE's can be applied. This transformation has to be carried out sufficiently smoothly. Otherwise, the Lipschitz conditions on the coefficients of the martingale problem do not carry over to the corresponding SDE. This is difficult for the jump part. In a sense, the key idea underlying this part of the proof is an application of the simple result that for any $\lambda|_{[0,1]}$-distributed random variable X and any probability measure Q on $(\mathbb{R}, \mathcal{B})$, the random variable $F_Q^{-1}(X)$ is Q-distributed, where F_Q^{-1} here denotes the pseudo inverse of the cumulative distribution function F_Q of Q.

Corollary 2.38 *For any discrete-time martingale problem $(\Theta, \eta, 0, 0, 0, K)^M$ in $\mathbb{R}^d$ there exists a unique solution-measure.*

The following corollary considers the case that the process is constant between its jumps and the jump intensity is bounded.

Corollary 2.39 *Let $(\Theta, \eta, b, 0, F, K)^M$ be a martingale problem (in $\mathbb{R}^d$) and $M \in \mathbb{R}_+$ such that for any $(\bar{\omega}, t) \in \mathbb{D}^d \times \mathbb{R}_+$ we have $F((\bar{\omega}, t), \mathbb{R}^d) \leq M$ and $b_t(\bar{\omega}) = \int x\, F((\bar{\omega}, t), dx)$. Then there exists a unique solution-measure.*

If the situation is basically as in Corollary 2.39, but the jump intensity is only locally bounded, one has to add a growth condition.

Corollary 2.40 *Let $(\Theta, \eta, b, 0, F, K)^M$ be a martingale problem in $\mathbb{R}^d$ with $b_t(\bar{\omega}) = \int x\, F((\bar{\omega}, t), dx)$ for any $(\bar{\omega}, t) \in \mathbb{D}^d \times \mathbb{R}_+$. Suppose that for any $\beta \in \mathbb{R}_+$ there is an increasing mapping $M_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $(\bar{\omega}, t) \in \mathbb{D}^d \times \mathbb{R}_+$ with $\|\bar{\omega}\|_t^* \leq \beta$ we have $F((\bar{\omega}, t), \mathbb{R}^d) \leq M_1(t)$. Moreover, assume that there exists a measurable mapping $M_2 : \mathbb{R}_+ \to \mathbb{R}_+$ with $\int_0^t M_2(s)\, ds < \infty$ for any $t \in \mathbb{R}_+$ such that for any $(\bar{\omega}, t) \in \mathbb{D}^d \times \mathbb{R}_+$ we have $\int |x|\, F((\bar{\omega}, t), dx) \leq M_2(t)(1 + \|\bar{\omega}\|_t^*)$. Then there exists a unique solution-measure.*

If there are no jumps present (or only at fixed times), one may apply the following

Corollary 2.41 *Let $(\Theta, \eta, b, c, 0, K)^M$ be a martingale problem in $\mathbb{R}^d$. Suppose that $c = uu^\top$ for some $\mathcal{P}^d$-measurable mapping $u : \mathbb{D}^d \times \mathbb{R}_+ \to \mathbb{R}^{d \times d}$. Moreover, we assume*

Linear growth conditions: *There are measurable mappings $M_1, M_2 : \mathbb{R}_+ \to \mathbb{R}_+$ with $\int_0^t M_i(s)\, ds < \infty$ for $i = 1, 2$ and any $t \in \mathbb{R}_+$ such that for any $(\bar{\omega}, t) \in \mathbb{D}^d \times \mathbb{R}_+$ we have*

$$|b_t(\bar{\omega})| \leq M_1(t)(1 + \|\bar{\omega}\|_t^*),$$

$$\sum_{i,j=1}^d |u_t^{ij}(\bar{\omega})|^2 \leq M_2(t)(1 + \|\bar{\omega}\|_t^*)^2.$$

Local Lipschitz conditions: *For any $\beta \in \mathbb{R}_+$ there exist measurable mappings $L_1 : \mathbb{R}_+ \to \mathbb{R}_+$, $L_2 : \mathbb{R}_+ \to \mathbb{R}_+$ with $\int_0^t L_i(s)\, ds < \infty$ for $i = 1, 2$ and any $t \in \mathbb{R}_+$ such that the following condition holds. For any $t \in \mathbb{R}_+$ and any $\omega, \bar{\omega} \in \mathbb{D}^d$ with $\|\omega\|_t^* \leq \beta$, $\|\bar{\omega}\|_t^* \leq \beta$ we have*

$$|b_t(\omega) - b_t(\bar{\omega})| \leq L_1(t)\|\omega - \bar{\omega}\|_t^*,$$

$$\sum_{i,j=1}^d |u_t^{ij}(\omega) - u_t^{ij}(\bar{\omega})|^2 \leq L_2(t)(\|\omega - \bar{\omega}\|_t^*)^2.$$

Then there exists a unique solution-measure.

The following corollary applies to quite general situations where the jump intensity is locally bounded. Observe that no Lipschitz condition on the jumps is needed here.

Corollary 2.42 *Let $(\Theta, \eta, b, c, F, K)^M$ be a martingale problem in $\mathbb{R}^d$. We make the following assumptions.*

Existence: *Suppose that $c = uu^\top$ for some $\mathcal{P}^d$-measurable mapping $u : \mathbb{D}^d \times \mathbb{R}_+ \to \mathbb{R}^{d \times d}$. Moreover, assume that for any $\beta \in \mathbb{R}_+$ there is an increasing mapping $M_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $(\bar{\omega}, t) \in \mathbb{D}^d \times \mathbb{R}_+$ with $\|\bar{\omega}\|_t^* \leq \beta$ we have $F((\bar{\omega}, t), \mathbb{R}^d) \leq M_1(t)$.*

Linear growth conditions: *There are measurable mappings $M_2, M_3, M_4 : \mathbb{R}_+ \to \mathbb{R}_+$ with $\int_0^t M_i(s)\, ds < \infty$ for $i = 2, 3, 4$ and any $t \in \mathbb{R}_+$ such that for any $(\bar{\omega}, t) \in \mathbb{D}^d \times \mathbb{R}_+$ we have*

$$\left| b_t(\bar{\omega}) - \int x\, F((\bar{\omega}, t), dx) \right| \leq M_2(t)(1 + \|\bar{\omega}\|_t^*),$$

$$\sum_{i,j=1}^{d} |u_t^{ij}(\bar\omega)|^2 \leq M_3(t)(1+\|\bar\omega\|_t^*)^2,$$

$$\int |x|\, F((\bar\omega,t),dx)| \leq M_4(t)(1+\|\bar\omega\|_t^*).$$

Local Lipschitz conditions: *For any $\beta \in \mathbb{R}_+$ there exist measurable mappings $L_1 : \mathbb{R}_+ \to \mathbb{R}_+$, $L_2 : \mathbb{R}_+ \to \mathbb{R}_+$ with $\int_0^t L_i(s)\,ds < \infty$ for $i = 1,2$ and any $t \in \mathbb{R}_+$ such that the following condition holds. For any $t \in \mathbb{R}_+$ and any $\omega, \bar\omega \in \mathbb{D}^d$ with $\|\omega\|_t^* \leq \beta$, $\|\bar\omega\|_t^* \leq \beta$ we have*

$$\left| b_t(\omega) - \int x\, F((\omega,t),dx) - b_t(\bar\omega) + \int x\, F((\bar\omega,t),dx) \right| \leq L_1(t)\|\omega-\bar\omega\|_t^*,$$

$$\sum_{i,j=1}^{d} |u_t^{ij}(\omega) - u_t^{ij}(\bar\omega)|^2 \leq L_2(t)(\|\omega-\bar\omega\|_t^*)^2.$$

Then there exists a unique solution-measure to the given martingale problem.

The following last corollary leads to processes with independent increments (cf. Remark 5 below).

Corollary 2.43 *Let $(\Theta,\eta,b,c,F,K)^M$ be a martingale problem in $\mathbb{R}^d$ such that b,c,F,K are deterministic (i.e. they do not depend on $\bar\omega \in \mathbb{D}^d$). Then there exists a unique solution-measure.*

Remarks.

1. By Corollary 2.38 any discrete-time martingale problem (i.e. $b = 0$, $c = 0$, $F = 0$), e.g. for Markov chains and ARMA(p,q) time series (cf. Examples 1 and 3 in Section 2.7) has a unique solution-measure. But one should be aware that in the case of time series models this tells us nothing about the existence of *stationary* solutions.

2. For ODE's (cf. Example 2 in Section 2.7) we have, by Corollary 2.41 in the case $c = 0$, $\Theta = \varnothing$, $K = 0$, existence of a unique solution if f fulfills local Lipschitz and linear growth conditions. This is in line with Picard-Lindelöf type theorems in real analysis.

3. For diffusions (cf. Example 4 in Section 2.7) we obtain (by Corollary 2.41) the usual existence and uniqueness results under local Lipschitz and linear growth conditions. Note that there is a much stronger result by Stroock and Varadhan in the case of non-vanishing diffusion coefficient (cf. JS, Theorem III.2.34).

4. In a martingale problem of the form $(\Theta,\varepsilon_0,(\int x\,F_t(dx))_{t\in\mathbb{R}_+},0,F,K)^M$ there is no diffusion and no real drift part. The term $(\int x\,F_t(dx))_{t\in\mathbb{R}_+}$ just means that the jumps are not compensated as it would be done in the case $b = 0$. Such a martingale problem corresponds to a *multivariate point process*. By Corollaries 2.39 and 2.40 we know that a unique solution-measure exists if F has globally bounded jump intensity or if it has locally bounded jump intensity and meets an additional growth condition.

5. By Corollary 2.43 martingale problems with deterministic coefficients have a unique solution-measure. If $\eta = \varepsilon_0$, then the solution-processes are processes with independent (and in the case of time-independent coefficients also stationary) increments.

If the jump measure F_t is infinite, one cannot apply Corollaries 2.38 to 2.42, but has to fall back directly on Theorem 2.37. We present two examples where the role of the mappings g and ρ becomes apparent in the proof.

Examples. Both of the following martingale problems may be considered as stock price models, where the same events not only change the return process X_t^1 but also increase the volatility X_t^2 of the stock. The difference between the two models is that in Example 1 volatility can be interpreted as an *arrival rate* of price changes, whereas in Example 2 it is a measure of the average *size* of stock price jumps.

1. Fix parameters $\mu \in \mathbb{R}$, $\alpha, \beta, \sigma, x_0 \in \mathbb{R}_+^*$, $x_1 \in (\alpha, \infty)$. Let φ be the measure on $(\mathbb{R}, \mathcal{B})$ with λ-density $x \mapsto \frac{1}{|x|} e^{-|x|}$. For any $((\bar{\omega}^1, \bar{\omega}^2), t) \in \mathbb{D}^2 \times \mathbb{R}_+$ we define the measure $F(((\bar{\omega}^1, \bar{\omega}^2), t), \cdot)$ as the image of $h(\bar{\omega}_t^2)\varphi$ under the mapping $\mathbb{R} \to \mathbb{R}^2$, $x \mapsto (\sigma x, |x|)$, where $h : \mathbb{R} \to \mathbb{R}$ is a C^1-function such that $h(x) \geq \alpha/2$ and $h'(x) \in [0, 1]$ for any $x \in \mathbb{R}$, and $h(x) = x$ for any $x \in (\alpha, \infty)$. Moreover, define the drift by $b_t(\bar{\omega}) := (\mu, -(\bar{\omega}_t^2 - \alpha)\beta + \int x_2 F((\bar{\omega}, t), d(x_1, x_2)))$ for any $(\bar{\omega}, t) = ((\bar{\omega}^1, \bar{\omega}^2), t) \in \mathbb{D}^2 \times \mathbb{R}_+$. Then the martingale problem $(\varnothing, \varepsilon_{(x_0, x_1)}, b, 0, F, 0)^M$ in $\mathbb{R}^2$ has a unique solution-measure.

   For a solution-process (X^1, X^2), we interpret X^2 as a volatility. It increases due to positive jumps (which also affect X^1) and is pulled back towards the lower bound α by the drift term $-(\bar{\omega}_t^2 - \alpha)\beta$. The term $\int x_2 F((\bar{\omega}, t), d(x_1, x_2))$ just offsets the compensation of the jumps. Therefore, X^2 always stays above α. The function h (which is the identity for values above α) is only introduced to make the martingale problem meet the conditions of Theorem 2.37.

2. Fix parameters $\mu \in \mathbb{R}$, $\alpha, \beta, \sigma, x_0 \in \mathbb{R}_+^*$, $x_1 \in (\alpha, \infty)$, and let φ, h be as in the previous example. For any $((\bar{\omega}^1, \bar{\omega}^2), t) \in \mathbb{D}^2 \times \mathbb{R}_+$ we define the measure $F(((\bar{\omega}^1, \bar{\omega}^2), t), \cdot)$ as the image of φ under the mapping $\mathbb{R} \to \mathbb{R}^2$, $x \mapsto (\sigma h(\bar{\omega}_t^2)x, h(\bar{\omega}_t^2)|x|)$. Moreover, define b as in Example 1. Then the martingale problem $(\varnothing, \varepsilon_{(x_0, x_1)}, b, 0, F, 0)^M$ in $\mathbb{R}^2$ has a unique solution-measure. As in the previous example, the function h has been put in to allow for the application of Theorem 2.37.

Of course the existence of a unique solution can only be a first step if you want to apply martingale problems to real-world phenomena. You also need efficient numerical algorithms for explicit calculations. More specifically, one may ask for procedures yielding $E(f(X))$ if X is a solution-process to a given martingale problem and $f : \mathbb{D}^d \to \mathbb{R}$ some continuous mapping. Whereas there is extensive literature for martingale problems without jumps (i.e. for SDE's driven by a Wiener process, cf. e.g. Kloeden & Platen (1992)), there is less dealing with the jump case. Since discontinuous models are of equal interest especially in finance applications, we hope to address this question in future research.

Proofs

PROOF OF THEOREM 2.37. The proof will be broken down into many steps.

Definition 2.44 A *stochastic differential equation (SDE)* $(\Theta, \eta, a, u, w, Q)^{SDE}$ is given by

- a discrete set $\Theta \subset \mathbb{R}_+^*$,
- a probability measure η on $(\mathbb{R}^d, \mathcal{B}^d)$,
- a $\mathcal{P}^d$-measurable mapping $a : \mathbb{D}^d \times \mathbb{R}_+ \to \mathbb{R}^d$,
- a $\mathcal{P}^d$-measurable mapping $u : \mathbb{D}^d \times \mathbb{R}_+ \to \mathbb{R}^{d\times d}$,
- a Lusin space $E = \dot{\bigcup}_{i=1}^4 E_i$ with its Borel-σ-field $\mathcal{E}$, where E_1, E_2, E_3, E_4 are open subsets of E (cf. Jacod (1979), p. 66),
- a $(\mathcal{P}^d \otimes \mathcal{E})$-measurable mapping $w : \mathbb{D}^d \times \mathbb{R}_+ \times E \to \mathbb{R}^d$,
- a measure Q on E such that Q_1, Q_2 are σ-finite, Q_3 is finite, and $Q_4(E) \leq 1$, where $Q_i(\cdot) := Q(\cdot \cap E_i)$ for $i = 1, 2, 3, 4$.

Remark. By the previous definition we refer to the SDE

$$dX_t = a_t(X)\,dt + u_t(X)\cdot dW_t + \int_E w(X,t,x)(p_1 - q_1)(dt,dx) + \sum_{i=2}^4 \int_E w(X,t,x)\,p_i(dt,dx), \tag{2.14}$$

where p_i is an extended Poisson random measure on $\mathbb{R}_+ \times E$ with compensator $q_i(dt,dx) := Q_i(dx)dt$ for $i = 1, 2, 3$ and $q_4(dt,dx) := Q_4(dx)\sum_{s\in\Theta} \varepsilon_s(dt)$.

Definition 2.45 1. A tupel $((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P, W, p), X)$ is called *solution-process to the SDE (2.14) on* $[0,T]$ (or, more exactly, to the SDE $(\Theta, \eta, a, u, w, Q)^{SDE}$ on $[0,T]$) if

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ is a filtered probability space,
- W is a $\mathbb{R}^d$-valued standard Wiener process (on that space),
- p is an extended Poisson measure on $\mathbb{R}_+ \times E$ with intensity measure $q = \sum_{i=1}^4 q_i$, where $q_i(dt,dx) := Q_i(dx)\,dt$ for $i = 1, 2, 3$ and $q_4(dt,dx) := Q_4(dx)$ $(\sum_{s\in\Theta} \varepsilon_s)(dt)$ (Moreover, define the extended Poisson measures p_i on $\mathbb{R}_+ \times E$ by $p_i(dt,dx) = 1_{E_i}(x)\,p(dt,dx)$ for $i = 1, 2, 3, 4$),
- X is a $\mathbb{R}^d$-valued semimartingale on the above space with $P^{X_0} = \eta$,
- $\int_0^t |a_s(X)|\,ds < \infty$ P-almost surely for any $t \in \mathbb{R}_+$,
- $\int_0^t |u_s^{ij}(X)|^2\,ds < \infty$ P-almost surely for any $t \in \mathbb{R}_+$ and any $i, j \in \{1, \ldots, d\}$,
- $\int_0^t \int |w(X,s,x)|^2\,Q_1(dx)\,ds < \infty$ P-almost surely for any $t \in \mathbb{R}_+$,
- $\int_{[0,t]\times E} |w(X,s,x)|\,p_2(dx,ds) < \infty$ P-almost surely for any $t \in \mathbb{R}_+$,

- T is a stopping time,
- X is a solution to Equation (2.14) on $[0,T]$ (i.e. we have

$$\begin{aligned} X_{T\wedge t} &= X_0 + \int_0^{T\wedge t} a_s(X)\,ds + \sum_{j=1}^d \int_0^{T\wedge t} u_s^{\cdot j}(X)\,dW_s^j \\ &\quad + \int_{[0,T\wedge t]\times E} w(X,s,x)\,(p_1 - q_1)(ds,dx) \\ &\quad + \sum_{i=2}^4 \int_{[0,T\wedge t]\times E} w(X,s,x)\,p_i(ds,dx) \end{aligned} \tag{2.15}$$

for any $t \in \mathbb{R}_+$ P-almost surely).

2. The tupel is called *solution-process to the SDE* if $T = \infty$ P-almost surely.
3. The law P^X on $(\mathbb{D}^d, \mathcal{D}^d)$ of a solution-process (on $\mathbb{R}_+$) of the SDE is called *solution-measure* to the SDE.

Let $(\Theta, \eta, a, u, w, Q)^{SDE}$ be a SDE in $\mathbb{R}^d$. We introduce three kinds of conditions.

Integrability conditions (I) For any $\bar{\omega} \in \mathbb{D}^d$ and any $t \in \mathbb{R}_+$ we have

$$\int_0^t |a_s(\bar{\omega})|\,ds < \infty,$$

$$\sum_{i,j=1}^d \int_0^t |u_s^{ij}(\bar{\omega})|^2\,ds < \infty,$$

$$\int_0^t \int |w(\bar{\omega},s,x)|^2\,Q_1(dx)\,ds < \infty.$$

Lipschitz conditions (L) For any $\beta \in \mathbb{R}_+$ there exists a measurable mapping $L : \mathbb{R}_+ \to \mathbb{R}_+$ with $\int_0^t L(s)\,ds < \infty$ for any $t \in \mathbb{R}_+$, such that for any $t \in \mathbb{R}_+$ and any $\omega, \bar{\omega} \in \mathbb{D}^d$ with $\|\omega\|_t^* \le \beta$, $\|\bar{\omega}\|_t^* \le \beta$, we have

$$|a_t(\omega) - a_t(\bar{\omega})| \le L(t)\|\omega - \bar{\omega}\|_t^*,$$

$$\sum_{i,j=1}^d |u_t^{ij}(\omega) - u_t^{ij}(\bar{\omega})|^2 \le L(t)(\|\omega - \bar{\omega}\|_t^*)^2,$$

$$\int_E |w(\omega,t,x) - w(\bar{\omega},t,x)|^2\,Q_1(dx) \le L(t)(\|\omega - \bar{\omega}\|_t^*)^2.$$

Moreover, for any $\beta \in \mathbb{R}_+$ there are increasing mappings $M_2 : \mathbb{R}_+ \to \mathbb{R}_+$ and $H : \mathbb{R}_+ \to \mathcal{E}$ (i.e. $H(s) \subset H(t)$ for $s \le t$) with $Q_2(H(t)) \le M_2(t)$ for any $t \in \mathbb{R}_+$ and such that for any $t \in \mathbb{R}_+$, any $\bar{\omega} \in \mathbb{D}^d$ with $\|\bar{\omega}\|_t^* \le \beta$ and any $x \in E_2 \setminus H(t)$, we have $w(\bar{\omega},t,x) = 0$.

Growth conditions (G) There exists a measurable mapping $M : \mathbb{R}_+ \to \mathbb{R}_+$ with $\int_0^t M(s)\, ds < \infty$ for any $t \in \mathbb{R}_+$ such that for any $(\bar{\omega}, t) \in \mathbb{D}^d \times \mathbb{R}_+$ we have

$$|a_t(\bar{\omega})| \leq M(t)(1 + \|\bar{\omega}\|_t^*),$$

$$\sum_{i,j=1}^d |u_t^{ij}|^2 \leq M(t)(1 + \|\bar{\omega}\|_t^*)^2,$$

$$\int |w(\bar{\omega}, t, x)|^2\, Q_1(dx) \leq M(t)(1 + \|\bar{\omega}\|_t^*)^2,$$

$$\int |w(\bar{\omega}, t, x)|\, Q_2(dx) \leq M(t)(1 + \|\bar{\omega}\|_t^*),$$

Remark. The growth conditions (G) imply the integrability conditions (I).

Lemma 2.46 *Assume that the Lipschitz conditions (L) hold. Moreover, suppose that the processes X and $\widetilde{X}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P, W, p)$ are solution-processes to the SDE (2.14) on $[0, T]$, and that $X_0 = \widetilde{X}_0$ P-almost surely. Then X^T and $\widetilde{X}^T$ are indistinguishable.*

Proposition 2.47 *Let M be a locally square-integrable martingale with $M_0 = 0$ and T a stopping time. Then there is a constant $c \in \mathbb{R}_+$, independent of M and T, such that*

$$\|M^T\|_{S^1} \leq c\|(\langle M, M\rangle_T)^{\frac{1}{2}}\|_{L^1}.$$

PROOF. Since M^T is a local martingale, there is, by Jacod (1979), (2.34), a constant $c/2 \in \mathbb{R}_+$ such that $\|M^T\|_{S^1} \leq c/2\|([M^T, M^T]_\infty)^{\frac{1}{2}}\|_{L^1}$. Since $[M^T, M^T] \in \mathscr{A}^+_{\text{loc}}$ (cf. JS, I.4.50c), it follows from Lenglart et al. (1980), Théorème 4.1 and Dellacherie & Meyer (1982), VII.41.3 that $E(([M^T, M^T]_\infty)^{\frac{1}{2}}) \leq 2E((\langle M^T, M^T\rangle_\infty)^{\frac{1}{2}}) = 2E((\langle M, M\rangle_\infty^T)^{\frac{1}{2}})$, which yields the claim. □

PROOF OF LEMMA 2.46. Define the stopping time $S := \inf\{t \in \mathbb{R}_+ : X_t \neq \widetilde{X}_t\}$. By definition we have that $X^{S-} = \widetilde{X}^{S-}$. By predictability $\bar{\omega} \mapsto a_t(\bar{\omega})$, $\bar{\omega} \mapsto u_t(\bar{\omega})$, $\bar{\omega} \mapsto w(\bar{\omega}, t, x)$ are $\mathcal{D}^d_{t-}$-measurable mappings for any $t \in \mathbb{R}_+$ (cf. JS, I.2.4a). Since $\mathcal{D}^d_{t-}$ is generated by the projections strictly before t, it follows that $a_t(\bar{\omega})$ etc. depend only on $\bar{\omega}|_{[0,t)}$. Hence, we have $a_{t\wedge S}(X) = a_{t\wedge S}(\widetilde{X})$, $u_{t\wedge S}(X) = u_{t\wedge S}(\widetilde{X})$, $w(X, t\wedge S, x) = w(\widetilde{X}, t\wedge S, x)$ for any $t \in \mathbb{R}_+$, $x \in E$. By Equation (2.14) this implies $X^{T\wedge S} = \widetilde{X}^{T\wedge S}$. Assume now that $P(S < T) > 0$. Then one can find $\beta \in \mathbb{N}$, $N \in \mathbb{N}$ such that $P(\|X^S\|_\infty^* \leq \beta - 1, S < N, S < T) \geq \frac{1}{2}P(S < T)$. Now choose L and H as in the Lipschitz conditions (L) relative to this β. We define

$$\begin{aligned} R \ := \ T \wedge \Big(S \vee \inf \Big\{ & t > S : (|X_t| \vee |\widetilde{X}_t|) > \beta \text{ or } \int_S^t L(s)\, ds > \frac{1}{4} + \frac{1}{16c^2(d^2+d)^2} \\ & \text{or } p_2(\{t\} \times H(N)) + p_3(\{t\} \times E) > 0 \text{ or } t \in \Theta \cup [N, \infty) \Big\}\Big). \end{aligned}$$

Observe that, since $q_2([0,t]\times H(N))<\infty$ and $q_3([0,t]\times E)<\infty$ for any $t\in\mathbb{R}_+$, there are P-almost surely only finitely many t in any compact interval such that $p_2(\{t\}\times H(N))+p_3(\{t\}\times E)>0$ (cf. JS, II.4.10). Therefore, R is a stopping time with $R\geq S$ P-almost surely and $P(R>S)>0$. By Equation (2.15), the triangular inequality, Proposition 2.47, JS, II.1.33a and (L), we have

$$\begin{aligned}
\|X^{R-}-\widetilde{X}^{R-}\|_{S^1} &= \Big\| \Big(\int_{[0,\cdot)\times E} (w(X,s,x)-w(\widetilde{X},s,x))\,(p_1-q_1)(ds,dx) \Big)^{R-} \\
&\quad + \int_0^{\cdot} \Big(a_s(X)-a_s(\widetilde{X}))\,ds\Big)^{R-} + \sum_{j=1}^{d} \Big(\int_0^{\cdot} (u_s^{\cdot j}(X)-u_s^{\cdot j}(\widetilde{X}))\,dW_s^j\Big)^{R-} \Big\|_{S^1} \\
&\leq \Big\| \Big(\int_0^{\cdot} (a_s(X)-a_s(\widetilde{X}))\,ds\Big)^{R} \Big\|_{S^1} + \sum_{i,j=1}^{d} \Big\| \Big(\int_0^{\cdot} (u_s^{ij}(X)-u_s^{ij}(\widetilde{X}))\,dW_s^j\Big)^{R} \Big\|_{S^1} \\
&\quad + \sum_{i=1}^{d} \Big\| \Big(\int_{[0,\cdot)\times E} (w^i(X,s,x)-w^i(\widetilde{X},s,x))\,(p_1-q_1)(ds,dx) \Big)^{R} \Big\|_{S^1} \\
&\leq \Big\| \int_0^R |a_s(X)-a_s(\widetilde{X})|\,ds \Big\|_{L^1} + c\sum_{i,j=1}^{d} \Big\| \Big(\int_0^R (u_s^{ij}(X)-u_s^{ij}(\widetilde{X}))^2\,ds\Big)^{\frac{1}{2}} \Big\|_{L^1} \\
&\quad + c\sum_{i=1}^{d} \Big\| \Big(\int_0^R \int (w^i(X,s,x)-w^i(\widetilde{X},s,x))^2\,Q_1(dx)\,ds\Big)^{\frac{1}{2}} \Big\|_{L^1} \\
&\leq E\Big(\|(X-\widetilde{X})^{R-}\|_\infty^* \Big(\int_S^R L(s)\,ds + c(d^2+d)\Big(\int_S^R L(s)\,ds\Big)^{\frac{1}{2}}\Big)\Big) \\
&\leq \frac{1}{2}\|X^{R-}-\widetilde{X}^{R-}\|_{S^1},
\end{aligned}$$

where $\|\cdot\|_{S^1}$ is defined in Definition A.8. This clearly is impossible, since $\|X^{R-}-\widetilde{X}^{R-}\|_{S^1}\leq 2\beta$ and $P(X^{R-}\neq \widetilde{X}^{R-})>0$. Therefore $P(S<T)=0$, and the claim follows. $\square$

Lemma 2.48 *Suppose that the integrability conditions (I) and the Lipschitz conditions (L) hold. Moreover, fix $\beta\in\mathbb{R}_+$ and a space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{R}_+},P,W,p)$ as in Definition 2.45. Assume that there exists a $\mathcal{F}_0$-measurable, $\mathbb{R}^d$-valued random variable X_0 with $P^{X_0}=\eta$. Then there is a solution-process X to SDE (2.14) (on that space) on $[0,T^{\beta,X}]$, where $T^{\beta,X}:=\inf\{t\in\mathbb{R}_+:\|X\|_t^*\geq\beta\}$.*

PROOF. We define a sequence of stopping times $(R_n)_{n\in\mathbb{N}}$ recursively by $R_0=0$ and

$$\begin{aligned}
R_{n+1} := \inf\{t>R_n : t\in\Theta \text{ or } t>R_n+1 \text{ or } \int_{R_n}^t L(s)\,ds > \frac{1}{4}+\frac{1}{16c^2(d^2+d)^2} \\
\text{or } p_2(\{t\}\times H(R_n+1))+p_3(\{t\}\times E)>0\}.
\end{aligned}$$

As in the proof of Lemma 2.46, we have for any $M\in\mathbb{R}_+$ that $p_2(\{t\}\times H(M))+p_3(\{t\}\times E)>0$ pathwise only for finitely many t in any compact interval. Hence, $R_n\uparrow\infty$ P-almost surely. By assumption there is a solution-process to the SDE on $[0, R_0]$. In order to prove the lemma, it suffices to show that, given a solution-process X on $[0, T^{\beta,X} \wedge R_n]$ for some $n \in \mathbb{N}$, there exists a process $\widetilde{X}$ that coincides with X on $[0, T^{\beta,X} \wedge R_n]$ and solves the SDE on $[0, T^{\beta,\widetilde{X}} \wedge R_{n+1}]$.

Fix $n \in \mathbb{N}$ and let X denote a solution on $[0, T^{\beta,X} \wedge R_n]$. We define an operator $F : S^1 \to S^1$ (cf. Definition A.8 in the appendix) by

$$\begin{aligned} F(Y)_t \quad := \quad & \Big(X^{R_n} + \int_0^\cdot 1_{[0,R_n]^C}(s) a_s(Y \triangle \beta)\, ds + \sum_{i,j=1}^d \int_0^\cdot 1_{[0,R_n]^C}(s) u_s^{ij}(Y \triangle \beta)\, dW_s^j \\ & + \int_{[0,\cdot]\times E} 1_{[0,R_n]^C}(s) w(Y \triangle \beta, s, x)\, (p_1 - q_1)(ds, dx) \Big)_t^{R_{n+1}-} \triangle \beta \end{aligned}$$

for any $Y \in S^1$, $t \in \mathbb{R}_+$, where we define $x \triangle \beta \in \mathbb{R}^d$ by $(x \triangle \beta)^i := (x^i \vee (-\beta)) \wedge \beta$ for any $x \in \mathbb{R}^d$, $i \in \{1, \dots, d\}$. Now let $Y, \widetilde{Y} \in S^1$. By basically the same calculation as in the proof of Lemma 2.46, we have $\|F(Y) - F(\widetilde{Y})\|_{S^1} \leq \frac{1}{2} \|(Y \triangle \beta)^{R_{n+1}-} - (\widetilde{Y} \triangle \beta)^{R_{n+1}-}\|_{S^1} \leq \frac{1}{2}\|Y - \widetilde{Y}\|_{S^1}$. Banach's fixed point theorem yields that there is a unique fixed point $Y \in S^1$ of F. Let $S := R_{n+1} \wedge T^{\beta,Y}$ and define the adapted, càdlàg process

$$\widetilde{X} := Y^{S-} + \sum_{i=1}^4 \int_E w(Y, S, x)\, p_i(\{S\} \times dx) 1_{[S,\infty)}.$$

By the fixed point property of Y we have that $Y^{R_n} = F(Y)^{R_n} = X^{R_n} \triangle \beta$. Hence, Y, X are indistinguishable on $[0, R_n] \cap [0, T^{\beta, X^{R_n}}) = [0, R_n] \cap [0, T^{\beta,Y})$. By $Y = F(Y)$, by $Y_t = Y_t \triangle \beta$ and $X^{R_n} = Y^{R_n}$ on $[0, T^{\beta,Y})$ and by $p_i((R_n, R_{n+1}) \times E) = 0$ for $i = 2, 3, 4$, we obtain that Y solves SDE (2.14) on $[0, R_{n+1} \wedge T^{\beta,Y}) = [0, S)$. As in the proof of Lemma 2.46, one shows that $\widetilde{X}^{S-} = Y^{S-}$ implies $a_s(\widetilde{X}) = a_s(Y)$, $u_s(\widetilde{X}) = u_s(Y)$, $w(\widetilde{X}, s, x) = w(Y, s, x)$ for any $s \leq S$. Hence, $\widetilde{X}$ also solves the SDE (2.14) on $[0, S)$, and, by its definition, also on $[0, S]$. Since $[0, R_n \wedge T^{\beta,X}] \subset [0, S]$, both X and $\widetilde{X}$ are solutions to SDE (2.14) on $[0, R_n \wedge T^{\beta,X}]$. By Lemma 2.46 it follows that $\widetilde{X}$ coincides with X on $[0, R_n \wedge T^{\beta,X}]$. It remains to be shown that $[0, R_{n+1} \wedge T^{\beta,\widetilde{X}}] \subset [0, S]$. Observe that on $[0, R_{n+1}) \cap [0, S]$ we have $Y = \widetilde{X} \triangle \beta$, and therefore $[0, R_{n+1}) \cap [0, T^{\beta,\widetilde{X}}] = [0, R_{n+1}) \cap [0, T^{\beta,Y}]$. From here, $[0, R_{n+1} \wedge T^{\beta,\widetilde{X}}] \subset [0, S]$ easily follows. $\square$

Lemma 2.49 *Suppose that the Lipschitz conditions (L) and the growth conditions (G) hold. Moreover, fix a space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P, W, p)$ as in Definition 2.45. Assume that there exists a $\mathcal{F}_0$-measurable, $\mathbb{R}^d$-valued random variable X_0 with $P^{X_0} = \eta$. Then there is a solution-process X to SDE (2.14) (on this space and on $\mathbb{R}_+$).*

Proposition 2.50 *For a probability space $(\Omega, \mathcal{F}, P)$, let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be an increasing sequence of sub-σ-fields and $(a_n)_{n \in \mathbb{N}}$ a sequence in $\mathbb{R}_+$ with $\sum_{n=0}^\infty a_n = \infty$. For any $n \in \mathbb{N}$ let $A_n \in \mathcal{F}_n$ such that $P(A_{n+1} | \mathcal{F}_n) \geq a_n$ P-almost surely. Then we have $P(\limsup_{n \to \infty} A_n) = 1$.*

PROOF. We will show by induction on m (for fixed n) that $P(\cap_{k=n}^{n+m} A_k^C) \leq \prod_{k=n}^{n+m} (1 - a_k)$ for any $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{-1\}$. The claim then follows as in the proof of the usual Borel-Cantelli lemma (cf. Bauer (1978), Lemma 35.1). There is nothing to prove for $m = -1$. The induction step follows from $P(\cap_{k=n}^{n+m+1} A_k^C) = E(1_{\cap_{k=n}^{n+m} A_k^C} E(1_{A_{n+m+1}^C}|\mathcal{F}_{n+m})) \leq \prod_{k=n}^{n+m}(1-a_k)(1-a_{n+m+1})$. $\square$

Proposition 2.51 *For any square-integrable martingale M and any stopping time T we have $E(\sup_{t\in[T,\infty)}(M-M^T)_t^2|\mathcal{F}_T) \leq 4E((M-M^T)_\infty^2|\mathcal{F}_T)$ P-almost surely.*

PROOF. Fix $C \in \mathcal{F}_T$. We have to show that $E(1_C \sup_{t\in[T,\infty)}(M-M^T)_t^2) \leq E(1_C 4(M-M^T)_\infty^2)$. This inequality follows immediately if we apply Doob's inequality $E(\sup_{t\in\mathbb{R}_+} X_t^2) \leq 4E(X_\infty^2)$ (cf. JS, I.1.43) to the square-integrable martingale $X = (X_t)_{t\in\mathbb{R}_+}$ on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{G}_t)_{t\in\mathbb{R}_+}, P)$, where $X_t := (M_{T+t} - M_T)1_C$ and $\mathcal{G}_t := \mathcal{F}_{T+t}$ for any $t \in \mathbb{R}_+$. $\square$

Corollary 2.52 *For any locally square-integrable martingale M and any stopping time T we have $E(\sup_{t\in[T,\infty)}(M-M^T)_t^2|\mathcal{F}_T) \leq 4E(\langle M-M^T, M-M^T\rangle_\infty|\mathcal{F}_T)$ P-almost surely.*

PROOF. By localization it is enough to consider the case that M is a square-integrable martingale. The previous proposition implies that we have $E(\sup_{t\in[T,\infty)}(M-M^T)_t^2|\mathcal{F}_T) \leq 4E((M-M^T)_\infty^2|\mathcal{F}_T) = 4E(\langle M-M^T, M-M^T\rangle_\infty|\mathcal{F}_T)$. $\square$

Proposition 2.53 *Let μ be a random measure on $\mathbb{R}_+ \times E$ with compensator ν and $w : \Omega \times \mathbb{R}_+ \times E \to \mathbb{R}$ a non-negative predictable mapping. Then we have for any stopping time T that*

$$E((w1_{[0,T]^C}) * \mu_\infty|\mathcal{F}_T) = E((w1_{[0,T]^C}) * \nu_\infty|\mathcal{F}_T) \ \ P\text{-almost surely}.$$

PROOF. It suffices to show that $E((w1_{[0,T]^C}) * \mu_\infty 1_F) = E((w1_{[0,T]^C}) * \nu_\infty 1_F)$ for any $F \in \mathcal{F}_T$. This follows from the definition of the compensator, since by JS, I.2.5 the mapping $(\omega, t, x) \mapsto w(\omega, t, x)1_F(\omega)1_{[0,T]^C}(t)$ is predictable and non-negative. $\square$

PROOF OF LEMMA 2.49. By Lemma 2.48 there is a solution-process $X^{(N)}$ on $[0, T^{N,X^{(N)}}]$ for any $N \in \mathbb{N}$. For any $N, N' \in \mathbb{N}$ with $N \leq N'$ we have by Lemma 2.46 that $X^{(N)}, X^{(N')}$ coincide on $[0, T^{N,X^{(N)}} \wedge T^{N',X^{(N')}}]$, and hence $T^{N,X^{(N)}} \leq T^{N',X^{(N')}}$ P-almost surely. We can therefore define a process X by $X|_{[0,T^{N,X^{(N)}}]} := X^{(N)}|_{[0,T^{N,X^{(N)}}]}$ for any $N \in \mathbb{N}$ and $X|_{(\cup_{N\in\mathbb{N}}[0,T^{N,X^{(N)}}])^C} := 0$. It remains to be shown that $\cup_{N\in\mathbb{N}}[0, T^{N,X}] = \cup_{N\in\mathbb{N}}[0, T^{N,X^{(N)}}] = \mathbb{R}_+$. We define a sequence $(R_n)_{n\in\mathbb{N}}$ of stopping times recursively by $R_0 := 0$, $R_{n+1} := \inf\{t > R_n : t \in \Theta \cup [n+1, \infty) \text{ or } p_3(\{t\} \times E) > 0\}$. As in the proof of Lemma 2.48, we have $R_n \uparrow \infty$ P-almost surely for $n \to \infty$. Therefore, it suffices to show that $[0, R_n] \subset \cup_{n\in\mathbb{N}}[0, T^{N,X}]$ (up to an evanescent set) for any $n \in \mathbb{N}$. We proceed by induction. For $n = 0$ there is nothing to prove. Now fix $n \in \mathbb{N}$ and assume that $[0, R_n] \subset \cup_{n\in\mathbb{N}}[0, T^{N,X}]$. We define another sequence $(S_m)_{m\in\mathbb{N}}$ of stopping times recursively by $S_0 := R_n$, $S_{m+1} := \inf\{t > R_n \vee S_m : \|X\|_t^* \geq 2\|X\|_{S^m}^* + 1\} \wedge R_{n+1}$. By induction on m it follows that $[0, S_m] \subset \cup_{N\in\mathbb{N}}[0, T^{N,X}]$ and hence that X is a solution-process on $[0, S_m]$ for any $m \in \mathbb{N}$. By $t_0 := 0$, $t_{k+1} := \inf\{t > t_k : \int_{t_k}^t M(s)\, ds > \frac{1}{2048d^6}\}$ we define a sequence $(t_k)_{k\in\mathbb{N}}$ in $\mathbb{R}_+$ with $t_k \uparrow \infty$. Moreover, let $A_m := \{S_m < R_{n+1}\} \cap \{\text{There is a } k \in \mathbb{N} \text{ such that } t_k \leq S_{m-1} \leq S_m < t_{k+1} \text{ for any } m \in \mathbb{N}\}$. If $\omega \in A_m^C$ for infinitely many

$m \in \mathbb{N}$, then we have $S_m = R_{n+1}$ for some $m \in \mathbb{N}$. Therefore, it remains to be shown that $P(\limsup_{m\to\infty} A_m^C) = 1$. By Proposition 2.50 it suffices to prove that $P(A_{m+1}|\mathfrak{F}_{S_m}) \leq \frac{1}{2}$ for any $m \in \mathbb{N}$. Fix $m \in \mathbb{N}$ and note that $\widetilde{X} := X^{S_{m+1}}$ is a solution-process on $[0, S_{m+1}]$. Since p_3, p_4 have no mass on (R_n, R_{n+1}), we have that the following inequality holds on A_{m+1}.

$$\begin{aligned}
2\|X^{S_m}\|_\infty^* + 1 &\leq \|\widetilde{X}\|_\infty^* && (2.16)\\
&\leq \|X^{S_m}\|_\infty^* + \int_0^{S_{m+1}} 1_{[0,S_m]^C}(s)|a_s(\widetilde{X})|\,ds \\
&\quad + \sum_{i,j=1}^d \Big\| \int_0^\cdot 1_{]S_m,S_{m+1}]}(s) u_s^{ij}(\widetilde{X})\,dW_s^j \Big\|_\infty^* \\
&\quad + \sum_{i=1}^d \Big\| \int_{[0,\cdot]\times E} 1_{]S_m,S_{m+1}]}(s) w^i(\widetilde{X},s,x)\,(p_1 - q_1)(ds,dx) \Big\|_\infty^* \\
&\quad + \int_{[0,S_{m+1}]\times E} 1_{[0,S_m]^C}(s)|w(\widetilde{X},s,x)|\,p_2(ds,dx). && (2.17)
\end{aligned}$$

On A_{m+1}, we have that $t_k \leq S_m \leq S_{m+1} < t_{k+1}$ for some $k \in \mathbb{N}$ and hence that $\int_{S_m}^{S_{m+1}} M(s)\,ds < \frac{1}{2048d^6}$. Therefore, we obtain

$$\begin{aligned}
&E\Big(1_{A_{m+1}} \int_0^{S_{m+1}} 1_{[0,S_m]^C}(s)|a_s(\widetilde{X})|\,ds \Big| \mathfrak{F}_{S_m}\Big) \\
&\leq E\Big(1_{A_{m+1}} \int_{S_m}^{S_{m+1}} M(s)\,ds(\|X^{S_{m+1}-}\|_\infty^* + 1) \Big| \mathfrak{F}_{S_m}\Big) \\
&\leq \frac{1}{32}(\|X^{S_m}\|_\infty^* + 1).
\end{aligned}$$

By making use of Corollary 2.52, we have

$$\begin{aligned}
&E\Big(1_{A_{m+1}} \Big(\Big\| \int_0^\cdot 1_{]S_m,S_{m+1}]}(s) u_s^{ij}(\widetilde{X})\,dW_s^j \Big\|_\infty^*\Big)^2 \Big| \mathfrak{F}_{S_m}\Big) \\
&\leq 4E\Big(1_{A_{m+1}} \int_0^\infty 1_{]S_m,S_{m+1}]}(s)(u_s^{ij}(\widetilde{X}))^2\,ds \Big| \mathfrak{F}_{S_m}\Big) \\
&\leq 4E\Big(1_{A_{m+1}} \int_{S_m}^{S_{m+1}} M(s)\,ds(\|X^{S_{m+1}-}\|_\infty^* + 1)^2 | \mathfrak{F}_{S_m}\Big) \\
&\leq \frac{1}{128d^6}(\|X^{S_m}\|_\infty^* + 1)^2.
\end{aligned}$$

Similarly, we obtain by Corollary 2.52 and JS, II.1.33a

$$\begin{aligned}
&E\Big(1_{A_{m+1}} \Big(\Big\| \int_{[0,\cdot]\times E} 1_{]S_m,S_{m+1}]}(s) w^i(\widetilde{X},s,x)\,(p_1 - q_1)(ds,dx) \Big\|_\infty^*\Big)^2 \Big| \mathfrak{F}_{S_m}\Big) \\
&\leq 4E\Big(1_{A_{m+1}} \int_0^\infty \int 1_{]S_m,S_{m+1}]}(s)(w^i(\widetilde{X},s,x))^2\,Q_1(dx)\,ds \Big| \mathfrak{F}_{S_m}\Big) \\
&\leq \frac{1}{128d^3}(\|X^{S_m}\|_\infty^* + 1)^2.
\end{aligned}$$

Finally, it follows from Proposition 2.53 that

$$\begin{aligned}
&E\Big(1_{A_{m+1}}\int_{[0,S_{m+1}]\times E} 1_{[0,S_m]^C}(s)|w(\widetilde{X},s,x)|\,p_2(ds,dx)\Big|\mathcal{F}_{S_m}\Big)\\
&= E\Big(1_{A_{m+1}}\int_{[0,S_{m+1}]\times E} 1_{[0,S_m]^C}(s)|w(\widetilde{X},s,x)|\,q_2(ds,dx)\Big|\mathcal{F}_{S_m}\Big)\\
&\le \frac{1}{32}(\|X^{S_m}\|_\infty^*+1).
\end{aligned}$$

By Inequality (2.16), at least one of the last four terms in (2.17) has to be greater than $\frac{1}{4}(\|X^{S_m}\|_\infty^*+1)$. The conditional probability that this is the case for the first of these equals

$$\begin{aligned}
&P\Big(A_{m+1}\cap\Big\{\int_0^{S_{m+1}} 1_{[0,S_m]^C}(s)|a_s(\widetilde{X})|\,ds > \frac{1}{4}(\|X^{S_m}\|_\infty^*+1)\Big\}\Big|\mathcal{F}_{S_m}\Big)\\
&\le \frac{4}{(\|X^{S_m}\|_\infty^*+1)}E\Big(1_{A_{m+1}}\int_0^{S_{m+1}} 1_{[0,S_m]^C}(s)|a_s(\widetilde{X})|\,ds\Big|\mathcal{F}_{S_m}\Big)\le\frac{1}{8}.
\end{aligned}$$

For the other terms in (2.17) we get similar estimations. Altogether, we have $P(A_{m+1}|\mathcal{F}_{S_m})$ $= P(A_{m+1}$ and Inequality (2.16) holds $|\mathcal{F}_{S_m}) \le \frac{1}{2}$. □

Lemma 2.54 *Suppose that $Q_4 = 0$ and $\eta = \varepsilon_{x_0}$ for a $x_0 \in \mathbb{R}^d$. Moreover, assume that the integrability conditions (I) and the Lipschitz conditions (L) hold. Then there is at most one solution-measure to SDE (2.14).*

PROOF. By $Q_4 = 0$ we have that p is a homogeneous Poisson measure. We define the $\mathcal{P}^d$-measurable mapping $\widehat{a} : \mathbb{D}^d \times \mathbb{R}_+ \to \mathbb{R}^d$ by

$$\widehat{a}_t = a_t - \int_E (w(t,x)-h(w(t,x)))\,Q_1(dx) + \sum_{i=2}^3\int_E h(w(t,x))\,Q_i(dx),$$

where $h : \mathbb{R}^d \to \mathbb{R}^d$ is given by $h(x) := x1_{\{|x|\le 1\}}$. A straightforward calculation shows that $((\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{R}_+},P,W,p),X)$ is a solution-process to SDE (2.14) if and only if it is a solution-process to the SDE

$$dX_t = \widehat{a}_t\,dt + u_t\,dW_t + h(w_t)\,(dp_t - dq_t) + (w_t - h(w_t))\,dp_t \tag{2.18}$$

in the sense of Jacod (1979), (14.73). (There is in fact a small difference. In Jacod (1979), (14.73), a solution X is assumed to be càdlàg, but only $(\mathcal{F}_t^P)_{t\in\mathbb{R}_+}$-adapted, whereas for us, it is only P-almost surely càdlàg, but $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted. But by Jacod (1979), (1.1), it is easy to transform either type of solution into the other.) Let $((\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{R}_+},P,W,p),X)$ and $((\widetilde{\Omega},\widetilde{\mathcal{F}},(\widetilde{\mathcal{F}}_t)_{t\in\mathbb{R}_+},\widetilde{P},\widetilde{W},\widetilde{p}),\widetilde{X})$ be solution-processes to SDE (2.14). Then they are both solution-processes to SDE (2.18) as well. This implies that P^X and $\widetilde{P}^{\widetilde{X}}$ are solution-measures to SDE (2.18) in the sense of Jacod (1979), (14.79). By Jacod (1979), (14.94), we have $P^X = \widetilde{P}^{\widetilde{X}}$ if we can prove pathwise uniqueness for SDE (2.18). Therefore, it remains to be shown that if $((\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{R}_+},P,W,p),X)$ and $((\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{R}_+},P,W,p),\widetilde{X})$ are two solution-processes to SDE (2.18) (on the same space), then we have $X = \widetilde{X}$ up to indistinguishability. By the above equivalence two such processes X and $\widetilde{X}$ are solution-processes to SDE (2.14) as well. The claim now follows from Lemma 2.46. □

Lemma 2.55 *Under the assumptions of Theorem 2.37, there is a solution-measure to the martingale problem.*

PROOF. The idea of the proof is to define a SDE $(\Theta, \eta, a, u, w, Q)^{SDE}$ having solutions which also solve the martingale problem. To this end, let Θ, η, u be as in Theorem 2.37. Moreover, define $a_t(\bar{\omega}) := b_t(\bar{\omega}) - \int x\,(\kappa_2 + \kappa_1)((\bar{\omega}, t), dx)$ for any $(\bar{\omega}, t) \in \mathbb{D}^d \times \mathbb{R}_+$. Define $E := \dot{\bigcup}_{i=1}^4 E_i$, where $E_1 := S^p \times \mathbb{R}_+$ and E_2, E_3, E_4 are disjoint copies of $\mathbb{R}$. It is straightforward to show that the topological sum E (i.e. the disjoint union of the E_i endowed with the sum topology, cf. Querenburg (1973), Definition 3.28) is a Lusin space whose Borel sets are the unions of Borel subsets of E_1, E_2, E_3, E_4. For $i = 1, 2, 3, 4$, we define the measure Q_i on E_i by $Q_1 := \Gamma \otimes \lambda|_{\mathbb{R}_+}$, $Q_2 := \lambda|_{\mathbb{R}_+}$, $Q_3 := \lambda|_{[0,M]}$, $Q_4 := \lambda|_{[0,1]}$. As in Definition 2.44, $Q := \sum_{i=1}^4 Q_i$. Moreover, define the $(\mathcal{P}^d \otimes \mathcal{B}(S^p) \otimes \mathcal{B}_+)$-measurable mapping $w_1 : \mathbb{D}^d \times \mathbb{R}_+ \times S^p \times \mathbb{R}_+ \to \mathbb{R}^d$ by $w_1(\bar{\omega}, t, n, \zeta) := g(\bar{\omega}, t, n, \Phi^{-1}(\bar{\omega}, t, n, \zeta))$, where $\Phi : \mathbb{D}^d \times \mathbb{R}_+ \times S^p \times \mathbb{R}_+ \to \overline{\mathbb{R}}_+$, $(\bar{\omega}, t, n, r) \mapsto \int_r^\infty \rho(\bar{\omega}, t, n, \widetilde{r})\, d\widetilde{r}$ and $\Phi^{-1} : \mathbb{D}^d \times \mathbb{R}_+ \times S^p \times \mathbb{R}_+ \to \mathbb{R}_+$,

$$(\bar{\omega}, t, n, \zeta) \mapsto \begin{cases} \sup\{r \in \mathbb{R}_+ : \Phi(\bar{\omega}, t, n, r) \geq \zeta\} & \text{if this set is non-empty} \\ 0 & \text{else.} \end{cases}$$

Let the predictable mapping $a : \mathbb{D}^d \times \mathbb{R}_+ \to \mathbb{R}^d$ be given by $a_t = b_t - \int x(\kappa_1 + \kappa_2)_t(dx)$. For $i = 2, 3, 4$, choose $(\mathcal{P}^d \otimes \mathcal{B})$-measurable mappings $w_i : \mathbb{D}^d \times \mathbb{R}_+ \times E_i \to \mathbb{R}^d$ such that $Q_2^{w_2(\bar{\omega},t,\cdot)}|_{\mathbb{R}^d\setminus\{0\}} = \kappa_2(\bar{\omega}, t)$, $Q_3^{w_3(\bar{\omega},t,\cdot)}|_{\mathbb{R}^d\setminus\{0\}} = \kappa_1(\bar{\omega}, t)$ for any $(\bar{\omega}, t) \in \mathbb{D}^d \times \mathbb{R}_+$, and $Q_4^{w_4(\bar{\omega},t,\cdot)}|_{\mathbb{R}^d\setminus\{0\}} = K(\bar{\omega}, t)|_{\mathbb{R}^d\setminus\{0\}}$ for any $(\bar{\omega}, t) \in \mathbb{D}^d \times \Theta$. By M_2^β we denote the mapping M_2 in Theorem 2.37, chosen relative to β. W.l.o.g., M_2^β is increasing in β as well. Assume that for any $\beta, n \in \mathbb{N}$, any $(\bar{\omega}, t) \in \mathbb{D}^d \times [0, n]$ with $\|\bar{\omega}\|_t^* \leq \beta$ and any $x \notin [0, M_2^\beta(n)]$ we have $w_2(\bar{\omega}, t, x) = 0$. Finally, define $w : \mathbb{D}^d \times \mathbb{R}_+ \times E \to \mathbb{R}$ by $w := \sum_{i=1}^4 w_i 1_{E_i}$. We have to show that w_2, w_3, w_4 actually exist. We will do this only for w_2, since the argumentation is similar for w_3, w_4. Firstly, define $A^{\beta,n} := \{(\bar{\omega}, t) \in \mathbb{D}^d \times \mathbb{R}_+ : \|\bar{\omega}\|_{t-}^* \in]\beta - 1, \beta], t \in]n - 1, n]\}$ for any $\beta, n \in \mathbb{N}$. The $A^{\beta,n}$ are predictable sets (cf. JS, I.2.6), and we have $\mathbb{D}^d \times \mathbb{R}_+ = \dot{\bigcup}_{n,\beta\in\mathbb{N}} A^{\beta,n}$. For $\beta, n \in \mathbb{N}$, define the transition kernel $\kappa_2^{\beta,n}$ from $(\mathbb{D}^d \times \mathbb{R}_+, \mathcal{P}^d)$ into $(\mathbb{R}^d, \mathcal{B}^d)$ by $\kappa_2^{\beta,n}((\bar{\omega}, t), G) := 1_{A^{\beta,n}}(\bar{\omega}, t)\kappa_2((\bar{\omega}, t), G)$. Fix $\beta, n \in \mathbb{N}$ for the moment. Since $\kappa_2^{\beta,n}((\bar{\omega}, t), \mathbb{R}^d) \leq M_2^\beta(n)$ for any $(\bar{\omega}, t) \in \mathbb{D}^d \times \mathbb{R}_+$, there is, by Jacod (1979), (14.50) and Exercise (14.4), a $(\mathcal{P}^d \otimes \mathcal{B})$-measurable mapping $w_2^{\beta,n} : \mathbb{D}^d \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^d$ with $w_2^{\beta,n}(\bar{\omega}, t, x) = 0$ for $x \notin [0, M_2^\beta(n)]$ and $\kappa_2^{\beta,n}((\bar{\omega}, t), G) = \int 1_{G\setminus\{0\}}(w_2^{\beta,n}(\bar{\omega}, t, x))\, Q_2|_{[0,M_2(n)]}(dx)$ for any $(\bar{\omega}, t) \in \mathbb{D}^d \times \mathbb{R}_+$ and any $G \in \mathcal{B}^d$. Now define $w_2 : \mathbb{D}^d \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^d$ by $w_2(\bar{\omega}, t, x) := \sum_{\beta,n\in\mathbb{N}} 1_{A^{\beta,n}}(\bar{\omega}, t) w_2^{\beta,n}(\bar{\omega}, t, x)$. It is easy to verify that w_2 has the above properties.

The rest of the proof of Lemma 2.55 will be broken down into several propositions.

Proposition 2.56 *Fix $(\bar{\omega}, t, n) \in \mathbb{D}^d \times \mathbb{R}_+ \times S^p$ and $G \in \mathcal{B}_+$. Then $\int_{\mathbb{R}_+} 1_{G\setminus\{0\}}(\Phi^{-1}(\bar{\omega}, t, n, \zeta))\, d\zeta = \int_{\mathbb{R}_+} 1_G(r)\rho(\bar{\omega}, t, n, r)\, dr$.*

PROOF. By a Dynkin argument it suffices to prove the proposition for any $G = [r_0, \infty)$ with $r_0 > 0$. One easily verifies that for any $\zeta \in \mathbb{R}_+$ we have equivalence between

$\Phi^{-1}(\bar{\omega}, t, n, \zeta) \geq r_0$ and $\Phi(\bar{\omega}, t, n, r_0) \geq \zeta$. Therefore $\int_{\mathbb{R}_+} 1_G(\Phi^{-1}(\bar{\omega}, t, n, \zeta))\, d\zeta = \Phi(\bar{\omega}, t, n, r_0) = \int_{\mathbb{R}_+} 1_G(r)\rho(\bar{\omega}, t, n, r)\, dr$. □

Proposition 2.57 *Fix* $(\bar{\omega}, t) \in \mathbb{D}^d \times \mathbb{R}_+$. *Then* $Q_1^{w_1(\bar{\omega},t,\cdot)}|_{\mathbb{R}^d\setminus\{0\}} = \kappa_3(\bar{\omega}, t)$.

PROOF. By the previous proposition we have for any $G \in \mathcal{B}^d$ with $0 \notin G$ that

$$\begin{aligned} Q_1^{w_1(\bar{\omega},t,\cdot)}(G) &= \int\int_{\mathbb{R}_+} 1_G(g(\bar{\omega}, t, n, \Phi^{-1}(\bar{\omega}, t, n, \zeta)))\, d\zeta\, \Gamma(dn) \\ &= \int\int_{\mathbb{R}_+} 1_G(g(\bar{\omega}, t, n, r))\rho(\bar{\omega}, t, n, r)\, dr\, \Gamma(dn) \\ &= \int_{S^p\times\mathbb{R}_+} 1_G(g(\bar{\omega}, t, n, r))\, \mu((\bar{\omega}, t), d(n, r)) \\ &= \kappa_3((\bar{\omega}, t), G). \end{aligned}$$

□

Proposition 2.58 *Let* $((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P), X)$ *be a solution-process to the SDE* $(\Theta, \eta, a, u, w, Q)^{SDE}$. *Then it is a solution-process to the martingale problem.*

PROOF. By $Q_2^{w_2(\bar{\omega},t,\cdot)}|_{\mathbb{R}^d\setminus\{0\}} = \kappa_2(\bar{\omega}, t)$ etc. we have that

$$\begin{aligned} &\sum_{i=2}^{4} \int_{[0,t]\times E} |w(\bar{\omega}, t, x)|\, q_i(ds, dx) \\ &= \int_0^t \int |x|\, (\kappa_2 + \kappa_1)((\bar{\omega}, t), dx)\, ds + \sum_{s\in\Theta\cap[0,t]} \int |x|\, K((\bar{\omega}, t), dx) \\ &\leq \int_0^t \int (|x|^2 \wedge |x|)\, F((\bar{\omega}, s), dx)\, ds + \sum_{s\in\Theta\cap[0,t]} \int |x|\, K((\bar{\omega}, t), dx) \\ &\quad + t \sup_{s\in[0,t]} (\kappa_2 + \kappa_1)((\bar{\omega}, s), \mathbb{R}^d). \end{aligned} \tag{2.19}$$

By Definition 2.28 and the assumptions in Theorem 2.37, we have that this expression is finite for any $(\bar{\omega}, t) \in \mathbb{D}^d \times \mathbb{R}_+$. We may therefore rewrite the solution to Equation (2.14) as $X = X_0 + B + X^c + X^d$, where $B_t = \int_0^t a_s(X)\, ds + \sum_{i=2}^4 \int_{[0,t]\times E} w(X, s, x)\, q_i(ds, dx)$, $X_t^c := \sum_{j=1}^d \int_0^t u_s^{\cdot j}(X)\, dW_s^j$, $X_t^d := \sum_{i=1}^4 \int_{[0,t]\times E} w(X, s, x)\, (p_i - q_i)(ds, dx)$ for any $t \in \mathbb{R}_+$. Since B is predictable and of finite variation, X^c is a continuous local martingale and X^d is a discountinuous local martingale, we have that X is a special martingale. Denote its integral characteristics by $(B, C, \nu)^I$. From the above equations we see that B and C are as in Lemma 2.16 with $b_t = a_t(X) + \sum_{i=2}^3 \int w(X, t, x)\, Q_i(dx) = a_t(X) + \int x\, (\kappa_2 + \kappa_1)((X, t), dx) = b_t(X)$ and $c_t = u_t(X)u_t(X)^\top = c_t(X)$ for any $t \in \mathbb{R}_+$. By the continuity of q_1 we have that $\Delta(w_1 * (p_1 - q_1))_t = \int w(X, t, x)p_1(\{t\} \times dx)$ up to indistinguishability. Therefore, Equation (2.14) yields that we have $\Delta X_t = \int w(X, t, x)\, p(\{t\} \times dx)$ for any $t \in \mathbb{R}_+$. Since p is an integer-valued random measure, we have $\Delta X_t(\omega) \in G \setminus \{0\}$ if and only if $p(\omega, \{t,x\}) = 1$ for some x with $w(X,t,x) \in G \setminus \{0\}$. This yields $\mu^X([0,t]\times G) = \int_{[0,t]\times E} 1_{G\setminus\{0\}}(w(X,s,x))\, p(ds,dx)$ for any $t \in \mathbb{R}_+$, $G \in \mathcal{B}^d$. It follows that

$$\begin{aligned}\nu([0,t]\times G) &= \int_{[0,t]\times E} 1_{G\setminus\{0\}}(w(X,s,x))\, q(ds,dx)\\ &= \sum_{i=1}^{3}\int_0^t\int 1_{G\setminus\{0\}}(w_i(X,s,x))\, Q_i(dx)\, ds + \sum_{s\in\Theta\cap[0,t]}\int 1_{G\setminus\{0\}}(w_4(X,s,x))\, Q_4(dx)\end{aligned}$$

for any $t \in \mathbb{R}_+$, $G \in \mathcal{B}^d$. From the definition of the w_i's and by Proposition 2.57, it follows that we have $\nu([0,t]\times G) = \int_0^t F_s(G)\, ds + \sum_{s\in\Theta\cap[0,t]} K_s(G\setminus\{0\})$, where $F_t = (\kappa_3+\kappa_2+\kappa_1)(X,t)$ and $K_t = K_t(X)$. Therefore, X is indeed a solution to the martingale problem. $\square$

Proposition 2.59 *Fix* $(t,n) \in \mathbb{R}_+\times S^p$. *Then the mapping* $\mathbb{D}_t^d \times (0, R(t,n)) \to \mathbb{R}$, $(\bar\omega, r) \mapsto \Phi(\bar\omega,t,n,r)$ *is continuously Fréchet-differentiable with partial derivatives* $D_1\Phi(\bar\omega,t,n,r) = \int_r^{R(t,n)} D_1\rho(\bar\omega,t,n,\widetilde{r})\, d\widetilde{r} \in \mathscr{L}(\mathbb{D}_t^d,\mathbb{R})$ *and* $D_4\Phi(\bar\omega,t,n,r) = -\rho(\bar\omega,t,n,r) \in \mathscr{L}(\mathbb{R},\mathbb{R})$.

PROOF. Firstly, observe that $\mathbb{D}_t^d$ is a Banach space relative to the norm $\bar\omega \mapsto \|\bar\omega\|_t^*$ (for completeness, cf. Billingsley (1968), Section 18 and JS, Subsection VI.1a). Secondly, integrals of $\mathscr{L}(\mathbb{D}_t^d,\mathbb{R})$-valued functions (as in $D_1\Phi$ in the proposition) are meant for any single argument $\bar\omega \in \mathbb{D}_t^d$. This interpretation is consistent with the usual integral for Banach-space valued functions on an interval (cf. Flett (1980), Section 1.9, in particular Exercise 4). In order to prove that the mapping $(\bar\omega, r) \mapsto \Phi(\bar\omega,t,n,r)$ is continuously differentiable, it suffices to show that the partial derivatives $D_1\Phi$ and $D_4\Phi$ exist and that they are continuous (cf. Lang (1993), Theorem XIII.7.1). Fix $r > 0$. For any $N \in \mathbb{N}$, the mapping $\mathbb{D}_t^d \times [r, R(t,n)\wedge N] \to \mathbb{R}$, $(\bar\omega, r) \mapsto \rho(\bar\omega,t,n,\widetilde{r})$ is, by assumption, continuous, $D_1\rho$ exists and is also continuous. Hence (cf. Lang (1993), Theorem XIII.8.1), the mapping $\Phi^N : \mathbb{D}_t^d \to \mathbb{R}$, $\bar\omega \mapsto \int_{[r,R(t,n)\wedge N]} \rho(\bar\omega,t,n,\widetilde{r})\, d\widetilde{r}$ is differentiable with derivative $D\Phi^N(\bar\omega) = \int_r^{R(t,n)\wedge N} D_1\rho(\bar\omega,t,n,\widetilde{r})\, d\widetilde{r}$. Since $D_1\rho(\cdot,t,n,\widetilde{r})$ is continuous on $\mathbb{D}_t^d$ and by (2.13), dominated convergence yields that $D\Phi^N : \mathbb{D}_t^d \to \mathscr{L}(\mathbb{D}_t^d,\mathbb{R})$ is continuous, i.e. Φ^N is of class C^1. Also by dominated convergence, one shows that, for $N\to\infty$, $\Phi^N(\bar\omega)$ converges to $\Phi(\bar\omega,t,n,r)$ for any $\bar\omega \in \mathbb{D}_t^d$ and, moreover, $D\Phi^N$ converges uniformly on any ball $\{\bar\omega \in \mathbb{D}_t^d : \|\bar\omega\|_t^* \le \beta\}$ to the mapping $\mathbb{D}_t^d \to \mathscr{L}(\mathbb{D}_t^d,\mathbb{R})$, $\bar\omega \mapsto \int_r^{R(t,n)} D_1\rho(\bar\omega,t,n,\widetilde{r})\, d\widetilde{r}$. By Lang (1993), Theorem XIII.9.1, it follows that the mapping $\mathbb{D}_t^d \to \mathbb{R}$, $\bar\omega \mapsto \Phi(\bar\omega,t,n,r)$ is differentiable and its derivative is as claimed. The same dominated convergence argument as for Φ^N shows that $D_1\Phi$ is continuous in $\bar\omega$. The statement concerning $D_4\Phi$ simply follows from the fundamental theorem of calculus. $\square$

Proposition 2.60 *Fix* $(t,n) \in \mathbb{R}_+ \times S^p$. *Then for any* $\zeta > 0$, *the mapping* $(\Phi(\cdot,t,n,0))^{-1}((\zeta,\infty)) \to \mathbb{R}_+$, $\bar\omega \mapsto \Phi^{-1}(\bar\omega,t,n,\zeta)$ *is continuously Fréchet-differentiable with derivative*

$$D_1\Phi^{-1}(\bar\omega,t,n,\zeta) \in \mathscr{L}(\mathbb{D}_t^d,\mathbb{R}),\quad \bar\omega \mapsto \frac{\int_{\Phi^{-1}(\bar\omega,t,n,\zeta)}^{R(t,n)} D_1\rho(\bar\omega,t,n,r)\, dr}{\rho(\bar\omega,t,n,\Phi^{-1}(\bar\omega,t,n,\zeta))}.$$

Moreover, the mapping $\mathbb{D}_t^d \to \mathbb{R}_+$, $\bar\omega \mapsto \Phi^{-1}(\bar\omega,t,n,\zeta)$ *is continuous.*

PROOF. In the previous proposition we show that the mapping $\bar{\omega} \mapsto \Phi(\bar{\omega}, t, n, r)$ is continuous for any $r > 0$. Hence, the set $(\Phi(\cdot, t, n, 0))^{-1}((\zeta, \infty)) = \cup_{k\in\mathbb{N}}(\Phi(\cdot, t, n, \frac{1}{k}))^{-1}((\zeta, \infty))$ is open. Define a mapping $F : \mathbb{D}_t^d \times (0, R(t, n)) \times \mathbb{R}_+^* \to \mathbb{R}$ by $F(\bar{\omega}, r, \zeta) := \Phi(\bar{\omega}, t, n, r) - \zeta$. Fix $\zeta > 0$. For any $\bar{\omega} \in (\Phi(\cdot, t, n, 0))^{-1}((\zeta, \infty))$, there is a $r > 0$ such that $\int_r^\infty \rho(\bar{\omega}, t, n, \widetilde{r})\, d\widetilde{r} > \zeta$. By continuity of Φ we have $\Phi(\bar{\omega}, t, n, \Phi^{-1}(\bar{\omega}, t, n, \zeta)) = \zeta$, hence $F(\bar{\omega}, \Phi^{-1}(\bar{\omega}, t, n, \zeta), \zeta) = 0$. Note that $0 < r < \Phi^{-1}(\bar{\omega}, t, n, \zeta) < R(t, n)$.

By the previous proposition, the mapping F is continuously Fréchet-differentiable. Moreover, for any $(\bar{\omega}, r, \zeta) \in \mathbb{D}_t^d \times (0, R(t, n)) \times \mathbb{R}_+$ the derivative $D_2F(\bar{\omega}, r, \zeta) = D_4\Phi(\bar{\omega}, t, n, r) = -\rho(\bar{\omega}, t, n, r) \neq 0$ is a toplinear isomorphism in the sense of Lang (1993), p.67. Fix $(\bar{\omega}, \zeta) \in \mathbb{D}_t^d \times \mathbb{R}_+^*$ with $\bar{\omega} \in (\Phi(\cdot, t, n, 0))^{-1}((\zeta, \infty))$. By the implicit function theorem (cf. Flett (1980), (3.8.1)) there is a neighbourhood U of $(\bar{\omega}, \zeta)$ and a continuous mapping $h : U \to (0, R(t, n))$ such that $h(\bar{\omega}, \zeta) = \Phi^{-1}(\bar{\omega}, t, n, \zeta)$ and $F(\widehat{\omega}, h(\widehat{\omega}, \widehat{\zeta}), \widehat{\zeta}) = 0$ for any $(\widehat{\omega}, \widehat{\zeta}) \in U$. Moreover, h is continuously differentiable. Since $\rho(\cdot, \cdot, \cdot, r) > 0$ for $0 < r < R(t, n)$, we have that $\widetilde{r} > \Phi^{-1}(\widetilde{\omega}, t, n, \widetilde{\zeta})$ if and only if $\Phi(\widetilde{\omega}, t, n, \widetilde{\zeta}) > \widetilde{r}$ (and likewise for "$<$"). Thus, $h(\widetilde{\omega}, \widetilde{\zeta}) = \Phi^{-1}(\widetilde{\omega}, t, n, \widetilde{\zeta})$ for any $(\widetilde{\omega}, \widetilde{\zeta}) \in U$. By the implicit function theorem (cf. Flett (1980), (3.8.1)) we have

$$Dh(\bar{\omega}, \zeta) = -(D_2F(\bar{\omega}, \Phi^{-1}(\bar{\omega}, t, n, \zeta), \zeta))^{-1} \circ D_{(1,3)}F(\bar{\omega}, \Phi^{-1}(\bar{\omega}, t, n, \zeta), \zeta)$$

Hence by the previous proposition,

$$\begin{aligned} D_1\Phi^{-1}(\bar{\omega}, t, n, \zeta) &= D_1h(\bar{\omega}, \zeta) \\ &= -\frac{1}{\rho(\bar{\omega}, t, n, \Phi^{-1}(\bar{\omega}, t, n, \zeta))} D_1\Phi(\bar{\omega}, t, n, \Phi^{-1}(\bar{\omega}, t, n, \zeta)) \\ &= -\frac{1}{\rho(\bar{\omega}, t, n, \Phi^{-1}(\bar{\omega}, t, n, \zeta))} \int_{\Phi^{-1}(\bar{\omega},t,n,\zeta)}^{R(t,n)} D_1\rho(\bar{\omega}, t, n, \widetilde{r})\, d\widetilde{r} \end{aligned}$$

as claimed. It remains to be shown that the mapping $\mathbb{D}_t^d \to \mathbb{R}_+$, $\bar{\omega} \mapsto \Phi^{-1}(\bar{\omega}, t, n, \zeta)$ is continuous in any $\bar{\omega} \in \mathbb{D}_t^d$ with $\Phi(\bar{\omega}, t, n, 0) \leq \zeta$, i.e. with $\Phi^{-1}(\bar{\omega}, t, n, \zeta) = 0$. This follows by straightforward limit arguments from the continuity of Φ and the positivity of ρ. □

Proposition 2.61 *Fix $(t, n) \in \mathbb{R}_+ \times S^p$ as well as $\zeta > 0$ and $\omega, \bar{\omega} \in \mathbb{D}_t^d$. Then we have*

$$\begin{aligned} &|\Phi^{-1}(\omega, t, n, \zeta) - \Phi^{-1}(\bar{\omega}, t, n, \zeta)| \\ &\leq \int_0^1 \frac{1_{\{\Phi^{-1}(\omega+\lambda(\bar{\omega}-\omega),t,n,\zeta)\neq 0\}}}{\rho(\omega + \lambda(\bar{\omega} - \omega), t, n, \Phi^{-1}(\omega + \lambda(\bar{\omega} - \omega), t, n, \zeta))} \\ &\quad \cdot \int_{\Phi^{-1}(\omega+\lambda(\bar{\omega}-\omega),t,n,\zeta)}^{R(t,n)} \|D_1\rho(\omega + \lambda(\bar{\omega} - \omega), t, n, \widetilde{r})\|\, \|\omega - \bar{\omega}\|_t^*\, d\widetilde{r}\, d\lambda. \end{aligned}$$

PROOF. Firstly, observe that we have equivalence between $\Phi^{-1}(\widetilde{\omega}, t, n, \zeta) \neq 0$ and $\Phi(\widetilde{\omega}, t, n, 0) > \zeta$. Let $G := (\Phi(\cdot, t, n, 0))^{-1}((\zeta, \infty))$ and define

$$\begin{aligned} \lambda_0 &:= \sup\{\lambda \in [0, 1] : \text{ For any } \lambda' \in (0, \lambda_0), \text{ we have } \omega + \lambda(\bar{\omega} - \omega) \in G\}, \\ \lambda_1 &:= \inf\{\lambda \in [\lambda_0, 1] : \text{ For any } \lambda' \in (\lambda_1, 1), \text{ we have } \omega + \lambda(\bar{\omega} - \omega) \in G\}. \end{aligned}$$

Due to Proposition 2.60 and Lang (1993), p.337, the mapping $h : [0,1] \to \mathbb{R}_+$, $\lambda \mapsto \Phi^{-1}(\omega + \lambda(\bar\omega - \omega), t, n, \zeta)$ is continuous on $[0,1]$ and continuously differentiable on the intervals $(0, \lambda_0)$ and $(\lambda_1, 1)$ with derivative

$$\begin{aligned} h'(\lambda) &= D_1\Phi^{-1}(\omega + \lambda(\bar\omega - \omega), t, n, \zeta)(\bar\omega - \omega) \\ &= \frac{\int_{\Phi^{-1}(\omega+\lambda(\bar\omega-\omega),t,n,\zeta)}^{R(t,n)} D_1\rho(\omega + \lambda(\bar\omega - \omega), t, n, \widetilde{r})(\omega - \bar\omega)\, d\widetilde{r}}{\rho(\omega + \lambda(\bar\omega - \omega), t, n, \Phi^{-1}(\omega + \lambda(\bar\omega - \omega), t, n, \zeta))}. \end{aligned}$$

In the case $\lambda_0 \neq \lambda_1$, we have $h(\lambda_0) = 0 = h(\lambda_1)$. This implies $\Phi^{-1}(\omega, t, n, \zeta) - \Phi^{-1}(\bar\omega, t, n, \zeta) = h(0) - h(1) = -(h(1) - h(\lambda_1)) - (h(\lambda_0) - h(0))$. By the continuity of ρ and the fundamental theorem of calculus, we obtain

$$\begin{aligned} |\Phi^{-1}(\omega, t, n, \zeta) - \Phi^{-1}(\bar\omega, t, n, \zeta)| &\leq \lim_{\varepsilon\to 0}(|h(1-\varepsilon) - g(\lambda_1 + \varepsilon)| + |h(\lambda_0 - \varepsilon) - h(\varepsilon)|) \\ &= \lim_{\varepsilon\to 0}\left(\Big|\int_{\lambda_1-\varepsilon}^{1-\varepsilon} h'(\lambda)\, d\lambda\Big| + \Big|\int_{\varepsilon}^{\lambda_0-\varepsilon} h'(\lambda)\, d\lambda\Big|\right) \\ &\leq \int_0^1 |h'(\lambda)| 1_{\{\Phi^{-1}(\omega+\lambda(\bar\omega-\omega),t,n,\zeta)\neq 0\}}\, d\lambda. \end{aligned}$$

(Observe that the final estimate also holds if $\lambda_0 = 0$ or $\lambda_1 = 1$.) This implies the claim. □

Proposition 2.62 *Fix $(t,n) \in \mathbb{R}_+ \times S^p$ as well as $\omega, \bar\omega \in \mathbb{D}_t^d$ with $\|\omega\|_t^*, \|\bar\omega\|_t^* \leq \beta \in \mathbb{R}_+$. Then we have*

$$\begin{aligned} &\int_{\mathbb{R}_+} |\Phi^{-1}(\omega, t, n, \zeta) - \Phi^{-1}(\bar\omega, t, n, \zeta)|^2\, d\zeta \\ &\quad\leq \int_{\mathbb{R}_+} M_8(t, n, r)\, dr (\|\omega - \bar\omega\|_t^*)^2, \end{aligned}$$

where M_8 is chosen relative to β as in Theorem 2.37.

PROOF. By the previous proposition, the left-hand side is less than or equals the following expression:

$$\begin{aligned} &\int_0^1 \int_{\mathbb{R}_+} \Bigg(\frac{1_{\{\Phi^{-1}(\omega+\lambda(\bar\omega-\omega),t,n,\zeta)\neq 0\}}}{\rho(\omega + \lambda(\bar\omega - \omega), t, n, \Phi^{-1}(\omega + \lambda(\bar\omega - \omega), t, n, \zeta))} \\ &\qquad \int_{\Phi^{-1}(\omega+\lambda(\bar\omega-\omega),t,n,\zeta)}^{R(t,n)} \|D_1\rho(\omega + \lambda(\bar\omega - \omega), t, n, \widetilde{r})\|\, d\widetilde{r}\Bigg)^2 d\zeta\, d\lambda (\|\omega - \bar\omega\|_t^*)^2 \end{aligned} \tag{2.20}$$

By Proposition 2.56 we may replace the integration relative to ζ with an integration with respect to $r = \Phi^{-1}(\bar\omega, t, n, \zeta)$. Therefore, (2.20) is less than or equals

$$\begin{aligned} &(\|\omega - \bar\omega\|_t^*)^2 \int_0^1 \int_{\mathbb{R}_+} \rho(\omega + \lambda(\bar\omega - \omega), t, n, r) \\ &\quad \left(\int_r^{R(t,n)} \frac{\|D_1\rho(\omega + \lambda(\bar\omega - \omega), t, n, \widetilde{r})\|\, d\widetilde{r}}{\rho(\omega + \lambda(\bar\omega - \omega), t, n, r)}\right)^2 dr\, d\lambda \end{aligned}$$

In view of the definition of M_8, the claim easily follows. □

Proposition 2.63 *The Lipschitz conditions (L) and the growth conditions (G) hold for the SDE* $(\Theta, \eta, a, u, w, Q)^{SDE}$.

PROOF. Lipschitz and growth conditions for a and u are given in Theorem 2.37. Fix $t, \beta \in \mathbb{R}_+$. Let $\omega, \bar{\omega} \in \mathbb{D}^d$ with $\|\omega\|_t^*, \|\bar{\omega}\|_t^* \leq \beta$. By definition we have

$$\begin{aligned}
&\int |w(\omega,t,x) - w(\bar{\omega},t,x)|^2 \, Q_1(dx) \\
&\quad= \int\int_{\mathbb{R}_+} |g(\omega,t,n,\Phi^{-1}(\omega,t,n,\zeta)) - g(\bar{\omega},t,n,\Phi^{-1}(\bar{\omega},t,n,\zeta))|^2 \, d\zeta \, \Gamma(dn) \\
&\quad\leq 2\int\int_{\mathbb{R}_+} |g(\omega,t,n,\Phi^{-1}(\omega,t,n,\zeta)) - g(\bar{\omega},t,n,\Phi^{-1}(\omega,t,n,\zeta))|^2 \, d\zeta \, \Gamma(dn) \\
&\qquad + 2\int\int_{\mathbb{R}_+} |g(\bar{\omega},t,n,\Phi^{-1}(\omega,t,n,\zeta)) - g(\bar{\omega},t,n,\Phi^{-1}(\bar{\omega},t,n,\zeta))|^2 \, d\zeta \, \Gamma(dn).
\end{aligned} \tag{2.21}$$

By the Lipschitz conditions from Theorem 2.37 and by Proposition 2.56, the first term is dominated by

$$\begin{aligned}
&2\int\int_{\mathbb{R}_+} L_4^2(t,n,\Phi^{-1}(\omega,t,n,\zeta)) \, d\zeta \, \Gamma(dn)(\|\omega - \bar{\omega}\|_t^*)^2 \\
&\quad\leq 2\int\int_{\mathbb{R}_+} L_4^2(t,n,r)\rho(\omega,t,n,r) \, dr \, \Gamma(dn)(\|\omega - \bar{\omega}\|_t^*)^2.
\end{aligned} \tag{2.22}$$

By the Lipschitz conditions from Theorem 2.37 and by Proposition 2.62, the second term in (2.21) is dominated by

$$\begin{aligned}
&2\int L_3^2(t,n) \int_{\mathbb{R}_+} |\Phi^{-1}(\omega,t,n,\zeta) - \Phi^{-1}(\bar{\omega},t,n,\zeta)|^2 \, d\zeta \, \Gamma(dn) \\
&\quad\leq 2\int L_3^2(t,n) \int_{\mathbb{R}_+} M_8(t,n,r) \, dr \, \Gamma(dn)(\|\omega - \bar{\omega}\|_t^*)^2.
\end{aligned} \tag{2.23}$$

Adding the terms (2.22) and (2.23) up, one obtains $\int |w(\omega,t,x) - w(\bar{\omega},t,x)|^2 \, Q_1(dx) \leq L(t)$ for some $L : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^t L(s) \, ds < \infty$ for any $s \in \mathbb{R}_+$.

Keep $\beta \in \mathbb{N}$ fixed. For any $t \in \mathbb{R}_+$, define $H(t) := [0, M_2^\beta([t+1])]$ (as a subset of E_2). Then we have $Q_2(t) \leq M_2^\beta([t+1])$ and $w(\bar{\omega},t,x) = 0$ for any $(\bar{\omega},t,x) \in \mathbb{D}^d \times [0,t] \times E_2$ with $\|\bar{\omega}\|_t^* \leq \beta$, $x \in E_2 \setminus H(t)$.

By definition and Proposition 2.57, we have $\int |w(\bar{\omega},t,x)| \, Q_2(dx) = \int |x| \, \kappa_2((\bar{\omega},t),dx)$ resp. $\int |w(\bar{\omega},t,x)|^2 \, Q_1(dx) = \int |x|^2 \, \kappa_3((\bar{\omega},t),dx)$ for any $(\bar{\omega},t) \in \mathbb{D}^d \times \mathbb{R}_+$. Hence, the growth conditions (G) follow from the assumptions in Theorem 2.37. □

Lemma 2.55 now follows from Proposition 2.63, Lemma 2.49 and Proposition 2.58. □

Lemma 2.64 *Under the assumptions of Theorem 2.37 there is at most one solution-measure to the martingale problem.*

PROOF. Let $((\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{R}_+},P),X)$ and $((\widetilde{\Omega},\widetilde{\mathcal{F}},(\widetilde{\mathcal{F}}_t)_{t\in\mathbb{R}_+},\widetilde{P}),\widetilde{X})$ be solution-processes to the martingale problem. By Statement 2 of Lemma 2.29 we may assume without loss of generality that $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{R}_+})$ and $(\widetilde{\Omega},\widetilde{\mathcal{F}},(\widetilde{\mathcal{F}}_t)_{t\in\mathbb{R}_+})$ both equal $(\mathbb{D}^d,\mathcal{D}^d,(\mathcal{D}^d_t)_{t\in\mathbb{R}_+})$ and $X=\widetilde{X}$ is the canonical process on $\mathcal{D}^d$. Let $(t_k)_{k\in\mathbb{N}}$ be an increasing sequence in $\mathbb{R}_+$ such that $t_0=0$, $\Theta\subset\{t_k:k\in\mathbb{N}\}$ and $t_k\uparrow\infty$ for $k\to\infty$. It suffices to prove that $P^{X^{t_k}}=\widetilde{P}^{X^{t_k}}$ for any $k\in\mathbb{N}$. We proceed by induction. We have $P^{X_0}=\eta=\widetilde{P}^{X_0}$. Assume that for given $k\in\mathbb{N}$, X^{t_k} has the same distribution under P and $\widetilde{P}$. Observe that both $((\mathbb{D}^d,\mathcal{D}^d,(\mathcal{D}^d_t)_{t\in\mathbb{R}_+},P),X^{t_{k+1}-})$ and $((\mathbb{D}^d,\mathcal{D}^d,(\mathcal{D}^d_t)_{t\in\mathbb{R}_+},\widetilde{P}),X^{t_{k+1}-})$ are solution-processes to the martingale problem $(\Theta\cap[0,t_k],\eta,\widehat{b},\widehat{c},\widehat{F},\widehat{K})^M$, where $\widehat{b}(\bar\omega):=1_{[0,t_{k+1})}b(\bar\omega)$ and similarly for $\widehat{c}$, $\widehat{F}$, $\widehat{K}$. By Lemma 2.33, $P^{(X^{t_{k+1}-}_{t_k+s})_{s\in\mathbb{R}_+}|X^{t_k}}$ and $\widetilde{P}^{(X^{t_{k+1}-}_{t_k+s})_{s\in\mathbb{R}_+}|X^{t_k}}$ are P- resp. $\widetilde{P}$-almost surely solution-measures to the (random) martingale problem $(\varnothing,\varepsilon_x,\widetilde{b},\widetilde{c},\widetilde{F},0)^M$ with $x:=X_{t_k}$, $\widetilde{b}_s(\bar\omega):=\widehat{b}_{t_k+s}(\iota(X^{t_k},\bar\omega))$ etc. for any $\bar\omega\in\mathbb{D}^d$, where

$$\iota(X^{t_k}(\omega),\bar\omega)_s:=\begin{cases}X^{t_k}_s(\omega) & \text{for } s\in[0,t_k)\\ \bar\omega_{s-t_k} & \text{for } s\in[t_k,\infty).\end{cases}$$

Fix $\omega\in\mathbb{D}^d$. By Lemma 2.29, $P^{(X^{t_{k+1}-}_{t_k+s})_{s\in\mathbb{R}_+}|X^{t_k}}(\omega)$ and $\widetilde{P}^{(X^{t_{k+1}-}_{t_k+s})_{s\in\mathbb{R}_+}|X^{t_k}}(\omega)$ are for P- resp. $\widetilde{P}$-almost all $\omega\in\mathbb{D}^d$ solution-measures to the (random) martingale problem $\mathscr{s}(\sigma(X_0),X|\varepsilon_x;\widetilde{B}(h),\widetilde{C},\widetilde{\nu})$ on the Skorohod space $((\mathbb{D}^d,\mathcal{D}^d,(\mathcal{D}^d_t)_{t\in\mathbb{R}_+},P),X)$, where $\widetilde{B}(h)$, $\widetilde{C}$, $\widetilde{\nu}$ are defined as in Statement 2 of Lemma 2.29, but relative to $\widetilde{b},\widetilde{c},\widetilde{F}$ and the truncation function $h:\mathbb{R}^d\to\mathbb{R}^d$, $x\mapsto x1_{\{|x|\le1\}}$. If we define the mappings $\widetilde{b}(h):\mathbb{D}^d\times\mathbb{R}_+\to\mathbb{R}^d$, $\widetilde{u}:\mathbb{D}^d\times\mathbb{R}_+\to\mathbb{R}^{d\times d}$, $\widetilde{w}:\mathbb{D}^d\times\mathbb{R}_+\times E\to\mathbb{R}^d$ by $\widetilde{b}(h)_s:=\widetilde{b}_s+\int(h(x)-x)\,\widetilde{F}_s(dx)$, $\widetilde{u}_s(\bar\omega):=1_{[0,t_{k+1}-t_k)}(s)u_{t_k+s}(\iota(X^{t_k},\bar\omega))$, $\widetilde{w}_s(\bar\omega):=1_{[0,t_{k+1}-t_k)}(s)1_{E_4^C}(x)w(\iota(X^{t_k},\bar\omega),t_k+s,x)$, then we have that $\widetilde{B}(h)_s=\int_0^s\widetilde{b}(h)_r\,dr$, $\widetilde{C}_s=\int_0^s\widetilde{u}_r\widetilde{u}_r^\top\,dr$, $\widetilde{\nu}([0,s]\times G)=\int_0^s\int 1_{G\setminus\{0\}}(\widetilde{w}(r,x))$ $(Q_1+Q_2+Q_3)(dx)\,dr$ for any $s\in\mathbb{R}_+$, $G\in\mathcal{B}^d$ (cf. the definition of w_1,w_2 and Proposition 2.57). By Jacod (1979), (14.80), $P^{(X^{t_{k+1}-}_{t_k+s})_{s\in\mathbb{R}_+}|X^{t_k}}(\omega)$ and $\widetilde{P}^{(X^{t_{k+1}-}_{t_k+s})_{s\in\mathbb{R}_+}|X^{t_k}}(\omega)$ are for P- resp. $\widetilde{P}$-almost all $\omega\in\mathbb{D}^d$ solution-measures to the SDE

$$dX_s=\widetilde{b}(h)_s\,ds+\widetilde{u}_s\,dW_s+h(\widetilde{w}_s)\,(dp_s-dq_s)+(\widetilde{w}_s-h(\widetilde{w}_s))\,dp_s \tag{2.24}$$

in the sense of Jacod (1979), (14.79), where p is a homogeneous Poisson random measure on $\mathbb{R}_+\times E$ with compensator $q(dx,ds)=(Q_1+Q_2+Q_3)(dx)\,ds$. As in the proof of Lemma 2.54, we have that a probability measure is a solution to the SDE (2.24) if and only if it is a solution-measure to the SDE $(\varnothing,\varepsilon_x,\widetilde{a},\widetilde{u},\widetilde{w},Q)^{SDE}$, where $\widetilde{a}:\mathbb{D}^d\times\mathbb{R}_+\to\mathbb{R}^d$ is given by

$$\widetilde{a}_s:=\widetilde{b}(h)_s+\int(\widetilde{w}(s,x)-h(\widetilde{w}(s,x))\,Q_1(dx)-\sum_{i=2}^3\int(h(\widetilde{w}(s,x))\,Q_i(dx).$$

We will now verify the conditions (I) and (L) for the coefficients $\widetilde{a},\widetilde{u},\widetilde{w}$, firstly to make sure that $\widetilde{a}$ is well-defined, and secondly to be able to apply Lemma 2.54. Note that, by definition

$$\begin{aligned}\widetilde{a}_s &= \widetilde{b}(h)_s+\int(x-h(x))\,(\widetilde{\kappa}_3)_s(dx)-\int h(x)\,(\widetilde{\kappa}_2+\widetilde{\kappa}_1)_s(dx)\\ &= \widetilde{b}_s-\int x\,(\widetilde{\kappa}_1+\widetilde{\kappa}_2)_s(dx),\end{aligned}$$

where the $\widetilde{\kappa}_i$ are defined (parallel to $\widetilde{b}, \widetilde{c}, \widetilde{F}$) by $\widetilde{\kappa}_i(\bar{\omega}, s, dx) := 1_{[0,t_{k+1}-t_k)}(s)\kappa_i((\iota(X^{t_k}, \bar{\omega}), t_k+s), x)$. Now observe that $\|\iota(X^{t_k}, \check{\omega})-\iota(X^{t_k}, \bar{\omega})\|^*_{t_k+s} \leq \|\check{\omega}-\bar{\omega}\|^*_s$ and $\|\iota(X^{t_k}, \bar{\omega})\|^*_{t_k+s} \leq \|X^{t_k}\|^*_{t_k} + \|\bar{\omega}\|^*_s$ for any $s \in \mathbb{R}_+$, $\check{\omega}, \bar{\omega} \in \mathbb{D}^d$. Hence, the Lipschitz and growth conditions (L), (G) and therefore also the implied integrability conditions (I) for the SDE $(\varnothing, \varepsilon_x, \widetilde{a}, \widetilde{u}, \widetilde{w}, Q)^{SDE}$ with

$$\widetilde{a}_s(\bar{\omega}) = 1_{[0,t_{k+1}-t_k)}(s)\left(b_{t_k+s}(\iota(X^{t_k}, \bar{\omega})) - \int x\,(\widetilde{\kappa}_1 + \widetilde{\kappa}_2)((\iota(X^{t_k}, \bar{\omega}), t_k+s), dx)\right)$$

etc. follow as in Proposition 2.63 from the conditions in Theorem 2.37. By Lemma 2.54 we can now conclude that $P^{(X^{t_{k+1}-}_{t_k+s})_{s\in\mathbb{R}_+}|X^{t_k}}(\omega) = \widetilde{P}^{(X^{t_{k+1}-}_{t_k+s})_{s\in\mathbb{R}_+}|X^{t_k}}(\omega)$ for $P|_{\sigma(X^{t_k})} = \widetilde{P}|_{\sigma(X^{t_k})}$-almost all $\omega \in \mathbb{D}^d$. Thus, $P^{X^{t_{k+1}-}} = \widetilde{P}^{X^{t_{k+1}-}}$, i.e. $P|_{\mathcal{D}^d_{t_{k+1}-}} = \widetilde{P}|_{\mathcal{D}^d_{t_{k+1}-}}$. By Remark 3 in Section 2.4 we have that $K((X(\omega), t_{k+1}), \cdot)$ is a version of $P^{\Delta X_{t_{k+1}}|\mathcal{D}^d_{t_{k+1}-}}(\omega)$ for $P|_{\mathcal{D}^d_{t_{k+1}-}}$-almost all $\omega \in \mathbb{D}^d$ and likewise for $\widetilde{P}$. This implies $P^{X^{t_k}} = \widetilde{P}^{X^{t_k}}$, and hence we are done. □

Theorem 2.37 now follows from the Lemmas 2.55 and 2.64. □

PROOF OF COROLLARY 2.43. We will not use Theorem 2.37 for the proof of Corollary 2.43 since, in the case of unbounded jump intensity, the jump measure F_t must have a continuous density around 0. For PII, however, this restriction is not necessary.

By Lemma 2.29, it suffices to show that the martingale problem $\mathscr{I}(\sigma(X_0), X|\eta; B(h), C, \nu)$ on $(\mathbb{D}^d, \mathcal{D}^d, (\mathcal{D}^d_t)_{t\in\mathbb{R}_+})$ has a unique solution, where $B(h), C, \nu$ are defined in that statement. By JS, III.2.16 this is indeed the case. Moreover, we know from JS, II.4.15 and JS, II.4.19 that for $\eta = \varepsilon_0$, any solution-process X is a process with independent increments, which, in addition, has stationary increments if and only if the coefficients of the martingale problem are constant (i.e. they do not depend on t, either). □

PROOF OF THE EXAMPLES. *1.* Choose $u = 0$, $\Gamma := \varepsilon_{-1} + \varepsilon_1$ on S^0, $\kappa_1 := \kappa_2 := 0$. Moreover, define $g : \mathbb{D}^2 \times \mathbb{R}_+ \times S^0 \times \mathbb{R}_+ \to \mathbb{R}^2$ and $\rho : \mathbb{D}^2 \times \mathbb{R}_+ \times S^0 \times \mathbb{R}_+ \to \mathbb{R}_+$ by $g(\bar{\omega}, t, n, r) = (\sigma n r, |r|)$ and $\rho(\bar{\omega}, t, n, r) = h(\bar{\omega}^2_t)\varphi(r)$, where $\bar{\omega}^2$ denotes the second component of $\bar{\omega}$ (*not* $\bar{\omega}$ squared). One easily verifies that the measure $\kappa_3((\bar{\omega}, t), \cdot)$ in Theorem 2.37 indeed equals $F((\bar{\omega}, t), \cdot)$ for any $(\bar{\omega}, t) \in \mathbb{D}^2 \times \mathbb{R}_+$. Observe that

$$\int |x_1| F((\bar{\omega}, t), d(x_1, x_2)) = \int \sigma |x| h(\bar{\omega}^2_t) \frac{1}{|x|} e^{-|x|}\, dx \leq 2\sigma(\alpha + \|\bar{\omega}\|^*_t)$$

and

$$\int |x|^2 F((\bar{\omega}, t), d(x_1, x_2)) = \int (\sigma^2 x^2 + x^2) h(\bar{\omega}^2_t) \frac{1}{|x|} e^{-|x|}\, dx \leq (1 + \sigma^2)\gamma(\alpha + \|\bar{\omega}\|^*_t)$$

for any $(\bar{\omega}, t) \in \mathbb{D}^2 \times \mathbb{R}_+$, where $\gamma := \int |x| e^{-|x|}\, dx < \infty$. From these inequalities the growth conditions easily follow. Now let $R(t, n) := \infty$ for any $(t, n) \in \mathbb{R}_+ \times S^0$.

We have $D_1\rho(\bar\omega,t,n,r)(\omega) = \frac{1}{r}e^{-r}h'(\bar\omega_t^2)\omega$ and hence $\|D_1\rho(\bar\omega,t,n,r)\| \le \frac{1}{r}e^{-r}$ for any $(\bar\omega,t,n,r) \in \mathbb{D}^2 \times \mathbb{R}_+ \times S^0 \times \mathbb{R}_+$ and any $\omega \in \mathbb{D}^2$. Since $\int_r^\infty \frac{1}{\tilde r}e^{-\tilde r}\,d\tilde r < \infty$, the regularity conditions on ρ hold. It is straightforward to verify the Lipschitz conditions for b. Moreover, we may take $L_3(t,n) := 1+\sigma$ and $L_4(t,n,r) := 0$ for any $(t,n,r) \in \mathbb{R}_+ \times S^0 \times \mathbb{R}_+$. It remains to show the integrability conditions involving L_3 in Theorem 2.37. Since $\rho(\bar\omega,s,n,r) \ge \frac{\alpha}{2}\frac{1}{r}e^{-r}$ and $\|D_1\rho(\bar\omega,s,n,r)\| \le \frac{1}{r}e^{-r}$ for any $(\bar\omega,s,n,r)$), it suffices to prove that $\int_0^\infty re^r(\int_r^\infty \frac{1}{\tilde r}e^{-\tilde r}\,d\tilde r)^2\,dr < \infty$. We denote the integrand by $\iota(r)$. By application of l'Hospital's rule (cf. Heuser (1990a), (50.1)), it follows that $\iota(r)$ converges to 0 for $r \to 0$. Therefore $\iota(r)$ is bounded on $[0,1]$, and we have $\int_0^1 \iota(r)\,dr < \infty$. Moreover, $\int_r^\infty re^{-r}\,dr < \infty$ for any $r \ge 1$. By Theorem 2.37 we can now conclude that the martingale problem has a unique solution-measure.

2. All definitions will be as in the previous example, except for g and ρ, which will now be given by $g(\bar\omega,t,n,r) := (\sigma nrh(\bar\omega_t^2), |r|h(\bar\omega_t^2))$ and $\rho(\bar\omega,t,n,r) := \varphi(r)$. The bulk of the proof follows as above, but note that this time ρ does not depend on $\bar\omega$ and hence $D_1\rho = 0$. Choose $L_4(t,n,r) := (1+\sigma)r$ for $(t,n,r) \in \mathbb{R}_+ \times S^0 \times \mathbb{R}_+$. The integrability condition containing L_4 in Theorem 2.37 now follows from $\int_0^\infty r^2\frac{1}{r}e^{-r}\,dr = \int_0^\infty re^{-r}\,dr < \infty$. $\square$

2.9 Martingale Representation

It is well-known that any local martingale can be written as a stochastic integral with respect to a Wiener process if the latter generates the underlying filtration. The situation is more complicated for arbitrary semimartingales instead of Brownian motion. Not only does one need two integrals instead of one (the first one with respect to the continuous local martingale part (as in the Brownian case) and the second with respect to the compensated measure of jumps of the semimartingale), but this reprensetation also holds only under conditions connected with martingale problems (cf. JS, Theorem III.4.29). A sufficient condition is given in the following

Theorem 2.65 *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ be a stochastic basis with $\mathcal{F} = \mathcal{F}_{\infty-}$, and let X be a $\mathbb{R}^d$-valued special semimartingale on that space. Assume that $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is the canonical filtration of X or its P-completion. Moreover, suppose that X is a solution-process to a martingale problem as in Definition 2.28, which has a unique solution-measure (e.g. by Theorem 2.37). Then for any local martingale M there is a process $H \in L^2_{\mathrm{loc}}(X^c)$ and a mapping $W \in G_{\mathrm{loc}}(\mu^X)$ such that*

$$M = M_0 + \int_0^{\cdot} H_s \cdot dX_s^c + \int_{[0,\cdot]\times\mathbb{R}^d} W(s,x)\,(\mu^X - \nu)(ds,dx)$$

(for notation cf. Appendix A). Moreover, M is an extended Grigelionis process. In particular, all local martingales have the representation property relative to X (cf. Appendix A, Definition A.9).

Corollary 2.66 *Under the conditions of the preceding theorem we have that for any $T \in \mathbb{R}_+$ and any $\mathcal{F}_T$-measurable, integrable random variable Y, there are H and W as in Theorem 2.65 such that for any $t \in [0,T]$ we have*

$$E(Y|\mathcal{F}_t) = E(Y|\mathcal{F}_0) + \int_0^t H_s \cdot dX_s^c + \int_{[0,t]\times\mathbb{R}^d} W(s,x)\,(\mu^X - \nu)(ds,dx).$$

Proofs

PROOF OF THEOREM 2.65. By assumption, $((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P), X)$ is the solution-process to a martingale problem $(\Theta, \eta, b, c, F, K)^M$ having a unique solution-measure. By Lemma 2.29, P is a solution to the martingale problem $\mathscr{s}(\sigma(X_0), X|(P|_{\sigma(X_0)}); B(h), C, \nu)$ relative to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ and X, where $B(h), C, \nu$ are defined in that lemma. Assume that $\widetilde{P}$ is another solution to this martingale problem. Again by Lemma 2.29, Statement 1 both P^X and $\widetilde{P}^X$ are solution-measures to the martingale problem $(\Theta, \eta, b, c, F, K)^M$, and hence $P^X = \widetilde{P}^X$. This implies that $P, \widetilde{P}$ coincide on the σ-field $\sigma(X)$ generated by X. Since $\mathcal{F}$ equals $\sigma(X)$ or its P-completion, we have that $P = \widetilde{P}$. From JS, III.4.29 we can now conclude that any local martingale has the representation property relative to X, which is to prove. By JS, III.4.7 and Proposition 2.24, M is an extended Grigelionis process. □

PROOF OF COROLLARY 2.66. The process M defined by $M_t := E(Y|\mathcal{F}_t)$ for any $t \in \mathbb{R}_+$ is a martingale. □

Chapter 3

Markets, Strategies, Prices

In this chapter we generalize the approach presented in Section 1.2 to a continuous-time setting. One should note that discrete-time models are always regarded here as a special case of this more general framework. Contrary to the introduction, we attach importance to mathematical rigour. We rely heavily on the notions of Chapter 2 (mainly Sections 2.2 – 2.4). The proofs are again located at the end of each section. For a discussion of the economical motivation, application and limitation of our approach we refer the reader to Section 1.2.

3.1 The Market Model

As in Subsection 1.2.1, we confine ourselves here to frictionless markets with a finite number of traded securities. We work mathematically with a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ in the sense of Section 2.2. Ω here denotes the set of possible states of the market in a more or less abstract sense. The σ-field $\mathcal{F}_t$ represents the information that is available to traders up to time t. It is assumed to be the same for any investor. We treat the market as a random system governed by some objective probability measure P, which is neither subject to personal beliefs nor usually a risk-neutral measure for contingent claim valuation. We assume that the probability of various events is, in principle, known to the investors, either intuitively by market experience or by statistical observation. As in the introduction, we consider securities termed $0, \dots, n$, which are modelled by their respective price processes $S^0, \dots, S^n$. As a *numeraire* by which all other securities are discounted, Security 0 can be interpreted as the benchmark for risklessness. We assume that its value S^0_t is positive for any $t \in \mathbb{R}_+$. By Z^i with $Z^i_t := S^i_t/S^0_t$ for any $t \in \mathbb{R}_+$ we denote the *discounted price process* of asset i. The $\mathbb{R}^{n+1}$-valued stochastic process $Z = (Z^0, \dots, Z^n)$ on the given filtered probability space is often called *market*. For the rest of this chapter we make the following weak

Assumption. The $\mathbb{R}^{n+1}$-valued stochastic process $Z = (Z^0, \dots Z^n)$ is an extended Grigelionis process with extended characteristics $(\Theta, P^{Z_0}, b, c, F, K)^E$.

Using the usual embedding (cf. Appendix A) one may treat discrete-time settings in a continuous-time framework.

Definition 3.1 Let Θ be a discrete set. We call the market $Z = (Z^0, \dots, Z^n)$ Θ-*discrete* if $t \mapsto \mathcal{F}_t$ is constant and $t \mapsto Z_t$ is P-almost surely constant on the open intervals between neighbouring points of $\Theta \cup \{0, \infty\}$.

As usual, trading means picking a $(n+1)$-dimensional, predictable stochastic process φ, termed the *trading strategy*. φ^i_t denotes the number of securities i you hold at time t. In continuous-time models an appropriate choice of the set of admitted trading strategies is not easy. If one allows too many portfolios, then even very decent models contain arbitrage opportunities, e.g. modified versions of the doubling strategy. If one restricts the set too strictly, one may lose a number of perfect hedging strategies of the Black-Scholes type. We do not follow the classical choice proposed in Harrison & Pliska (1981) for two reasons. Firstly, the value process of the portfolio is assumed to be bounded from below, which e.g. in a discrete-time stock price model with normal log-returns may prohibit even the short-sale of a single stock. In addition, the set of admissible strategies depends in a sophisticated way on some equivalent martingale measure, which is not very intuitive from an economic point of view. Instead, we introduce two kinds of portfolios for the market $(Z^0, \dots, Z^n)$.

Definition 3.2

1. We call any predictable $\mathbb{R}^{n+1}$-valued stochastic process $\varphi = (\varphi^0, \dots, \varphi^n)$ a *(trading) strategy* or *portfolio*. The set of all strategies is denoted $\mathfrak{A}$.

2. We call a strategy $\varphi \in \mathfrak{A}$ *feasible* if it is of the form

$$\varphi = \psi_0 1_{[T_0, T_1]} + \sum_{i=1}^{m-1} \psi_i 1_{]T_i, T_{i+1}]},$$

   where $m \in \mathbb{N}$, $0 = T_0 \leq T_1 \leq \ldots \leq T_m$ are stopping times and ψ_i is a bounded $\mathcal{F}_{T_i}$-measurable random variable for $i = 0, \dots, m$. The set of all feasible strategies is denoted by $\mathfrak{S}$.

We think that "real" trading resembles feasible portfolios. Nevertheless, we often consider general predictable strategies as limiting cases of feasible ones. As in the introduction, we distinguish between different kinds of traders. A *speculator* is free to choose his portfolio among the whole of $\mathfrak{A}$ (resp. $\mathfrak{S}$) whereas a *hedger* is confined to some subset $\mathfrak{M} \subset \mathfrak{A}$ (resp. $\mathfrak{M} \cap \mathfrak{S} \subset \mathfrak{S}$). We consider usually only fixed positions in certain securities as in Subsection 1.2.1, or alternatively, short-sale restrictions. Both situations correspond to convexly restricted sets of strategies in the sense of the following

Definition 3.3 Let $\mathfrak{M} \subset \mathfrak{A}$ be non-empty. We say that $\mathfrak{M}$ is *convexly restricted* if for some $q \in \mathbb{N}$ there are $(\mathcal{P} \otimes \mathcal{B}^{n+1})$-measurable mappings $g^1, \dots, g^q : \Omega \times \mathbb{R}_+ \times \mathbb{R}^{n+1} \to \mathbb{R}$ such that for some $p \in \{0, 1, \dots, q\}$ the following conditions hold.

1. For any $(\omega, t) \in \Omega \times \mathbb{R}_+$ the function $g^j(\omega, t) : \mathbb{R}^{n+1} \to \mathbb{R}$ is convex for any $j \in \{1, \ldots, p\}$ and affine for any $j \in \{p+1, \ldots, q\}$ (cf. e.g. Rockafellar (1970)).

2. For any $(\omega, t) \in \Omega \times \mathbb{R}_+$ we have $\{\psi \in \mathbb{R}^{n+1} : g^j(\psi) < 0$ for any $j \in \{1, \ldots, p\}\} \neq \varnothing$.

3. $\mathfrak{M} = \{\varphi \in \mathfrak{A}$: For any $t \in \mathbb{R}_+$ we have $g^j_t(\varphi_t) \leq 0$ for $j \in \{1, \ldots, p\}$ and $g^j_t(\varphi_t) = 0$ for $j \in \{p+1, \ldots, q\}\}$.

We call the constraints "$g^1, \ldots, g^p \leq 0$, $g^{p+1}, \ldots g^q = 0$" *fixed* if the mappings $(\omega, t, x) \mapsto g^j_t(x)(\omega) := g^j(\omega, t, x)$ do not depend on (ω, t).

Example. Let $J \subset \{0, \ldots, n\}$ and $\psi^j \in \mathbb{R}$ for any $j \in J$. Then the set $\mathfrak{M} := \{\varphi \in \mathfrak{A} : \varphi^j(\omega, t) = \psi^j$ for any $j \in J$, $(\omega, t) \in \Omega \times \mathbb{R}_+\}$ is convexly restricted with fixed constraints. This is the state of affairs for the hedger in the introduction.

Analogously to Section 1.2, we now define the corresponding gain processes.

Definition 3.4 Let $\varphi \in \mathfrak{A}$ be locally bounded (cf. Lemma A.1 in the appendix). The process $(G(\varphi)_t)_{t \in \mathbb{R}_+}$ defined by $G(\varphi)_t := \int_0^t \varphi_s \cdot dZ_s$ for any $t \in \mathbb{R}_+$ is called *discounted gain process* of φ.

Remark. By Lemma 2.22, $G(\varphi)$ is an extended Grigelionis process.

In the continuous-time setting arbitrage is defined as in the introduction, but relative to feasible portfolios.

Definition 3.5 We call the trading strategy $\varphi \in \mathfrak{S}$ *arbitrage* if there is a $T \in \mathbb{R}_+$ such that $G_T(\varphi) \geq 0$ P-almost surely and $P(G_T(\varphi) > 0) > 0$. If there exists such a strategy, we say that *the market allows arbitrage*.

Lemma 3.6 *We have equivalence between*

1. *The market allows arbitrage.*

2. *There are bounded stopping times $T_1 \leq T_2$ and a bounded, $\mathcal{F}_{T_1}$-measurable, $\mathbb{R}^{n+1}$-valued random variable ψ such that $\psi \cdot (Z_{T_2} - Z_{T_1}) \geq 0$ P-almost surely and $P(\psi \cdot (Z_{T_2} - Z_{T_1}) > 0) > 0$. (If the market is Θ-discrete, one can even choose $T_1 = s$, $T_2 = t$ for two neighbouring points s, t in $\Theta \cup \{0\}$.)*

The following lemma expresses the well-known fact that the existence of an *equivalent martingale measure (EMM)* implies that the market allows no arbitrage.

Lemma 3.7 *If for any $T \in \mathbb{R}_+$ there is a probability measure P^* on $\mathcal{F}_T$ such that $P^* \sim P|_{\mathcal{F}_T}$ and Z^T (or at least $Z^T - Z_0$) is a P^*-martingale, then the market allows no arbitrage.*

Note that no equivalence is claimed in the previous lemma. Observe also that $\mathfrak{A}$ may contain "arbitrage," but that we do not consider it as such as long as it is not feasible.

Proofs

PROOF OF LEMMA 3.6. $1 \Rightarrow 2$: Let $\varphi = \psi_0 1_{[0,T_1]} + \sum_{i=1}^{m-1} \psi_i 1_{]T_i,T_{i+1}]} \in \mathfrak{S}$ and $T \in \mathbb{R}_+$ with $G_T(\varphi) \geq 0$ and $P(G_T(\varphi) > 0) > 0$. W.l.o.g., $T_m \leq T$. Moreover, let $k \in \{1, \ldots, m\}$ be maximal with the property that $P(G_{T_{k-1}}(\varphi) < 0) > 0$ or $G_{T_{k-1}}(\varphi) \leq 0$ P-almost surely. In the second case we have $\psi_{k-1} \cdot (Z_{T_k} - Z_{T_{k-1}}) = G_{T_k}(\varphi) - G_{T_{k-1}}(\varphi) \geq G_{T_k}(\varphi)$ which is, by assumption, non-negative and positive with positive probability. In the case $P(G_{T_{k-1}}(\varphi) < 0) > 0$ define $A := \{G_{T_{k-1}}(\varphi) < 0\} \in \mathfrak{F}_{T_{k-1}}$. Then we have that $P(A) > 0$ and $(1_A \psi_{k-1})(Z_{T_k} - Z_{T_{k-1}}) = 1_A(G_{T_k}(\varphi) - G_{T_{k-1}}(\varphi))$ is strictly positive on A. Now we consider the case that the market is Θ-discrete. Since any term $1_{]T_i,T_{i+1}]}$ in the definition of φ can be written as $\sum_{l\in\mathbb{N}} 1_{]T_i \vee t_l, T_{l+1} \wedge t_{l+1}]}$ where $\Theta \cup \{0, \infty\} = \{t_0, t_1, \ldots\}$, it follows from the above proof that ψ, T_1, T_2 in Statement 2 can be chosen such that $t_l \leq T_1 \leq T_2 \leq t_{l+1}$ for some $l \in \mathbb{N}$. If we set $\widetilde{\psi} := \psi 1_{\{T_1 < t_{l+1}\} \cap \{T_2 < t_{l+1}\}^C}$, then we have $\widetilde{\psi}$ is $\mathfrak{F}_{t_{l+1}-} = \mathfrak{F}_{t_l}$-measurable and $\widetilde{\psi} \cdot (Z_{t_{l+1}} - Z_{t_l}) = \psi \cdot (Z_{T_2} - Z_{T_1})$.

$2 \Rightarrow 1$: The strategy $\psi \cdot 1_{]T_1,T_2]}$ is an arbitrage. □

PROOF OF LEMMA 3.7. Assume that there exist ψ, T_1, T_2 as in Statement 2 of Lemma 3.6. Moreover, let $T \in \mathbb{R}_+$ with $T \geq T_2$. By Doob's stopping theorem (cf. JS, I.1.39) we have $E^*(\psi \cdot (Z_{T_2} - Z_{T_1})) = \psi \cdot E^*(E^*(Z_{T_2} - Z_0 | \mathfrak{F}_{T_1}) - (Z_{T_1} - Z_0)) = 0$, where E^* denotes expectation relative to P^*. Since $\psi \cdot (Z_{T_2} - Z_{T_1}) \geq 0$ P- and hence P^*-almost surely, this implies $\psi \cdot (Z_{T_2} - Z_{T_1}) = 0$ P^*- and hence P-almost surely, in contradiction to the assumption. □

3.2 Optimal Strategies

As in Subsection 1.2.2, we define optimal strategies in terms of local maximization of expected utility. We begin by defining utility functions as in the introduction.

Definition 3.8 $u : \mathbb{R} \to \mathbb{R}$ is called *utility function* if

1. u is three times continuously differentiable.
2. The derivatives u', u'', u''' are bounded and $\lim_{x\to\infty} u'(x) = 0$.
3. $u(0) = 0$, $u'(0) = 1$
4. $u'(x) > 0$ for any $x \in \mathbb{R}$
5. $u''(x) < 0$ for any $x \in \mathbb{R}$

$\kappa := -u''(0)$ is called *risk aversion*.

Alhough all we do can be done with any utility function in the sense of the previous definition, we usually focus on standard utility functions.

Definition 3.9 For any $\kappa \in \mathbb{R}_+^*$ the function $u_\kappa : \mathbb{R} \to \mathbb{R},\ x \mapsto \frac{1}{\kappa}(1 + \kappa x - \sqrt{1 + \kappa^2 x^2})$ is called *standard utility function* with risk aversion κ.

Remarks.

1. For any $\kappa \in \mathbb{R}_+^*$, $x \in \mathbb{R}$ we have
$$u'_\kappa(x) = 1 - \frac{\kappa x}{\sqrt{1 + \kappa^2 x^2}},\quad u''_\kappa(x) = \frac{-\kappa}{(1 + \kappa^2 x^2)^{\frac{3}{2}}},$$
$$u'_\kappa(x) = u'_1(\kappa x).$$
In particular, u_κ is a utility function.

2. One may wonder why we claim $\lim_{x \to -\infty} u'(x)$ to be finite, which rules out utility functions as e.g. $u(x) := 1 - e^{-x}$. One reason is that we would otherwise have to impose strong moment conditions in order to obtain hedging strategies, derivative prices etc. Such a limitation of the set of models under consideration contradicts our intentions. Secondly, observe that the expected utility of the gain $E(u(\Delta G_t(\varphi)))$ in Subsection 1.2.2 has an easy interpretation, especially for standard utility functions. For small κ it is close to the expected gain $E(\Delta G_t(\varphi))$, whereas for large risk aversion it approximates twice the expected loss $E(0 \wedge \Delta G_t(\varphi))$. For arbitrary κ it is something in between.

For the rest of this section, the utility function u and its risk aversion κ is fixed (unless otherwise stated). In Chapter 1 we define optimal strategies in terms of the local gains ΔG_t over one period. Since there is no shortest possible time-span in a continuous-time framework, a transfer of this approach is not evident. However, by means of a limiting argument, we will be able to define a natural counterpart. To begin with, we define the expected utility of a strategy for arbitrary (short) time intervals.

Definition 3.10 For any $\varphi \in \mathfrak{S}$, $t, t' \in \mathbb{R}_+$ with $t < t'$ we define the *expected utility of φ in the interval* $[t, t']$ by
$$U(\varphi, t, t') := \begin{cases} E(u(G_{t'}(\varphi) - G_{t-}(\varphi))) & \text{if } E(|u(G_{t'}(\varphi) - G_{t-}(\varphi))|) < \infty \\ -\infty & \text{else.} \end{cases}$$
(We set $G_{0-}(\varphi) := 0$.)

The limiting behaviour of the expected utility for small time intervals will later be expressed in terms of local utility in the sense of the following

Definition 3.11 For any $\psi \in \mathbb{R}^{n+1}$, $t \in \mathbb{R}_+$ we call the $\mathbb{R}^2$-valued random variable $(\Gamma_t(\psi), \gamma_t(\psi))$ *local utility* of ψ in t, where
$$\Gamma_t(\psi) := \int u(\psi \cdot x)\, K_t(dx)$$
$$\gamma_t(\psi) := \psi \cdot b_t - \frac{1}{2}\kappa \psi^\top c_t \psi + \int (u(\psi \cdot x) - \psi \cdot x)\, F_t(dx).$$

Remarks.

1. Note that $\Gamma_t(\psi) = E(u(\psi \cdot \Delta Z_t)|\mathcal{F}_{t-})$ P-almost surely by Remark 3 in Section 2.4.

2. $\Gamma_t(\psi)$ and $\gamma_t(\psi)$ do not depend on ψ^0 because Z^0 is constant.

The following lemma shows that the local utility is well-defined and unique outside some null sets.

Lemma 3.12 *Let $\varphi \in \mathfrak{A}$.*

1. *There exists a version $(\Gamma_t(\varphi_t), \gamma_t(\varphi_t))_{t\in\mathbb{R}_+}$ of the local utility of φ_t in t for any $t \in \mathbb{R}_+$.*

2. *Let $(\Gamma_t(\varphi_t), \gamma_t(\varphi_t))_{t\in\mathbb{R}_+}$ and $(\widetilde{\Gamma}_t(\varphi_t), \widetilde{\gamma}_t(\varphi_t))_{t\in\mathbb{R}_+}$ be two versions of the local utility of φ_t in t for any $t \in \mathbb{R}_+$. Then we have*

   (a) $\Gamma_t(\varphi_t(\omega))(\omega) = \widetilde{\Gamma}_t(\varphi_t(\omega))(\omega)$ *up to indistinguishability.*

   (b) There is some $(P \otimes \lambda)$-null set $N \in \mathcal{P}$ such that $\gamma_t(\varphi_t(\omega))(\omega) = \widetilde{\gamma}_t(\varphi_t(\omega))(\omega)$ for any $(\omega, t) \in N^C$.

3. *Up to an evanescent set for Γ and a $(P \otimes \lambda)$-null set for γ, Definition 3.11 does not depend on the choice of Θ in the extended characteristics of X.*

For Theorem 3.14 below, we need the following integrability conditions.

Definition 3.13 1. We say that the market $Z = (Z^0, \dots, Z^n)$ meets *regularity condition (RC 1)* if there is a $\varepsilon > 0$ such that for any $t \in \mathbb{R}_+$ we have

$$
\begin{aligned}
&E\Big(\int_0^t |b_s|^{1+\varepsilon}\, ds\Big) < \infty,\\
&\sum_{i,j=0}^{n} E\Big(\int_0^t |c_s^{ij}|^{1+\varepsilon}\, ds\Big) < \infty,\\
&E\bigg(\int_0^t \Big(\int (|x|^2 \wedge |x|)\, F_s(dx)\Big)^{1+\varepsilon} ds\bigg) < \infty,\\
&E\bigg(\Big(\int |x|\, K_t(dx)\Big)^{1+\varepsilon}\bigg) < \infty.
\end{aligned}
$$

2. We say that the market meets *regularity condition (RC 1')* if it meets (RC 1) or if it is Θ-discrete.

The following theorem states that for small time intervals, the expected utility of a feasible strategy can be approximated by an expression that depends only on the local utility.

Theorem 3.14 *Assume that regularity condition (RC 1) holds. Let $\varphi \in \mathfrak{S}$. For any $t \in \mathbb{R}_+$ we have*

$$U(\varphi, t, t') = E(\Gamma_t(\varphi_t)) + o(1),$$

where $o(1) \to 0$ *for* $t' \downarrow t$. *Moreover, for* λ*-almost all* $t \in \mathbb{R}_+ \setminus \Theta$ *we have*

$$U(\varphi, t, t') = E\Big(\int_t^{t'} \gamma_s(\varphi_s)\, ds \Big) + o(t' - t),$$

where $\frac{o(t'-t)}{t'-t} \to 0$ *for* $t' \downarrow t$.

Remarks.

1. Definition 3.10 can be extended to abitrary locally bounded strategies. Then the previous theorem holds for any bounded $\varphi \in \mathfrak{A}$.

2. If Z (but not necessarily the market) is Θ-discrete, then we even have $U(\varphi, t, t') = E(\Gamma_t(\varphi_t))$ if $t' - t$ is sufficiently small.

In discrete-time models we can do without the regularity condition (RC 1) if we consider a slightly different notion of expected utility.

Definition 3.15 For any $\varphi \in \mathfrak{S}$, $t, t' \in \mathbb{R}_+$ with $t < t'$, we define the *conditional expected utility of* φ *in the interval* $[t, t']$ by

$$\widetilde{U}(\varphi, t, t') := \begin{cases} E(u(G_{t'}(\varphi) - G_{t-}(\varphi))|\mathcal{F}_{t-}) & \text{if } E(|u(G_{t'}(\varphi) - G_{t-}(\varphi))||\mathcal{F}_{t-}) < \infty \\ -\infty & \text{else.} \end{cases}$$

Lemma 3.16 *Assume that the market is* Θ*-discrete. Let* $\varphi \in \mathfrak{S}$, $t \in \mathbb{R}_+$. *Then we have*

$$\widetilde{U}(\varphi, t, t') = \Gamma_t(\varphi_t)$$

if $t' \geq t$ *is small enough.*

As in the discrete-time setting in the introduction, we want to call a strategy φ *optimal* (relative to a given set of strategies $\mathfrak{M} \subset \mathfrak{A}$ and a utility function u) if it maximizes the expected utility for very short time intervals, where here *very short* is to be understood in a limiting sense. By Theorem 3.14 (or Lemma 3.16) we know that, up to a small error $o(1)$ resp. $o(t' - t)$, the expected utility depends monotonically on $\Gamma_t(\varphi_t)$, $\gamma_t(\varphi_t)$. Therefore, it makes sense to call a strategy optimal if its local utility is maximal compared to all strategies in $\mathfrak{M}$.

Definition 3.17 Let $\mathfrak{M} \subset \mathfrak{A}$. We call a strategy $\varphi \in \mathfrak{M}$ u*-optimal for* $\mathfrak{M}$ if the following conditions hold:

1. P-almost surely and for any $t \in \mathbb{R}_+$ we have

$$\Gamma_t(\varphi_t) \geq \Gamma_t(\widetilde{\varphi}_t) \text{ for any } \widetilde{\varphi} \in \mathfrak{M}.$$

2. Outside some $(P \otimes \lambda)$-null set $N \in \mathcal{P}$ we have

$$\gamma_t(\varphi_t) \geq \gamma_t(\widetilde{\varphi}_t) \text{ for any } \widetilde{\varphi} \in \mathfrak{M}.$$

A u-optimal strategy is generally not feasible. Hence, from a practical point of view it is only useful as a limiting object, i.e. if we can approximate the optimal portfolio and its local utility by feasible strategies.

Definition 3.18 Let $\varphi \in \mathfrak{A}$ and $(\varphi^m)_{m\in\mathbb{N}}$ a sequence in $\mathfrak{S}$. We call $(\varphi^m)_{m\in\mathbb{N}}$ an *approximating sequence* for $\varphi \in \mathfrak{A}$ if

1. P-almost surely we have
$$\Gamma_t(\varphi_t^m) \overset{m\to\infty}{\longrightarrow} \Gamma_t(\varphi_t) \text{ for any } t \in \mathbb{R}_+.$$
2. Outside some $(P \otimes \lambda)$-null set $N \in \mathcal{P}$ we have
$$\gamma_t(\varphi_t^m) \overset{m\to\infty}{\longrightarrow} \gamma_t(\varphi_t).$$
3. Outside some $(P \otimes (\lambda + \sum_{s\in\Theta} \varepsilon_s))$-null set we have
$$\varphi^m \overset{m\to\infty}{\longrightarrow} \varphi.$$

Definition 3.19 A set $\mathfrak{M} \subset \mathfrak{A}$ is called *regular* if, for any $\varphi \in \mathfrak{M}$, there exists an approximating sequence $(\varphi^m)_{m\in\mathbb{N}}$ in $\mathfrak{S} \cap \mathfrak{M}$.

The following lemma states that the set $\mathfrak{A}$ of all strategies, which corresponds to the speculator, is regular.

Lemma 3.20 *For any $\varphi \in \mathfrak{A}$ there exists an approximating sequence $(\varphi^m)_{m\in\mathbb{N}}$. If φ is locally bounded, then the approximating sequence can be chosen such that P-almost surely we have $G(\varphi^m) \overset{m\to\infty}{\longrightarrow} G(\varphi)$ uniformly on any interval $[0,t]$.*

Corollary 3.21 *Let $\mathfrak{M} \subset \mathfrak{A}$ be convexly restricted with fixed constraints. Then $\mathfrak{M}$ is regular. More precisely, for $\varphi \in \mathfrak{M}$ the sequence $(\varphi^m)_{m\in\mathbb{N}}$ in Lemma 3.20 can be chosen in $\mathfrak{S} \cap \mathfrak{M}$.*

The following theorem gives necessary and sufficient conditions for u-optimal strategies, which permits explicit calculations. Its corollary focuses on the hedger from Subsection 1.2.2 and represents a continuous-time counterpart of Lemma 1.2.

Theorem 3.22 *Let $\mathfrak{M} \subset \mathfrak{A}$ be convexly restricted by constraints $g^1, \dots, g^p \le 0$, $g^{p+1}, \dots, g^q = 0$ such that the mappings $g^j(\omega,t) : \mathbb{R}^{n+1} \to \mathbb{R}$ are differentiable for any $(\omega,t) \in \Omega \times \mathbb{R}_+$, $j \in \{1,\dots,q\}$. Moreover, let $\varphi \in \mathfrak{M}$. Then φ is u-optimal for $\mathfrak{M}$ if and only if the following two conditions hold.*

1. *P-almost surely and for any $t \in \mathbb{R}_+$ there exist $\lambda_1, \dots, \lambda_q \in \mathbb{R}$ with $\lambda_j \ge 0$ and $\lambda_j g^j(\varphi_t) = 0$ for $j = 1, \dots, p$ such that*
$$\int x^i u'(\varphi_t \cdot x)\, K_t(dx) - \sum_{j=1}^{q} \lambda_j D_i g^j(\varphi_t) = 0 \text{ for } i = 0, \dots, n.$$

2. *Outside some $(P \otimes \lambda)$-null set $N \in \mathcal{P}$, there exist $\lambda_1, \dots, \lambda_q \in \mathbb{R}$ with $\lambda_j \geq 0$ and $\lambda_j g^j(\varphi_t) = 0$ for $j = 1, \dots, p$ such that*

$$b_t^i - \kappa c_t^{i\cdot} \cdot \varphi_t + \int x^i (u'(\varphi_t \cdot x) - 1)\, F_t(dx) - \sum_{j=1}^q \lambda_j D_i g^j(\varphi_t) = 0 \text{ for } i = 0, \dots, n.$$

Corollary 3.23 *Let $\mathfrak{M} = \{\varphi \in \mathfrak{A} : \varphi_t^i = \psi^i$ for any $i \in \{k, \dots, n\}$, $t \in \mathbb{R}_+\}$ for some $k \in \{0, \dots, n+1\}$ and some $\psi^k, \dots, \psi^n \in \mathbb{R}$. Moreover, let $\varphi \in \mathfrak{M}$. Then φ is u-optimal for $\mathfrak{M}$ if and only if the following two conditions hold.*

1. *P-almost surely and for any $t \in \mathbb{R}_+$ we have*

$$\int x^i u'(\varphi_t \cdot x)\, K_t(dx) = 0 \text{ for } i = 1, \dots, k-1.$$

2. *Outside some $(P \otimes \lambda)$-null set $N \in \mathcal{P}$ we have*

$$b_t^i - \kappa c_t^{i\cdot} \cdot \varphi_t + \int x^i (u'(\varphi_t \cdot x) - 1)\, F_t(dx) = 0 \text{ for } i = 1, \dots, k-1.$$

Remark. The preceding corollary also holds for predictable processes $\psi^k, \dots, \psi^n$ instead of fixed real numbers.

Especially in markets with redundant securities, optimal strategies are far from unique. However, the following result shows that they do not differ by much as far as their financial gains are concerned.

Lemma 3.24 *Let $\mathfrak{M} \subset \mathfrak{A}$ as in Theorem 3.22. Moreover let $\varphi, \widetilde{\varphi} \in \mathfrak{M}$ be locally bounded u-optimal strategies for $\mathfrak{M}$. Then we have $G(\varphi) = G(\widetilde{\varphi})$ up to indistinguishability.*

So far we have not shown that optimal strategies actually exist. Sufficient conditions are given below.

Definition 3.25 We say that the market $Z = (Z^0, \dots, Z^n)$ meets *regularity condition (RC 2)* if the following two conditions hold.

1. P-almost surely and for any $t \in \mathbb{R}_+$, there exists a $\psi \in \mathbb{R}^{n+1}$ such that

$$\int x^i u'(\psi \cdot x)\, K_t(dx) = 0 \text{ for } i = 0, \dots, n.$$

2. Outside some $(P \otimes \lambda)$-null set $N \in \mathcal{P}$, there exists a $\psi \in \mathbb{R}^{n+1}$ such that

$$b_t^i - \kappa c_t^{i\cdot} \cdot \psi + \int x^i (u'(\psi \cdot x) - 1)\, F_t(dx) = 0 \text{ for } i = 0, \dots, n.$$

Remark. Although u and κ appear in the above definition, condition (RC 2) does not depend on the chosen value of κ if one works with standard utility functions.

Theorem 3.26 *Assume that regularity condition (RC 2) holds. Let $\mathfrak{M} \subset \mathfrak{A}$ be as in Theorem 3.22 with the additional condition that all g^j are affine functions. Then there exists a u-optimal strategy $\varphi \in \mathfrak{M}$ for $\mathfrak{M}$.*

Corollary 3.27 *We have equivalence between*

1. *The market meets regularity condition (RC 2).*
2. *$\mathfrak{A}$ contains a u-optimal strategy for $\mathfrak{A}$. In other words, there exists a strategy that is u-optimal for the speculator.*

Let us turn to discrete markets as in the introduction.

Remark. If the market is $\mathbb{N}^*$-discrete, then Lemma 2.20 yields that

$$\Gamma_t(\psi) = \int u(\psi \cdot x)\, P^{\Delta Z_t|\mathfrak{F}_{t-1}} = E\Big(u\Big(\sum_{i=1}^n \psi^i \Delta Z_t^i\Big)\Big|\mathfrak{F}_{t-1}\Big) \quad P\text{-almost surely.}$$

Hence, maximization of $\psi \mapsto \Gamma_t(\psi)$ is exactly what is done in Subsection 1.2.2. Moreover, we have

$$\int x^i u'(\varphi_t \cdot x)\, K_t(dx) = E\Big(u'\Big(\sum_{j=1}^n \varphi_t^j \Delta Z_t^j\Big)\Delta Z_t^i\Big|\mathfrak{F}_{t-1}\Big).$$

Therefore the conditions in Corollary 3.23 and Lemma 1.2 coincide as well.

One may wonder whether regularity condition (RC 2) means a serious restriction of the class of markets under consideration. For practical purposes this is not the case. The following theorem shows that in discrete markets (RC 2) is equivalent to the absence of arbitrage.

Theorem 3.28 *For Θ-discrete markets we have equivalence between*

1. *For any $T \in \mathbb{R}_+$ there is a probability measure P^* on $\mathfrak{F}_T$ such that $P^* \sim P|_{\mathfrak{F}_T}$ and $(Z - Z_0)^T$ is a P^*-local martingale.*
2. *The market meets regularity condition (RC 2).*
3. *P-almost surely we have for any $t \in \Theta$ and any $\psi \in \mathbb{R}^{n+1}$ the implication*

   $$K_t(-H^\psi) = 0 \Rightarrow K_t(H^\psi) = 0,$$

   where $H^\psi := \{x \in \mathbb{R}^{n+1} : \psi \cdot x > 0\}$.
4. *The market allows no arbitrage.*

Remark. In particular, we have 1' $\Rightarrow$ 2, where

1'. For any $T \in \mathbb{R}_+$ there is a probability measure P^* on $\mathcal{F}_T$ such that $P^* \sim P|_{\mathcal{F}_T}$ and Z^T is a P^*-martingale.

Unfortunately, we doubt that any of the above inclusions holds for continuous-time markets as well.

We need the following results for Section 3.4.

Lemma 3.29 *1. For any $\varphi \in \mathfrak{A}$, $\kappa > 0$ we have*

$$\varphi \in \mathfrak{A} \textit{ is } u_\kappa\textit{-optimal for } \mathfrak{A} \Leftrightarrow \kappa\varphi \in \mathfrak{A} \textit{ is } u_1\textit{-optimal for } \mathfrak{A}.$$

2. *Let $\kappa_1, \dots, \kappa_p > 0$ and $\kappa = (\sum_{j=1}^p \kappa_j^{-1})^{-1}$. If $\varphi^{(j)} \in \mathfrak{A}$ is u_{κ_j}-optimal for $\mathfrak{A}$ for any $j \in \{1, \dots, p\}$, then the sum $\sum_{j=1}^p \varphi^{(j)}$ is u_κ-optimal for $\mathfrak{A}$.*

3. *If $\varphi^{(1)}, \dots, \varphi^{(p)}$ are as in the second statement and additionally $\sum_{j=1}^p \varphi^{(j),i} = 0$ for $i = l+1, \dots, n$, then there exists a u_1-optimal strategy $\varphi \in \mathfrak{A}$ with $\varphi^i = 0$ for $i = l+1, \dots, n$.*

4. *If $\varphi^{(1)}, \dots, \varphi^{(p)}$ are u-optimal strategies for $\mathfrak{A}$ with $\sum_{j=1}^p \varphi^{(j),i} = 0$ for $i = l+1$, $\dots, n$, then there exists a u-optimal strategy $\varphi \in \mathfrak{A}$ with $\varphi^i = 0$ for $i = l+1, \dots, n$.*

Proofs

PROOF OF LEMMA 3.12. *1.* We have to show that the integrals exist. By Lemma 2.18, there are versions of F, K such that on $\Omega \times \mathbb{R}_+$ we have identically $\int |x|\, K_t(dx) < \infty$ and $\int (|x|^2 \wedge |x|)\, F_t(dx) < \infty$. Since $|u(\psi \cdot x)| \leq \sup_{y \in \mathbb{R}} |u'(y)||\psi \cdot x| \leq \sup_{y \in \mathbb{R}} |u'(y)||\psi||x|$ for any $\psi, x \in \mathbb{R}^{n+1}$ we have $\int |u(\psi \cdot x)|\, K_t(dx) \leq \sup_{y \in \mathbb{R}} |u'(y)||\psi| \int |x|\, K_t(dx) < \infty$. For any $\psi, x \in \mathbb{R}^{n+1}$, there exist $\vartheta_1, \vartheta_2 \in [0,1]$ such that $u(\psi \cdot x) - \psi \cdot x = \psi \cdot x(u'(\vartheta_1 \psi \cdot x) - 1) = (\psi \cdot x)^2 \vartheta_1 u''(\vartheta_1 \vartheta_2 \psi \cdot x)$. Therefore, $|u(\psi \cdot x) - \psi \cdot x| \leq (|x|^2 \wedge |x|)(|\psi|^2 \sup_{y \in \mathbb{R}} |u''(y)| + |\psi| \sup_{y \in \mathbb{R}} |u'(y)|)$. Since $\int (|x|^2 \wedge |x|)\, F_t(dx) < \infty$, it follows that $\int |u(\psi \cdot x) - \psi \cdot x|\, F_t(dx) < \infty$ as well.

2. and 3. This follows immediately from Statement 2 in Lemma 2.18. $\square$

PROOF OF THEOREM 3.14. We prove the theorem for any bounded $\varphi \in \mathfrak{A}$. Fix $t \in \mathbb{R}_+$. For the proof of the first statement, observe that

$$u(G_{t'}(\varphi) - G_{t-}(\varphi)) = u\Big(\varphi_t \cdot \Delta Z_t + \int_0^{t'} 1_{[0,t]^C}(s) \varphi_s \cdot dZ_s\Big).$$

Since the mean value theorem implies

$$\Big|u\Big(\varphi_t \cdot \Delta Z_t + \int_0^{t'} 1_{[0,t]^C}(s) \varphi_s \cdot dZ_s\Big) - u(\varphi_t \cdot \Delta Z_t)\Big| \leq \sup_{x \in \mathbb{R}} |u'(x)| \Big| \int_0^{t'} 1_{[0,t]^C}(s) \varphi_s \cdot dZ_s \Big|,$$

it suffices to prove that $E(u(\varphi_t \cdot \Delta Z_t)) = E(\Gamma_t(\varphi_t))$ and $E(|\int_0^{t'} 1_{[0,t]^C}(s)\varphi_s \cdot dZ_s|) \to 0$ for $t' \to t$. The assumption $E((\int |x| \, K_t(dx))^{1+\varepsilon}) < \infty$ implies that $\int |x| \, \nu(\{t\} \times dx)$ and hence $\int |x| \, \mu^Z(\{t\} \times dx)$ is integrable, where μ^Z, ν denote the jump measure of Z and its compensator. Therefore $|u(\varphi_t \cdot \Delta Z_t)| = \int |u(\varphi_t \cdot x)| \, \mu^Z(\{t\} \times dx) \leq \sup_{x \in \mathbb{R}_+} |u'(x)||\varphi_t| \int |x| \, \mu^Z(\{t\} \times dx)$ and $\Gamma_t(\varphi_t) \leq \sup_{x \in \mathbb{R}_+} |u'(x)||\varphi_t| \int |x| \, \nu(\{t\} \times dx)$ are integrable as well. Moreover, we have

$$u(\varphi_t \cdot \Delta Z_t) = \Gamma_t(\varphi_t) + \int_{[0,t] \times \mathbb{R}^{n+1}} 1_{\{t\}}(s) u(\varphi_s \cdot x) \, (\mu^Z - \nu)(ds, dx). \tag{3.1}$$

Integrability of the first two terms implies that the third term is integrable as well. From JS, I.2.27, it follows that $E(\int_{[0,t] \times \mathbb{R}^{n+1}} 1_{\{t\}}(s) u(\varphi_s \cdot x) \, (\mu^Z - \nu)(ds, dx) | \mathcal{F}_{t-}) = 0$ P-almost surely and hence $E(u(\varphi_t \cdot \Delta Z_t)) = E(\Gamma_t(\varphi_t))$. If $(B, C, \nu)^I$ denotes the integral characteristics of Z, then

$$\begin{aligned}\int_0^{t'} 1_{[0,t]^C}(s)\varphi_s \cdot dZ_s &= \int_0^{t'} 1_{[0,t]^C}(s)\varphi_s \cdot dB_s + \int_0^{t'} 1_{[0,t]^C}(s)\varphi_s \cdot dZ_s^c \\ &\quad + \int_{[0,t'] \times \mathbb{R}^{n+1}} 1_{[0,t]^C}(s)\varphi_s \cdot x \, (\mu^Z - \nu)(ds, dx) \end{aligned} \tag{3.2}$$

for any $t' \in \mathbb{R}_+$. The first term on the right-hand side of Equation 3.2 equals $\int_t^{t'} \varphi_s \cdot b_s \, ds$ for $t' > t$ small enough, which implies its uniform integrability on $[0, T]$ for $T > t$ small enough. The second term is a square-integrable martingale on any compact interval $[0, T]$, since φ is bounded and $E(\sum_{i,j=0}^n \int_0^t |c_s^{ij}| \, ds) < \infty$ by assumption (cf. JS, III.4.5d). Moreover, the last term is uniformly integrable on any interval $[0, T]$ by Proposition 2.8. Hence, $\int_0^{\cdot} 1_{[0,t]^C}(s)\varphi_s \cdot dZ_s$ is uniformly integrable as well. By right-continuity of the stochastic integral this implies $E(|\int_0^{t'} 1_{[0,t]^C}(s)\varphi_s \cdot dZ_s|) \to 0$ for $t' \to t$.

We will now turn to the proof of the second statement. Let $t \in \mathbb{R}_+ \setminus \Theta$. Define the process $Y = (Y_{t'})_{t' \in \mathbb{R}_+}$ by $Y_{t'} := \int_0^{t'} 1_{[0,t]^C}(s)\varphi_s \cdot dZ_s$. By Lemma 2.22, Y is a special semimartingale, and, by Remark 2 in Section 2.5, so is $u(Y)$. Moreover, we have $\Delta Y_{t'} = 1_{[0,t]^C}(t')\varphi_{t'} \cdot \Delta Z_{t'}$, and hence $\Delta Y_{t'}(\omega) = 1_{[0,t]^C}(t')\varphi_{t'}(\omega) \cdot x$ for $M^P_{\mu^Z}$-almost all $(\omega, t', x) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^{n+1}$ (in the sense of Jacod (1979), (3.10)). From $\Delta Z_t = 0$ P-almost surely and by Itô's formula (cf. Jacod (1979), (3.89)), we have

$$\begin{aligned} & u(G_{t'}(\varphi) - G_{t-}(\varphi)) = u(Y_{t'}) \\ &= \int_0^{t'} u'(Y_{s-}) \, dY_s^c \\ &\quad + \int_{[0,t'] \times \mathbb{R}^{n+1}} (u(Y_{s-} + \varphi_s \cdot x) - u(Y_{s-})) \cdot 1_{[0,t]^C}(s) \, (\mu^Z - \nu)(ds, dx) \\ &\quad + \int_0^{t'} u'(Y_{s-}) \, dA_s + \frac{1}{2} \int_0^{t'} u''(Y_{s-}) \, d\langle Y^c, Y^c \rangle_s \\ &\quad + \int_{[0,t'] \times \mathbb{R}^{n+1}} \Big(u(Y_{s-} + \varphi_s \cdot x) - u(Y_{s-}) - u'(Y_{s-})\varphi_s \cdot x \Big) 1_{[0,t]^C}(s) \, \nu(ds, dx) \end{aligned}$$

for any $t' \geq t$, where A denotes the predictable part of finite variation of the special semimartingale Y. As in the proof of Lemma 2.22, we conclude that $dY^c_s = 1_{[0,t]^C}(s)\varphi_s \cdot dZ^c_s$, $d\langle Y^c, Y^c\rangle_s = 1_{[0,t]^C}(s)\sum_{i,j=0}^n \varphi^i_s c^{ij}_s \varphi^j_s\, ds$ and $dA_s = 1_{[0,t]^C}(s)\varphi_s \cdot dB_s$. It follows that we have for any $t' \geq t$ with $[t,t'] \cap \Theta = \varnothing$:

$$u(G_{t'}(\varphi) - G_{t-}(\varphi)) = \int_t^{t'} \gamma_s(\varphi_s)\, ds \tag{3.3}$$

$$+\int_0^{t'} u'(Y_{s-})1_{[0,t]^C}(s)\varphi_s\, dZ^c_s \tag{3.4}$$

$$+\int_{[0,t']\times\mathbb{R}^{n+1}} (u(Y_{s-}+\varphi_s\cdot x) - u(Y_{s-}))\cdot 1_{[0,t]^C}(s)\,(\mu^Z - \nu)(ds,dx) \tag{3.5}$$

$$+\int_t^{t'} (u'(Y_{s-}) - 1)\varphi_s \cdot b_s\, ds \tag{3.6}$$

$$+\frac{1}{2}\int_t^{t'} (u''(Y_{s-}) + \kappa)\varphi_s^\top c_s \varphi_s\, ds \tag{3.7}$$

$$+\int_t^{t'}\int \Big(u(Y_{s-}+\varphi_s\cdot x) - u(Y_{s-}) - u(\varphi_s\cdot x) - (u'(Y_{s-})-1)\varphi_s\cdot x\Big)\, F_s(dx)\, ds. \tag{3.8}$$

Since φ is bounded, the integrand $|u(\varphi_s\cdot x) - \varphi_s\cdot x|$ in the definition of $\gamma_t(\varphi_t)$ is dominated by some multiple of $|x|^2 \wedge |x|$ (cf. the proof of Lemma 3.12). Therefore, the integrability of $\int_t^{t'} \gamma_s(\varphi_s)\, ds$ follows from the regularity condition (RC 1). It remains to be shown that the expectation of the remaining terms (3.4) - (3.8) is $o(t'-t)$ for $t' \downarrow t$. Since $u'(Y_{\cdot-})\varphi$ is bounded and $E(\int_0^t \sum_{i,j=0}^n |c^{ij}_s|\, ds) < \infty$, it follows from JS, III.4.5d that term (3.4) is a square-integrable martingale (on any compact interval $[0,T]$) starting in 0. Hence, its expectation equals 0. If we denote the upper bound of $|\varphi|$ by $M \in \mathbb{R}_+$, then $|u(Y_{s-} + \varphi_s\cdot x) - u(Y_{s-})| \leq \sup_{y\in\mathbb{R}} |u'(y)| M |x|$. Moreover, (RC 1) implies that $\int_{[t,t']\times\mathbb{R}^{n+1}} (|x|^2 \wedge |x|)\,\nu(ds,dx) = \int_t^{t'}\int (|x|^2\wedge|x|)\, F_s(dx)\, ds$ is integrable for $t' \geq t$ small enough. Together, we obtain from Proposition 2.8 that term (3.5) is a uniformly integrable martingale on $[0,T]$ for $T \geq t$ small enough. Hence, its expectation is 0 as well. Let $p := 1+\varepsilon$, $q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$, where ε is chosen as in regularity condition (RC 1). Define the increasing function $V : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$V_{t'} := \int_0^{t'} E\Big(|b_s|^p + \sum_{i,j=0}^n |c^{ij}_s|^p + \Big(\int (|x|^2\wedge|x|)\, F_s(dx)\Big)^p\Big)\, ds.$$

Since V is absolutely continuous, it is differentiable in λ-almost all $t \in \mathbb{R}_+$ (cf. Elstrodt (1996), VII.4.12). Assume for the rest of the proof that differentiability holds in t. Then we have $\frac{V_{t'} - V_t}{t'-t} = O(1)$ for $t' \downarrow t$. By Jensen's inequality, it follows that $E((\frac{1}{t'-t}\int_t^{t'} |b_s|\, ds)^p) \leq E(\frac{1}{t'-t}\int_t^{t'} |b_s|^p\, ds) = O(1)$ for $t' \downarrow t$ and likewise for $E((\frac{1}{t'-t}\int_t^{t'} \sum_{i,j=0}^n |c^{ij}_s|\, ds)^p)$ and $E((\frac{1}{t'-t}\int (|x|^2 \wedge |x|)\, F_s(dx)\, ds)^p)$. If $M \in \mathbb{R}_+$ denotes an upper bound of $|\varphi|$, then the triangular inequality and Hölder's inequality yield

$$\left\|\frac{1}{t'-t}\int_t^{t'}(u'(Y_{s-})-1)\varphi_s\cdot b_s\,ds\right\|_{L^1}\leq\left\|\sup_{s\in[t,t']}|u'(Y_{s-})-1|\cdot M\right\|_{L^q}\left\|\frac{1}{t'-t}\int_t^{t'}|b_s|\,ds\right\|_{L^p}.$$

By dominated convergence, the first factor converges to 0 for $t'\downarrow t$. Since the second factor is $O(1)$ for $t'\downarrow t$, we have that the expectation of term (3.6) is $o(t'-t)$ for $t'\downarrow t$. Similarly, it follows that term (3.7) is $o(t'-t)$ for $t'\downarrow t$. By Taylor's formula with integral remainder (cf. Heuser (1990b), p. 284-285) and the mean value theorem, we obtain

$$\begin{aligned}
&|(u(Y_{s-}+\varphi_s\cdot x)-u(Y_{s-}))-(u(\varphi_s\cdot x)-0)-(u'(Y_{s-})-1)\varphi_s\cdot x|\\
&= \left|\left(\int_0^1(u'(Y_{s-}+z\varphi_s\cdot x)-u'(z\varphi_s\cdot x))(1-z)\,dz-(u'(Y_{s-})-1)\right)\varphi_s\cdot x\right|\\
&\leq \left(\left(2\sup_{y\in\mathbb{R}}|u''(y)||Y_{s-}|\right)\wedge\left(3\sup_{y\in\mathbb{R}}|u'(y)|+1\right)\right)M|x|
\end{aligned}$$

Similarly, the second order Taylor formula and the mean value theorem yield that

$$\begin{aligned}
&|(u(Y_{s-}+\varphi_s\cdot x)-u(Y_{s-})-u'(Y_{s-})\varphi_s\cdot x)-(u(\varphi_s\cdot x)-0-\varphi_s\cdot x)|\\
&= \left|\int_0^1(u''(Y_{s-}+z\varphi_s\cdot x)-u''(z\varphi_s\cdot x))(1-z)\,dz(\varphi_s\cdot x)^2\right|\\
&\leq \left(\left(\sup_{y\in\mathbb{R}}|u'''(y)||Y_{s-}|\right)\wedge\left(2\sup_{y\in\mathbb{R}}|u''(y)|\right)\right)M^2|x|^2.
\end{aligned}$$

Together, we obtain by Hölder's inequality that

$$\begin{aligned}
&\left\|\frac{1}{t'-t}\int_t^{t'}\int\left(u(Y_{s-}+\varphi_s\cdot x)-u(Y_{s-})-u(\varphi_s\cdot x)-(u'(Y_{s-})-1)\varphi_s\cdot x\right)F_s(dx)ds\right\|_{L^1}\\
&\leq \left\||Y_{s-}|\sup_{y\in\mathbb{R}}(2M|u''(y)|+M^2|u'''(y)|)\wedge\sup_{y\in\mathbb{R}}(M(3|u'(y)|+1)+2M^2|u''(y)|)\right\|_{L^q}\\
&\quad\cdot\left\|\frac{1}{t'-t}\int_t^{t'}\int(|x|^2\wedge|x|)\,F_s(dx)\,ds\right\|_{L^p}.
\end{aligned}$$

As for (3.6), (3.7), it follows that the expectation of term (3.8) is $o(t'-t)$ for $t'\downarrow t$. □

PROOF OF REMARK 2. If Z is discrete, then $u(G_{t'}(\varphi)-G_{t-}(\varphi))=u(\Delta G_t(\varphi))=u(\varphi_t\Delta Z_t)$ P-almost surely for any $t'\geq t$ with $(t,t']\cap\Theta=\varnothing$. The claim now follows from $E(u(\varphi_t\cdot\Delta Z_t))=E(\Gamma_t(\varphi_t))$ (cf. the proof of the previous theorem). □

PROOF OF LEMMA 3.16. By Equation (3.1) we have

$$u(G_{t'}(\varphi)-G_{t-}(\varphi))=u(\varphi_t\cdot\Delta Z_t)=\Gamma_t(\varphi_t)+\int_{[0,t]\times\mathbb{R}^{n+1}}1_{\{t\}}(s)u(\varphi_s\cdot x)\,(\mu^Z-\nu)(ds,dx)$$

for any $t'\geq t$ with $(t,t']\cap\Theta=\varnothing$. From JS, I.2.27, it follows that the conditional expectation given $\mathcal{F}_{t-}$ of the last term equals 0 (cf. JS, I.2.27). This implies the claim. □

PROOF OF LEMMA 3.20. *First step:* W.l.o.g., we assume $\varphi_0 = 0$. Moreover, suppose that φ is locally bounded. The general case is considered in the fifth step. Let $Z = Z_0 + M + A$ be a decomposition of the semimartingale X such that the components of M are locally square-integrable martingales and $A \in \mathscr{V}^d$ (cf. JS, I.4.21, I.4.17). By $(S_k)_{k\in\mathbb{N}}$ denote a localizing sequence of all components of M such that, in addition, $|\varphi^{S_k}| \leq k$ for any $k \in \mathbb{N}$ (for existence cf. the proof of Lemma A.1). Moreover, define for any $k \in \mathbb{N}$ the stopping time $T_k := S_k \wedge \inf\{t > 0 : \sum_{i=0}^n \mathrm{Va}(A^i)_t \geq k\}$ and let Θ_k be the set of the k smallest elements of Θ. Obviously, we have $T_k \uparrow \infty$ P-almost surely for $k \to \infty$. For any $k \in \mathbb{N}$, we define the finite measure μ on $(\Omega \times \mathbb{R}_+, \mathcal{P})$ by

$$\begin{aligned} \mu(C) \;:=\; & \sum_{k=1}^{\infty} \frac{2^{-k}}{k^3}\Bigg(\sum_{i=0}^{n} \frac{E(\int_0^{S_k} 1_C \, d\langle M^i, M^i\rangle_s)}{(n+1)(E(\langle M^i, M^i\rangle_{S_k}) + 1)} + \sum_{i=0}^{n} \int_0^{T_k -} 1_C \, d\mathrm{Va}(A^i)_s \\ & + \Big(P \otimes \Big(\lambda|_{[0,k]} + \sum_{s\in\Theta_k} \varepsilon_s\Big)\Big)(C)\Bigg) \end{aligned}$$

for any $C \in \mathcal{P}$. Note that the first part contains the Doléans measure m of M^i in JS, p. 48. Since $|\varphi^{S_k}| \leq k$ for any $k \in \mathbb{N}$, we have $\int |\varphi^i|^2 \, d\mu \leq 4$ and hence $\varphi^i \in L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, \mu)$ for $i = 0, \ldots, n$.

Second step: We now show that there is a sequence $(\varphi^{(l)})_{l\in\mathbb{N}}$ in $\mathfrak{S}$ such that $\varphi^{(l),i} \to \varphi^i$ in $L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, \mu)$ for $i = 0, \ldots, n$. One easily sees that it suffices to find an approximating sequence separately for each component φ^i. Since any non-negative $\mathcal{P}$-measurable mapping $\psi : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ can be pointwise approximated from below by a linear combination of indicator functions, it follows from the dominated convergence theorem (e.g. Bauer (1978), Satz 15.4) that $\{\sum_{\alpha=1}^p a_i 1_{A^i} : p \in \mathbb{N},\ C_1, \ldots, C_p \in \mathcal{P},\ a_1, \ldots, a_p \in \mathbb{R}\}$ is dense in $L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, \mu)$. Therefore, it suffices to show that for any $C \in \mathcal{P}$, $\varepsilon > 0$, there exists a $\widetilde{C} \in \mathcal{P}$ such that $1_{\widetilde{C}} \in \mathfrak{S}$ and $\int (1_C - 1_{\widetilde{C}})^2 \, d\mu = \mu((C \setminus \widetilde{C}) \cup (\widetilde{C} \setminus C)) < \varepsilon$. Define the ring $\mathcal{R} := \{(H \times \{0\}) \,\dot{\cup} \dot{\bigcup}_{i\in\{1,\ldots,p\}}]T_{i-1}, T_i] : H \in \mathcal{F}_0, p \in \mathbb{N}, \text{ and } T_0 \leq \ldots \leq T_p \text{ bounded stopping times}\}$. Observe that $1_{\widetilde{C}} \in \mathfrak{S}$ for any $\widetilde{C} \in \mathcal{R}$. Since $\mathcal{R}$ generates $\mathcal{P}$ (cf. JS, I.2.2), there exists a $\widetilde{C} \in \mathcal{R}$ such that $\mu((C \setminus \widetilde{C}) \cup (\widetilde{C} \setminus C)) < \varepsilon$ (cf. Billingsley (1979), Theorem 11.4).

Third step: Denote by $(\varphi^{(l)})_{l\in\mathbb{N}}$ a sequence in $\mathfrak{S}$ as in the second step. Since $\varphi^{(l)}$ converges to φ in μ-measure, there is a subsequence, which we denote again by $(\varphi^{(l)})_{l\in\mathbb{N}}$, such that $\varphi^{(l)} \to \varphi$ μ-almost everywhere. By definition of μ this implies that $\varphi^{(l)} \to \varphi$ outside some $(P \otimes (\lambda + \sum_{s\in\Theta} \varepsilon_s))$-null set. Since the mappings $\psi \mapsto \Gamma_t(\psi)$ and $\psi \mapsto \gamma_t(\psi)$ are continuous, it follows that $(\varphi^{(l)})_{l\in\mathbb{N}}$ is an approximating sequence.

Fourth step: It remains to prove the convergence $G(\varphi^{(l)}) \to G(\varphi)$ P-almost surely uniformly on any interval $[0, T]$. By taking subsequences and by a diagonal procedure, it suffices to prove that, for any $k \in \mathbb{N}$, $\|G(\varphi^{(l)}) - G(\varphi)\|^*_{T_k -} \to 0$ in probability. Fix $k \in \mathbb{N}$. We have

$$E((\|G(\varphi^{(l)}) - G(\varphi)\|^*_{T_k -})^2)$$

$$\begin{aligned}
&\leq \quad 2\sum_{i=0}^{n} E\biggl(\Bigl(\Bigl\|\int_0^{\cdot} (\varphi_s^{(l),i} - \varphi_s^i)\, d(M^i)_s^{S_k}\Bigr\|_\infty^*\Bigr)^2\biggr) \\
&\qquad + 2\sum_{i=0}^{n} E\biggl(\Bigl(\Bigl\|\int_0^{\cdot} (\varphi_s^{(l),i} - \varphi_s^i)\, d(A^i)_s^{T_k-}\Bigr\|_\infty^*\Bigr)^2\biggr) \qquad (3.9)
\end{aligned}$$

The first term on the right-hand side of (3.9) is dominated by

$$\begin{aligned}
&8\sum_{i=0}^{n} E\Bigl(\int_0^{\infty} (\varphi_s^{(l),i} - \varphi_s^i)^2\, d\langle (M^i)^{S_k}, (M^i)^{S_k}\rangle_s\Bigr) \\
&\leq \quad 8\frac{k^3}{2^{-k}}(n+1)(E(\langle M^i, M^i\rangle_{S_k}) + 1)\int (\varphi^{(l)} - \varphi)^2\, d\mu
\end{aligned}$$

(cf. Doob's inequality, e.g. as in Corollary 2.52 (for $T \equiv 0$)). The second term on the right-hand side of (3.9) is dominated by

$$\begin{aligned}
&2\sum_{i=0}^{n} E\biggl(\Bigl(\int_0^{T_k-} |\varphi_s^{(l),i} - \varphi_s^i|\, d\mathrm{Va}(A^i)_s\Bigr)^2\biggr) \\
&\leq \quad 2k\sum_{i=0}^{n} E\Bigl(\int_0^{T_k-} (\varphi_s^{(l),i} - \varphi_s^i)^2\, d\mathrm{Va}(A^i)_s\Bigr) \\
&\leq \quad 2\frac{k^4}{2^{-k}}\int (\varphi^{(l)} - \varphi)^2\, d\mu.
\end{aligned}$$

Since $\int(\varphi^{(l)} - \varphi)^2\, d\mu$ converges to 0 for $l \to \infty$, it follows that $\|G(\varphi^{(l)}) - G(\varphi)\|_{T_k-}^* \to 0$ in $L^2(\Omega, \mathcal{F}, P)$ and thus in probability.

Fifth step: If φ is not locally bounded, define the bounded strategy $\widetilde{\varphi} \in \mathfrak{A}$ by $\widetilde{\varphi}^i := \arctan(\varphi^i)$ for $i = 0, \ldots, n$. Let $\widetilde{\varphi}^{(l)}$ be an approximating sequence of $\widetilde{\varphi}$. Using the continuity of $\tan, \gamma, T$, one easily shows that the sequence $\varphi^{(l)}$, defined by $\varphi^{(l),i} := \tan(\widetilde{\varphi}^{(l)})$, is an approximating sequence for $\varphi \in \mathfrak{A}$. $\square$

PROOF OF COROLLARY 3.21. We have to show that the approximating sequence $(\varphi^{(l)})_{l\in\mathbb{N}}$ in the second step of the previous proof can be chosen in $\mathfrak{S} \cap \mathfrak{M}$. Note that $\mathfrak{M} = \{\varphi \in \mathfrak{A} : \varphi(\omega, t) \in M$ for any $(\omega, t) \in \Omega \times \mathbb{R}_+\}$ for some convex set $M \subset \mathbb{R}^{n+1}$. For any $l \in \mathbb{N}$, let $(D_p^l)_{p\in\{1,\ldots,l^{2(n+1)}\}}$ be a partition of $\{x \in \mathbb{R}^{n+1} : |x^i| \leq l$ for $i = 0, \ldots, n\}$ into cubes of edge length $2/l$, and fix a point $x_p^l \in D_p^l \cap M$ for any $p, l \in \mathbb{N}$ with $D_p^l \cap M \neq \varnothing$. Now define for any $l \in \mathbb{N}$ a strategy $\varphi^{(l)} \in \mathfrak{A} \cap \mathfrak{M}$ by $\varphi^{(l)} := \sum_{p=1}^{l^{2(n+1)}} x_p^l \cdot 1_{\varphi^{-1}(D_p^l)}$. Dominated convergence yields that $\varphi^{(l),i} \to \varphi^i$ in $L^2(\mathbb{R}_+ \times \Omega, \mathcal{P}, \mu)$ for $i = 0, \ldots, n$. As in the second step of the previous proof, one shows that $\varphi^{-1}(D_p^l) \in \mathcal{P}$ can be replaced with some $\widetilde{C}^{l,p} \in \mathcal{R}$. So, one obtains an approximating sequence in $\mathfrak{S} \cap \mathfrak{M}$.

If φ is not locally bounded, one argues similarly as in the fifth step of the previous proof by substituting $\widetilde{\varphi}$ for φ and $\widetilde{M} := \{x \in \mathbb{R}^{n+1} : (\tan(x^0), \ldots, \tan(x^n)) \in M\}$ for M. $\square$

PROOF OF THEOREM 3.22. $\Rightarrow$: We will only show the second statement, because the proofs are very similar.

First step: For any $(\omega, t) \in \Omega \times \mathbb{R}_+$ define $M(\omega, t) \subset \mathbb{R}^{n+1}$ by $M(\omega, t) := \{\psi \in \mathbb{R}^{n+1} : g^j(\omega, t, \psi) \leq 0$ for $j = 1, \dots, p$ and $g^j(\omega, t, \psi) = 0$ for $j = p+1, \dots, q\}$. Then we have $\mathfrak{M} = \{\varphi \in \mathfrak{A} : \varphi(\omega, t) \in M(\omega, t)$ for any $(\omega, t)\}$. Let $N \in \mathcal{P}$ be a $(P \otimes \lambda)$-null set such that $\gamma_t(\varphi_t) \geq \gamma_t(\widetilde{\varphi}_t)$ for any $\widetilde{\varphi} \in \mathfrak{M}$. Fix $(\omega, t) \in N^C$. Since the mappings g^j are predictable and convex, one can find, for any $\psi \in M(\omega, t)$, a strategy $\widetilde{\varphi}_t \in \mathfrak{M}$ such that $\widetilde{\varphi}_t(\omega) = \psi$. Indeed, one may define $\widetilde{\varphi}$ by $\widetilde{\varphi}(\widetilde{\omega}, s) := \psi \cdot 1_{M(\widetilde{\omega}, s)}(\psi) + \xi(\widetilde{\omega}, s) \cdot 1_{M(\widetilde{\omega}, s)^C}(\psi)$, where $\xi(\widetilde{\omega}, s)$ is defined as the unique element of $M(\widetilde{\omega}, s)$ with the smallest Euclidean norm. For existence and predictability of ξ we refer to the last two steps of the proof of Theorem 3.26. Therefore, we have $\gamma_t(\varphi_t) \geq \gamma_t(\psi)$ for any $\psi \in M(\omega, t)$. Define a mapping $h : \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$h(\psi) := -\psi \cdot b_t + \frac{1}{2}\kappa \psi^\top c_t \psi - \int (u(\psi \cdot x) - \psi \cdot x) \, F_t(dx).$$

Note that $h(\psi) = -\gamma_t(\psi)$ for any $\psi \in \mathbb{R}^{n+1}$.

Second step: We will now show that h is a convex function. Since $\psi \mapsto -\psi \cdot b_t$ is linear and c_t is non-negative definite, this is evident for the first two terms (cf. Rockafellar (1970), Theorem 4.5). Moreover, the mapping $\psi \mapsto \psi \cdot x - u(\psi \cdot x)$ is convex for any $x \in \mathbb{R}^{n+1}$, since the matrix of its second partial derivatives $(-x^i x^j u''(\psi \cdot x))_{i,j \in \{0 \dots, n\}}$ is non-negative definite (cf. Rockafellar (1970), Theorem 4.5). Since integration is a linear operation, it follows that the mapping $\psi \mapsto \int (\psi \cdot x - u(\psi \cdot x)) \, F_t(dx)$ is convex as well (cf. Rockafellar (1970), Theorem 4.1). Hence, h is convex by Rockafellar (1970), Theorem 5.2.

Third step: We will show that $h : \mathbb{R}^{n+1} \to \mathbb{R}$ is differentiable with partial derivatives $D_i h(\psi) = -b_t^i + \frac{1}{2}\kappa \sum_{j=0}^n c_t^{ij} \psi^j - \int x^i (u'(\psi \cdot x) - 1) \, F_t(dx)$ for any $\psi \in \mathbb{R}^{n+1}$, $i \in \{0 \dots, n\}$. The claim follows at once if we have proven that we may interchange differentiation and integration in the integral relative to F_t. Observe that for any $\psi, x \in \mathbb{R}^{n+1}$ we have $|x^i(u'(\psi \cdot x) - 1)| \leq |x| \cdot (1 + \sup_{y \in \mathbb{R}} |u'(y)|)$ and, by the mean value theorem, $|x^i(u'(\psi \cdot x) - 1)| \leq |x||\psi \cdot x| \sup_{y \in \mathbb{R}} |u''(y)|$ and hence $|x^i(u'(\psi \cdot x) - 1)| \leq (1 + \sup_{y \in \mathbb{R}} |u'(y)| + |\psi| \cdot \sup_{y \in \mathbb{R}} |u''(y)|)(|x|^2 \wedge |x|)$, where the first factor is bounded in a neighbourhood of any $\psi \in \mathbb{R}^{n+1}$ and the second factor is F_t-integrable. By Billingsley (1979), Theorem 16.8 it follows that we may differentiate under the integral sign.

Fourth step: We define the *ordinary convex program* (P) in the sense of Rockafellar (1970), p.273 by the convex function $h : \mathbb{R}^{n+1} \to \mathbb{R}$, the set $C := \mathbb{R}^{n+1}$ and the constraints $g^j \leq 0$ for $j = 1, \dots, p$, $g^j = 0$ for $j = p+1, \dots, q$. From the first step, we know that φ_t is an optimal solution to (P). By Rockafellar (1970), Theorem 28.2 there exists a Kuhn-Tucker vector $(\lambda_1, \dots \lambda_q)$ for (P). It follows from Rockafellar (1970), Theorems 28.3 and 25.1 that $0 \in \partial h(\varphi_t) + \sum_{j=1}^q \lambda_j \partial g^j(\varphi_t) = \{\nabla h(\varphi_t) + \sum_{j=1}^q \lambda_j \nabla g^j(\varphi_t)\}$, where ∂f denotes the subdifferential and ∇f the gradient of a function f. This implies $0 = D_i h(\varphi_t) + \sum_{j=1}^q D_j \lambda_i g^j(\varphi_t)$ for any $i \in \{0, \dots, n\}$ and hence the claim.

$\Leftarrow$: Fix $(\omega, t) \in N^C$, where N denotes the $(P \otimes \lambda)$-null set in Theorem 3.22. By Condition 2 and Rockafellar (1970), Theorem 28.3, φ_t is an optimal solution to the above ordinary convex program (P). Therefore, $h(\varphi_t) \leq h(\psi)$ and hence $\gamma_t(\varphi_t) \geq \gamma_t(\psi)$ for any $\psi \in M(\omega, t)$. The statement concerning $\Gamma_t(\varphi_t)$ follows along the same lines. $\square$

PROOF OF COROLLARY 3.23. Apply Theorem 3.22 to fixed constraints $g^1 = 0, \dots, g^{n-k+1} = 0$ given by $g^j(\xi) := \xi^{k-1+j} - \psi^{k-1+j}$ for any $j \in \{1, \dots, n-k+1\}$, $\xi \in \mathbb{R}^{n+1}$. Elementary calculations yield that the conditions in Theorem 3.22 are for any fixed (ω, t) equivalent to those in Corollary 3.23. □

Proposition 3.30 *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a convex function and $x, y \in \mathbb{R}^{n+1}$. Then the mapping $\widetilde{f} : [0,1] \to \mathbb{R}$, $\lambda \mapsto f(\lambda x + (1-\lambda)y)$ is convex as well.*

PROOF. For any $\mu, \lambda_1, \lambda_2 \in [0,1]$ we have

$$
\begin{aligned}
\widetilde{f}((1-\mu)\lambda_1 + \mu\lambda_2) &= f((\lambda_1 x + (1-\lambda_1)y)(1-\mu) + (\lambda_2 x + (1-\lambda_2)y)\mu) \\
&\leq f(\lambda_1 x + (1-\lambda_1)y)(1-\mu) + f(\lambda_2 x + (1-\lambda_2)y)\mu \\
&= \widetilde{f}(\lambda_1)(1-\mu) + \widetilde{f}(\lambda_2)\mu,
\end{aligned}
$$

which implies that $\widetilde{f}$ is convex by Rockafellar (1970), Theorem 4.1. □

PROOF OF LEMMA 3.24. Fix $(\omega, t) \in \Omega \times \mathbb{R}_+$ (outside some $(\lambda \otimes P)$-null set as in the proof of Theorem 3.22). In the proof of Theorem 3.22 we show that $\varphi_t, \widetilde{\varphi}_t$ are solutions to the ordinary convex program (P). Moreover, we obtaine $0 \in \partial h(\varphi_t) + \sum_{j=1}^q \lambda_j \partial g^j(\varphi_t)$ and, using the same arguments, $0 \in \partial h(\widetilde{\varphi}_t) + \sum_{j=1}^q \lambda_j \partial g^j(\widetilde{\varphi}_t)$, where $(\lambda_1, \dots, \lambda_q)$ denotes a Kuhn-Tucker vector for (P) which, by definition, does not depend on the particular solution. By Rockafellar (1970), Theorem 23.5 (a)⇒(b) this implies that the convex mapping $\widetilde{h} : \mathbb{R}^{n+1} \to \mathbb{R}$, $\psi \mapsto h(\psi) + \sum_{j=1}^q \lambda_j g^j(\psi)$ achieves its infimum in φ_t and $\widetilde{\varphi}_t$, and hence by convexity of the minimum set, in any $\psi \in E := \{\lambda\varphi_t + (1-\lambda)\widetilde{\varphi}_t : \lambda \in [0,1]\}$. In particular, we have $\lambda_j g^j(\psi) = 0$ for any $j \in \{1, \dots, q\}$. Now define mappings $h_1, h_2, h_3, h_4 : [0,1] \to \mathbb{R}$ by

$$
\begin{aligned}
h_1(\lambda) &:= \widetilde{h}(\lambda\varphi_t + (1-\lambda)\widetilde{\varphi}_t) \\
h_2(\lambda) &:= -b_t \cdot (\lambda\varphi_t + (1-\lambda)\widetilde{\varphi}_t) \\
h_3(\lambda) &:= \frac{1}{2}\kappa(\lambda\varphi_t + (1-\lambda)\widetilde{\varphi}_t)^\top c_t(\lambda\varphi_t + (1-\lambda)\widetilde{\varphi}_t) \\
h_4(\lambda) &:= \int \Big((\lambda\varphi_t + (1-\lambda)\widetilde{\varphi}_t) \cdot x - u((\lambda\varphi_t + (1-\lambda)\widetilde{\varphi}_t) \cdot x) \Big) F_t(dx).
\end{aligned}
$$

Observe that $h_1 = h_2 + h_3 + h_4$ is constant. By Proposition 3.30, h_2, h_3, h_4 are convex. Therefore, $h_3 = h_1 - h_2 - h_4$ is also concave and hence affine. This implies that $0 = h_3''(\lambda) = \kappa(\varphi_t - \widetilde{\varphi}_t)^\top c_t(\varphi_t - \widetilde{\varphi}_t) = \kappa\psi^\top\psi$ for $\psi := c_t^{1/2}(\varphi_t - \widetilde{\varphi}_t)$, where $c_t^{1/2}$ denotes a symmetric matrix satisfying $c_t^{1/2} c_t^{1/2} = c_t$. Hence, we have $\psi = 0$ and therefore $h_3'(\lambda) = \kappa(\varphi_t - \widetilde{\varphi}_t)^\top c_t(\lambda\varphi_t + (1-\lambda)\widetilde{\varphi}_t) = 0$ for any $\lambda \in [0,1]$, which yields that h_3 is constant. Thus, $h_4 = h_1 - h_2 - h_3$ is concave and hence affine. In the proof of Theorem 3.22 we have shown that we may differentiate the mapping $\psi \mapsto \int (u'(\psi \cdot x) - \psi \cdot x)\, F_t(dx)$ under the integral sign. Since the derivative of an affine function is constant, this implies that $h_4'(1) = \int x \cdot (\varphi_t - \widetilde{\varphi}_t)(1 - u'(\varphi_t \cdot x))\, F_t(dx)$ and $h_4'(0) = \int x \cdot (\varphi_t - \widetilde{\varphi}_t)(1 - u'(\widetilde{\varphi}_t \cdot x))\, F_t(dx)$

are equal. We obtain $0 = \int(\varphi_t \cdot x - \widetilde{\varphi}_t \cdot x)(u'(\varphi_t \cdot x) - u'(\widetilde{\varphi}_t \cdot x))\, F_t(dx)$. The product under the integral sign is negative or 0, because u is a strictly decreasing function. Therefore, $(\varphi_t - \widetilde{\varphi}_t) \cdot x = 0$ for F_t-almost all $x \in \mathbb{R}^{n+1}$. In particular, h_4 is constant. This implies that $h_2 = h_1 - h_3 - h_4$ is constant as well. Hence $(\varphi_t - \widetilde{\varphi}_t)b_t = 0$. Similarly, one shows that P-almost surely and for any $t \in \mathbb{R}_+$ we have $(\varphi_t - \widetilde{\varphi}_t) \cdot x = 0$ for K_t-almost all $x \in \mathbb{R}^{n+1}$. Now observe that $G(\varphi) - G(\widetilde{\varphi}) = \int_0^{\cdot}(\varphi_s - \widetilde{\varphi}_s) \cdot dX_s$. By Lemma 2.22 and the preceding results, $G(\varphi) - G(\widetilde{\varphi})$ is an extended Grigelionis process with extended characteristics $(\Theta, \varepsilon_0, 0, 0, 0, (\varepsilon_0 1_\Theta(t))_{t \in \mathbb{R}_+})^E$. Thus, $G(\varphi) - G(\widetilde{\varphi}) = 0$ up to indistinguishability. $\square$

PROOF OF THE REMARK. This is true, since $u'_\kappa(x) = u'_1(\kappa x)$ for any $\kappa > 0, x \in \mathbb{R}$. $\square$

Proposition 3.31 *For $k, d \in \mathbb{N}^*$ and a measurable space $(\Gamma, \mathcal{G})$, let $\beta_1, \dots, \beta_k : \Gamma \times \mathbb{R}^d \to \mathbb{R}$ be $(\mathcal{G} \otimes \mathcal{B}^d)$-measurable mappings that are continuous in the second argument. Then there exists a $(\mathcal{G} \otimes \mathcal{B}^d)$-measurable mapping $\beta : \Gamma \times \mathbb{R}^d \to \mathbb{R}$ that is also continuous in the second argument and such that for any $(\omega, \psi) \in \Gamma \times \mathbb{R}^d$,*

$$\beta(\omega, \psi) \leq 0 \Leftrightarrow (\beta^i(\omega, \psi) \leq 0 \text{ for } i = 1, \dots, k).$$

PROOF. Define $\delta : \mathbb{R}^k \to \mathbb{R}$ by $\delta(x) := \inf\{|x - y| : y \in (\mathbb{R}_-)^k\}$. Then δ is a continuous mapping with $\delta(x^1, \dots, x^k) \leq 0 \Leftrightarrow (x^i \leq 0 \text{ for } i = 1, \dots, k)$. Now let $\beta(\omega, \psi) := \delta(\beta^1(\omega, \psi), \dots, \beta^k(\omega, \psi))$. $\square$

Proposition 3.32

1. *For $d \in \mathbb{N}$ and a measurable space $(\Gamma, \mathcal{G})$, let $\alpha, \beta : \Gamma \times \mathbb{R}^d \to \mathbb{R}$ be $(\mathcal{G} \otimes \mathcal{B}^d)$-measurable mappings that are continuous in the second argument. Then the mapping $\gamma : \Gamma \to \overline{\mathbb{R}}$, $\omega \mapsto \sup\{\alpha(\omega, \psi) : \psi \in \mathbb{R}^d \text{ with } \beta(\omega, \psi) \leq 0\}$ is $\mathcal{G}$-measurable.*

2. *If, moreover, for any $\omega \in \Gamma$ there is a unique $\delta(\omega) \in \mathbb{R}^d$ with $\alpha(\omega, \delta(\omega)) = \gamma(\omega)$ and $\beta(\omega, \psi) \leq 0$, then the mapping $\delta : \Gamma \to \mathbb{R}^d$ is $\mathcal{G}$-measurable as well.*

PROOF. *1.* W.l.o.g. $\alpha \leq 0$. Otherwise consider $\widetilde{\alpha}(\omega, \psi) := -\exp(-\alpha(\omega, \psi))$. Moreover, by letting $k \to \infty$, it suffices to prove $\mathcal{G}$-measurability of $\gamma^k : \Gamma \to \mathbb{R}$, $\omega \mapsto \sup\{\alpha(\omega, \psi) : \psi \in \mathbb{R}^d \text{ with } |\psi| \leq k \text{ and } \beta(\omega, \psi) \leq 0\}$ for any $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$. For any $l \in \mathbb{N}$ the mapping $\gamma^{k,l} : \Gamma \to \mathbb{R}$, $\omega \mapsto \sup\{\alpha(\omega, \psi) - l(0 \vee \beta(\omega, \psi)) : \psi \in \mathbb{R}^d \text{ with } |\psi| \leq k\}$ is $\mathcal{G}$-measurable because it suffices to take the supremum over all $\psi \in \mathbb{Q}^d$. Fix $\omega \in \Gamma$ for the moment. For any closed set $A \subset \mathbb{R}^d$ with $\beta(\omega, \cdot) > 0$ on A we have, by uniform continuity of β on the compact set $\{\psi \in \mathbb{R}^d : |\psi| \leq k\}$, that $\inf\{l(0 \vee \beta(\omega, \psi)) : \psi \in A \text{ with } |\psi| \leq k\} \uparrow \infty$ for $l \to \infty$. By continuity of α there exists for any $\varepsilon > 0$ an open set A^C containing $\{\psi \in \mathbb{R}_+ : \beta(\omega, \cdot) \leq 0\}$ and such that $\sup\{\alpha(\omega, \psi) : \psi \in A^C \text{ with } |\psi| \leq k\} \leq \gamma(\omega) + \varepsilon$. Together, we obtain that $\gamma^{k,l} \downarrow \gamma^k$ for $l \to \infty$. This implies that γ^k is $\mathcal{G}$-measurable.

2. Denote by $\widetilde{\beta} : \Gamma \times \mathbb{R}^d \to \mathbb{R}$ a $(\mathcal{G} \otimes \mathcal{B}^d)$-measurable mapping, continuous in the second argument and such that $\widetilde{\beta}(\omega, \psi) \leq 0$ if and only if $\beta(\omega, \psi) \leq 0$ and $\gamma(\omega) - \alpha(\omega, \psi) \leq 0$ (cf. Proposition 3.31). Then we have

$$\begin{aligned} \delta^i(\omega) &= \sup\{\psi^i : \psi \in \mathbb{R}^d \text{ with } \beta(\omega, \psi) \leq 0 \text{ and } \gamma(\omega) - \alpha(\omega, \psi) \leq 0\} \\ &= \sup\{\psi^i : \psi \in \mathbb{R}^d \text{ with } \widetilde{\beta}(\omega, \psi) \leq 0\}. \end{aligned}$$

for any $i \in \{1, \ldots, d\}$. By Statement 1, δ^i is a $\mathcal{G}$-measurable mapping. $\square$

PROOF OF THEOREM 3.26. *First step:* Fix $(\omega, t) \in N^C$ with $t \notin \Theta$, where N is a null set as in the second condition of Definition 3.25. If we define the mapping h as in the proof of Theorem 3.22, then (RC 2) implies that there is a $\psi \in \mathbb{R}^{n+1}$ with $\nabla h(\psi) = 0$, i.e. such that $0 \in \partial h(\psi)$. By Rockafellar (1970), Theorem 23.5 (a)⇒(b), this implies that h attains its finite infimum in ψ.

Second step: We now show that h is constant in its directions of recession. If the direction of $y \in \mathbb{R}^{n+1}$ is such a direction, then y is an element of the recession cone of h, which equals the recession cone of $\{x \in \mathbb{R}^{n+1} : h(x) \leq \min h\} = \{x \in \mathbb{R}^{n+1} : h(x) = \min h\}$ (cf. Rockafellar (1970), Theorem 8.7). By definition, this means that h attains its minimum in $\psi + \lambda y$ for any $\lambda \in \mathbb{R}_+$, in particular in $\widetilde{\psi} = \psi + y$. With the same arguments as in the proof of Lemma 3.24, it follows that $y \cdot b_t = 0$, $c_t y = 0$, $y \cdot x = 0$ for F_t-almost all $x \in \mathbb{R}^{n+1}$. An easy calculation shows that this implies $h(\rho + \lambda y) = h(\rho)$ for any $\rho \in \mathbb{R}^{n+1}, \lambda \in \mathbb{R}$. Thus, h is constant in the direction of y.

Third step: By Rockafellar (1970), Corollary 27.3.3, h attains its infimum subject to the given constraints. By Rockafellar (1970), p.264 the set of optimal solutions $\psi \in M(\omega, t)$ is convex ($M(\omega, t)$ is defined as in the proof of Theorem 3.22). Since it is also non-empty and bounded, the projection theorem (cf. Alt (1992), 2.17) yields that it contains a unique element of minimal Euclidean norm, which we denote by $\varphi_t(\omega)$. For $(\omega, t) \in N$ with $t \notin \Theta$, let $\varphi_t(\omega)$ be the point of minimal Euclidean norm in $M(\omega, t)$. For $(\omega, t) \in \Omega \times \Theta$ (or more exactly, outside the evanescent set of regularity condition (RC 2)), we define $\varphi_t(\omega)$ analogously as above, but with respect to the function $\overline{h}(\psi) := -\int u(\psi \cdot x)\, K_t(dx)$ instead of h. Hence, we have defined a mapping $\varphi : \Omega \times \mathbb{R}_+ \to \mathbb{R}^{n+1}$ meeting Conditions 1 and 2 in Definition 3.17.

Fourth step: It remains to show that φ is predictable. By Proposition 3.31 there exists a $(\mathcal{P} \otimes \mathcal{B}^{n+1})$-measurable mapping $\beta : \Omega \times \mathbb{R}_+ \times \mathbb{R}^{n+1} \to \mathbb{R}$ that is continuous in the last variable and such that $(\beta(\omega, t, \psi) \leq 0 \Leftrightarrow \psi \in M(\omega, t))$. Hence, the mapping $\widetilde{\gamma} : \Omega \times \mathbb{R}_+ \to \mathbb{R}$, $(\omega, t) \mapsto \sup\{\gamma_t(\psi) 1_{N^C} : \psi \in M(\omega, t)\}$ is $\mathcal{P}$-measurable (cf. Proposition 3.32). Again by Proposition 3.31, there exists a $(\mathcal{P} \otimes \mathcal{B}^{n+1})$-measurable mapping $\widetilde{\beta} : \Omega \times \mathbb{R}_+ \times \mathbb{R}^{n+1} \to \mathbb{R}$ that is continuous in the last variable and such that $(\widetilde{\beta}(\omega, t, \psi) \leq 0 \Leftrightarrow \psi \in M(\omega, t)$ and $\widetilde{\gamma}(\omega, t) - \gamma_t(\psi) 1_{N^C}(\omega, t) \leq 0)$. Observe that there is a unique $\psi \in \mathbb{R}^{n+1}$ (namely $\varphi_t(\omega)$ (for $t \notin \Theta$)) such that the following three conditions hold: $\psi \in M(\omega, t)$; $\gamma_t(\psi) 1_{N^C}(\omega, t) = \widetilde{\gamma}(\omega, t)$; $-|\psi| = \sup\{-|\widetilde{\psi}| : \widetilde{\psi} \in M(\omega, t)$ and $\gamma_t(\widetilde{\psi}) 1_{N^C}(\omega, t) = \widetilde{\gamma}(\omega, t)\}$. Now apply Statement 2 of Proposition 3.32 to the mappings $\alpha((\omega, t), \psi) := -|\psi|$ and $\widetilde{\beta}$. We obtain that $(\omega, t) \mapsto \varphi_t(\omega) 1_{\Theta^C}(t)$ is a predictable process. By considering Γ instead of γ one similarly shows that the mapping $(\omega, t) \mapsto \varphi_t(\omega) 1_{\Theta}(t)$ is predictable as well. $\square$

PROOF OF COROLLARY 3.27. This follows immediately from Theorem 3.26 and Corollary 3.23. $\square$

PROOF OF THEOREM 3.28. $4 \Rightarrow 3$: *First step:* If Statement 3 is violated, then there exists a $t \in \Theta$ such that the set $M(\omega) := \{\psi \in \mathbb{R}^{n+1} : K_t(-H^\psi) = 0, K_t(H^\psi) > 0\}$ is not P-almost surely empty. Observe that the set $\{\omega \in \Omega : M(\omega) \neq \varnothing\}$ equals $\{\omega \in \Omega :$ There is a $\psi \in \mathbb{R}^{n+1}$ with $(\omega, \psi) \in g^{-1}(\{0\} \times \mathbb{R}^*_+)\}$, where the mapping $g : \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}^2$ is defined by $(\omega, \psi) \mapsto (\int 1_{(-\infty,0)}(\psi \cdot x)\, K_t(dx), \int 1_{(0,\infty)}(\psi \cdot x)\, K_t(dx))$. Since g is $(\mathcal{F}_{t-} \otimes \mathcal{B}^{n+1})$-measurable, it follows by the projection theorem (cf. Sainte-Beuve (1974), Theorem 4) that $\{\omega \in \Omega : M(\omega) \neq \varnothing\}$ is $\mathcal{F}^P_{t-}$-measurable. This implies that the set $G := (\{\omega \in \Omega : M(\omega) = \varnothing\} \times \{0\}) \cup g^{-1}(\{0\} \times \mathbb{R}^*_+) \subset \Omega \times \mathbb{R}^{n+1}$ is $(\mathcal{F}^P_{t-} \otimes \mathcal{B}^{n+1})$-measurable. By Sainte-Beuve (1974), Theorem 3 there exists a $\mathcal{F}^P_{t-}$-measurable mapping $\widetilde{\xi} : \Omega \to \mathbb{R}^{n+1}$ with $(\omega, \widetilde{\xi}(\omega)) \in G$ for any $\omega \in \Omega$. Let $\xi : \Omega \to \mathbb{R}^{n+1}$ be a $\mathcal{F}_{t-}$-measurable mapping that P-almost surely equals $\widetilde{\xi}$. If s is the nearest neighbour to the left of t in Θ, then ξ is $\mathcal{F}_s$-measurable by definition of Θ-discrete markets.

Second step: We show that $\xi \cdot (Z_t - Z_s)$ is non-negative P-almost surely and positive with positive probability. By Lemma 3.6, this implies that Statement 4 is violated. Firstly, we have that

$$E\Big(1_{(-\infty,0)}(\xi \cdot (Z_t - Z_s))|\mathcal{F}_s\Big) = \int 1_{(-\infty,0)}(\xi \cdot x)\, K_t(dx) = 0,$$

where the first equality follows from Remark 3 in Section 2.4 (and the fact that $Z_s = Z_{t-}$, $\mathcal{F}_s = \mathcal{F}_{t-}$) and the second equality from the definition of ξ. Hence, we obtain $\xi \cdot (Z_t - Z_s) \geq 0$ P-almost surely. Similarly, we have $E(1_{(0,\infty)}(\xi \cdot (Z_t - Z_s))|\mathcal{F}_s) = \int 1_{(0,\infty)}(\xi \cdot x)\, K_t(dx) = K_t(H^\xi)$ P-almost surely, where $P(K_t(H^\xi) > 0) = P(M(\omega) \neq \varnothing) > 0$. Therefore, $P(\xi \cdot (Z_t - Z_s) > 0) = E(K_t(H^\xi)) > 0$.

$3 \Rightarrow 2$: Let $t \in \Theta$. We have to show that P-almost surely there exists a $\psi \in \mathbb{R}^{n+1}$ with $\int x^i u'(\psi \cdot x)\, K_t(dx) = 0$ for any $i \in \{0, \dots, n\}$. Fix $\omega \in \Omega$. This is equivalent to saying that P-almost surely there exists a $\psi \in \mathbb{R}^{n+1}$ with $0 \in \partial h(\psi)$, where the convex function $h : \mathbb{R}^{n+1} \to \mathbb{R}$ (for fixed $\omega \in \Omega$) is defined by $\psi \mapsto -\int u(\psi \cdot x)\, K_t(dx)$. It suffices to show that h is constant in all its directions of recession (cf. Rockafellar (1970), Theorems 23.5 (b)$\Rightarrow$(a) and Theorem 27.1(b)), for differentiability of h compare the proof of Theorem 3.22). If the direction of $\psi \in \mathbb{R}^{n+1}$ is a direction of recession of h, then for any $z \in \mathbb{R}^{n+1}$ the mapping $\mathbb{R}_+ \to \mathbb{R}$, $\lambda \mapsto h(z+\lambda\psi)$ is decreasing (cf. Rockafellar (1970), p.265). For $z = 0$ it follows that $g : \mathbb{R}_+ \to \mathbb{R}$, $\lambda \mapsto -\int u(\lambda\psi \cdot x) 1_{-H^\psi}(x)\, K_t(dx) - \int u'(\lambda\psi \cdot x) 1_{H^\psi}(x)\, K_t(dx)$ is a decreasing function. The utility function u is concave and its derivative is bounded from above. Hence, $\lambda \mapsto \frac{u(\lambda\psi\cdot x)}{\lambda\psi\cdot x}$ converges for $\psi \cdot x < 0$ and $\lambda \to \infty$ from below to $\lim_{y\to-\infty} u'(y) =: u'(-\infty)$. This implies that $\int \frac{u(\lambda\psi\cdot x)}{\lambda\psi\cdot x}(\psi \cdot x) 1_{-H^\psi}(x)\, K_t(dx)$ converges for $\lambda \to \infty$ to $u'(-\infty) \int (\psi \cdot x) 1_{-H^\psi}(x)\, K_t(dx)$, which is strictly negative if and only if $K_t(-H^\psi) \neq 0$. In this case, the first integral in the definition of g grows asymptotically linearly for $\lambda \to \infty$. Since the second integral is bounded from below by $-\sup_{y\in\mathbb{R}} u(y)$ and g is decreasing, this is impossible. Therefore, we must have $K_t(-H^\psi) = 0$, which by Statement 3 implies $K_t(H^\psi) = 0$ and hence $\psi \cdot x = 0$ for K_t-almost all $x \in \mathbb{R}^{n+1}$. This implies that the mapping $\lambda \mapsto h(z + \lambda\psi)$ is constant for any $z \in \mathbb{R}^{n+1}$. Hence, the claim follows.

$2 \Rightarrow 1$: This will be shown in Theorem 3.36 and the following Remark 3.

$1 \Rightarrow 4$: Assume that Statement 1 holds but the market allows arbitrage. For ease of notation assume that $\Theta = \mathbb{N}^*$. By Lemma 3.6 there is a $t \in \mathbb{N}^*$ and a bounded $\mathcal{F}_{t-1}$-measurable, $\mathbb{R}^{n+1}$-valued random variable ψ such that $\psi \cdot (Z_t - Z_{t-1})$ is non-negative P-almost surely and positive with positive probability. Take P^* relative to t as in Statement 1. By JS, I.1.64 we have $E^*(\psi \cdot (Z_t - Z_{t-1})|\mathcal{F}_{t-1}) = \psi \cdot (E^*(Z_t|\mathcal{F}_{t-1}) - Z_{t-1}) = 0$ P^*-almost surely. Since $\psi \cdot (Z_t - Z_{t-1}) \geq 0$ P^*-almost surely, this implies $\psi \cdot (Z_t - Z_{t-1}) = 0$ P^*- and hence P-almost surely. Thus we have obtained a contradiction. $\square$

PROOF OF LEMMA 3.29. *1.* This follows from Corollary 3.23 and $u'_\kappa(y) = u'_1(\kappa y)$ for any $\kappa > 0, y \in \mathbb{R}$.

2. In the proof of Theorem 3.22 we have shown that $\varphi \in \mathfrak{A}$ is u-optimal if and only if, for fixed $(\omega, t) \in \Omega \times \mathbb{R}_+$, $\varphi(\omega, t)$ is an optimal solution of some convex function h. Since the set of extremal points of a convex function is convex, it follows that any convex combination of u-optimal strategies (for $\mathfrak{A}$) is u-optimal for $\mathfrak{A}$.

Now if $\varphi^{(j)} \in \mathfrak{A}$ is u_{κ_j}-optimal for $\mathfrak{A}$, then, by Statement 1, $\kappa_j \varphi^{(j)}$ is u_1-optimal for $\mathfrak{A}$. Therefore, the convex combination $(\sum_{j=1}^p \kappa_j^{-1})^{-1} \sum_{j=1}^p \varphi^{(j)}$ is u_1-optimal for $\mathfrak{A}$, which, by Statement 1, yields the claim.

3. This follows immediately from Statements 1 and 2.

4. As in Statement 2, we conclude that $\frac{1}{p} \sum_{j=1}^p \varphi^{(j)}$ is a u-optimal strategy for $\mathfrak{A}$. $\square$

3.3 Trading Corridors

As in Subsection 1.2.3, we define regions of strategies whose local utility does not deviate too far from the optimal value. Since we are dealing with two kinds of local utility in the general framework (Γ_t for $t \in \Theta$ and γ_t for the quasi-continuous part between fixed jump times), we let the trading corridor also depend on two *utility bandwidths* $\varepsilon_1, \varepsilon_2$.

Definition 3.33 Let $\mathfrak{M} \subset \mathfrak{A}$ be convexly restricted with $g^1, \ldots, g^q$ as in Definition 3.3. Assume that a u-optimal strategy $\varphi \in \mathfrak{M}$ for $\mathfrak{M}$ exists and fix $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_+$. Let $J : \Omega \times \mathbb{R}_+ \to \mathfrak{P}(\mathbb{R}^{n+1})$ be a mapping with

$$\begin{aligned} J(\omega, t) \subset M(\omega, t) \quad := \quad & \{\psi \in \mathbb{R}^{n+1} : g_t^j(\psi) \leq 0 \text{ for } j \in \{1, \ldots, p\} \\ & \text{and } g_t^j(\psi) = 0 \text{ for } j \in \{p+1, \ldots, q\}\} \end{aligned}$$

for any $(\omega, t) \in \Omega \times \mathbb{R}_+$. We call J a $(u, \varepsilon_1, \varepsilon_2)$*-trading corridor* for $\mathfrak{M}$ if the following two conditions hold.

1. P-almost surely we have for any $t \in \mathbb{R}_+$ that

$$\psi \in J(\omega, t) \Leftrightarrow \Gamma_t(\psi) \geq \Gamma_t(\varphi_t) - \varepsilon_1.$$

2. Outside some $(P \otimes \lambda)$-null set we have

$$\psi \in J(\omega, t) \Leftrightarrow \gamma_t(\psi) \geq \gamma_t(\varphi_t) - \varepsilon_2.$$

Lemma 3.34 *Let $\mathfrak{M} \subset \mathfrak{A}$ as in Theorem 3.22. Assume that a u-optimal strategy $\varphi \in \mathfrak{M}$ for $\mathfrak{M}$ exists (e.g. by Theorem 3.26). Then we have outside some $(P \otimes (\lambda + \sum_{t \in \Theta} \varepsilon_t))$-null set:*

1. *$J(\omega, t)$ is a non-empty convex set.*

2. *If ψ is an element of the boundary of $J(\omega, t)$ in $M(\omega, t)$, then*

$$\begin{aligned} \Gamma_t(\psi) &= \Gamma_t(\varphi_t) - \varepsilon_1 \quad \textit{for } t \in \Theta \\ \gamma_t(\psi) &= \gamma_t(\varphi_t) - \varepsilon_2 \quad \textit{for } t \in \mathbb{R}_+ \setminus \Theta. \end{aligned}$$

Proofs

PROOF OF LEMMA 3.34. Fix $(\omega, t) \notin N$, where N is a $(P \otimes \lambda)$-null set as in the second condition of Definition 3.17. With the same notation as in the proof of Theorem 3.22, we have that $J(\omega, t) = M(\omega, t) \cap \{\psi \in \mathbb{R}^{n+1} : h(\psi) \leq h(\varphi_t) + \varepsilon_2\}$. Since $M(\omega, t)$ and h are convex, this is a convex set containing $\varphi_t(\omega)$. From the continuity of h, it follows that $h(\psi) = h(\varphi_t) + \varepsilon_2$ on the boundary of $J(\omega, t)$ in $M(\omega, t)$. For $t \in \Theta$, the proof works similarly. □

3.4 Derivative Pricing

While the definition of optimal trading in continuous-time was complicated by the lack of a minimal time span, we can easily transfer the approach concerning derivative pricing to our more general setting. In this section we assume that a market of *underlyings* $0, \ldots, l$ as in the previous sections is given. More specifically, suppose that the underlyings' price process $Z = (Z^0, \ldots, Z^l)$ is an extended Grigelionis process on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P)$ and meets regularity condition (RC 1'). Denote its extended characteristics by $(\Theta, P^{Z_0}, b, c, F, K)^E$. Moreover, *derivatives* $l+1, \ldots, n$ are given by their discounted terminal price X^i at time t_i for $i = l+1, \ldots, n$. The X^i are assumed to be $\mathcal{F}_{t_i}$-measurable random variables. We want to calculate derivative prices under the same assumptions as in Subsection 1.2.4. Suppose that the derivative market is almost exclusively dominated by speculators. More specifically, we assume that all speculators apply u_κ-optimal strategies (with possibly differing risk aversion κ) and that the union of the portfolios of these speculators contains 0 derivatives. By Statement 3 of Lemma 3.29 this implies that there exists a u_1-optimal portfolio for the speculator that has a zero position in any derivative. Thus, the derivative price processes have to be neutral in the sense of the following

Definition 3.35 We call the stochastic processes $Z^{l+1}, \ldots, Z^n$ *neutral price processes* for the derivatives $l+1, \ldots, n$ if the following conditions hold.

1. $\overline{Z} = (Z^0, \dots Z^n)$ is an extended Grigelionis process.

2. $Z^i_t = X^i$ P-almost surely for any $t \geq t_i$ and any $i \in \{l+1, \dots, n\}$.

3. The convexly restricted set $\mathfrak{M} := \{\varphi \in \mathfrak{A} : \varphi^i = 0 \text{ for any } i \in \{l+1, \dots, n\}\}$ contains a u_1-optimal strategy for $\mathfrak{A}$.

Remarks.

1. In particular, the market $\overline{Z}$ meets regularity condition (RC 2). Moreover, by Statement 1 of Lemma 3.29, $\mathfrak{M}$ contains a u_κ-optimal strategy for $\mathfrak{A}$ for any $\kappa > 0$.

2. If we substitute an arbitrary utility function u for the standard functions u_κ, then, by Statement 4 of Lemma 3.29 and the same reasoning as above, we also end up with neutral prices, but this time defined relative to u instead of u_1.

3. Regularity condition (RC 1') is only assumed to ensure that maximization of local utility is an intuitive concept (by Theorem 3.14 and Lemma 3.16). Mathematically, it is not necessary.

The following theorem corresponds to Lemma 1.7 in the introduction. It provides sufficient conditions for neutral derivative prices and allows one to compute them by means of an equivalent martingale measure.

Theorem 3.36 *Let $T := \sup\{t_{l+1}, \dots, t_n\}$ and fix $\kappa > 0$ (e.g. $\kappa = 1$, cf. Remark 2 below). Assume that the following conditions hold.*

1. *The market $Z = (Z^0, \dots Z^l)$ meets regularity condition (RC 2), i.e. there exists a u_κ-optimal strategy $\varphi \in \mathfrak{A}$ for $\mathfrak{A}$.*

2. *φ^T is locally bounded (cf. Lemma A.1 in the appendix).*

3. *The local martingale $L := \mathscr{E}(N)$ is a martingale, where $N = (N_t)_{t \in \mathbb{R}_+}$ is defined by*

$$\begin{aligned} N_t \;&=\; -\kappa \int_0^{t \wedge T} \varphi_s \cdot dZ^c_s + \int_{[0,t\wedge T]\times\mathbb{R}^{l+1}} \Bigg(1_{\Theta^C}(s)(u'_\kappa(\varphi_s \cdot x) - 1) \\ &\quad + 1_\Theta(s) \frac{u'_\kappa(\varphi_s \cdot x)}{\int u'_\kappa(\varphi_s \cdot \widetilde{x})\, K_s(d\widetilde{x})} \Bigg) (\mu - \nu)(ds, dx). \end{aligned}$$

Here, the random measure μ on $\mathbb{R}_+ \times \mathbb{R}^{n+1}$ is defined by

$$\mu([0,t] \times G) := \mu^Z([0,t] \times G) + \varepsilon_0(G) \sum_{s \in \Theta \cap [0,t]} (1 - \mu^Z(\{s\} \times \mathbb{R}^{n+1}))$$

(for any $t \in \mathbb{R}_+$, $G \in \mathcal{B}^{n+1}$) and ν denotes its compensator.

4. *The P^*-local martingale $Z^T - Z_0$ is a P^*-martingale, where the probability measure P^* is defined by $\frac{dP^*}{dP} := L_T$.*

5. *The market is Θ-discrete, or alternatively, $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is the canonical filtration (or its P-completion) of an extended Grigelionis process Y such that all local martingales have the martingale representation property relative to Y (e.g. a process as in Theorem 2.65).*

6. *For any $i \in \{l+1,\dots,n\}$, there exist $\xi,\eta \in \mathfrak{S}$, $M \in \mathbb{R}$ such that*

$$-M + \int_0^T \xi_s \cdot dZ_s \leq X^i \leq M + \int_0^T \eta_s \cdot dZ_s. \tag{3.10}$$

Then there exist up to indistinguishability unique neutral price processes $Z^{l+1},\dots,Z^n$ for the derivatives $l+1,\dots,n$ such that the market $\overline{Z} = (Z^0,\dots,Z^n)$ allows no arbitrage on $[0,T]$. These are given by

$$Z^i_t = E^*(X^i|\mathcal{F}_{t\wedge T}) \;\; \textit{for any } t \in \mathbb{R}_+, \tag{3.11}$$

where E^* denotes expectation with respect to P^*.

Remarks.

1. In Section 3.2 we justify the use of u-optimal strategies only in markets where regularity condition (RC 1') holds. For a satisfactory foundation of the derived prices, one should verify that (RC 1') holds in the enlarged market $\overline{Z} = (Z^0,\dots,Z^n)$ as well.

2. P^* does not depend on the choice of the u_κ-optimal strategy φ nor on the derivatives $l+1,\dots,n$. The conditional expectations $E^*(X^i|\mathcal{F}_{t\wedge T})$ depend neither on φ nor on T as long as $T > t_i$. Moreover, P^* and hence its conditional expectation is independent of the chosen risk aversion κ.

3. In Θ-discrete markets Conditions 2 and 3 in Theorem 3.36 automatically hold (at least φ can be chosen in that way). If, in addition, $Z^i - Z^i_0$ for $i = 0,\dots,l$ is bounded from below by a constant $D \in \mathbb{R}$, then Condition 4 holds as well.

4. Assumption 3 in the previous theorem holds if the following *Novikov*-type condition is fulfilled: For the random variable

$$\begin{aligned} C_T \;:=\; & \frac{\kappa^2}{2}\int_0^T \varphi_t^\top c_t \varphi_t \, dt + \int_0^T \int \Big(u'_\kappa(\varphi_t \cdot x)(\log(u'_\kappa(\varphi_t \cdot x)) - 1) + 1 \Big) F_t(dx)dt \\ & + \sum_{t\in\Theta\cap[0,T]} \int \frac{u'_\kappa(\varphi_t \cdot x)}{\int u'_\kappa(\varphi_t \cdot \widetilde{x})\, K_t(d\widetilde{x})} \log\left(\frac{u'_\kappa(\varphi_t \cdot x)}{\int u'_\kappa(\varphi_t \cdot \widetilde{x})\, K_t(d\widetilde{x})} \right) K_t(dx) \end{aligned}$$

we have $E(\exp(C_T)) < \infty$. (For the last integrand, we set $0/0 = 0$, $\log(0) = -\infty$, $0 \cdot \infty = 0$.)

5. Assumption 4 in Theorem 3.36 holds if $\Theta = \varnothing$ (resp. $K_t = \varepsilon_0$ for any $t \in \Theta$) and

$$E^*\Big(\sum_{i,j=1}^{l} \int_0^T |c_t^{ij}|\, dt + \int_0^T \int (|x| \wedge |x|^2)\, F_t(dx)dt\Big) < \infty$$

6. Assumption 6 in the previous theorem means that the derivatives can be *superhedged*. If we replace it with the following two weaker conditions, then the existence statement and Equation 3.11 (but not necessarily the uniqueness) still hold.
   (a) $E^*(|X^i|) < \infty$ for $i = l+1, \dots, n$.
   (b) The P-semimartingales $Z^{l+1}, \dots, Z^n$ defined by Equation 3.11 are P-special semimartingales.
7. For the proof of Theorem 3.36 it suffices to assume that Condition 6 holds with a $\mathfrak{F}_0$-measurable, integrable random variable M instead of $M \in \mathbb{R}$.

In discrete-time models the previous theorem looks a little easier.

Corollary 3.37 *Let $T := \sup\{t_{l+1}, \dots, t_n\}$ and assume that the following conditions hold.*

1. *The market $Z = (Z^0, \dots, Z^l)$ is Θ-discrete and allows no arbitrage. Moreover, $Z^i - Z_0^i$ is bounded from below by a constant $D \in \mathbb{R}$ for $i = 0, \dots, l$.*
2. *For any $i \in \{l+1, \dots, n\}$ there exist $\xi, \eta \in \mathfrak{S}$, $M \in \mathbb{R}$ such that*

$$-M + \int_0^T \xi_s \cdot dZ_s \le X^i \le M + \int_0^T \eta_s \cdot dZ_s.$$

Define the probability measure P^* (the same as in Theorem 3.36) by its Radon-Nikodým density

$$\frac{dP^*}{dP} := \prod_{s \in \Theta \cap [0,T]} \frac{u'(\varphi_s \cdot \Delta Z_s)}{\int u'(\varphi_s \cdot x)\, K_s(dx)},$$

where $\varphi \in \mathfrak{A}$ is a u_κ-optimal strategy for $\mathfrak{A}$. Then there exist up to indistinguishability unique neutral price processes $Z^{l+1}, \dots, Z^n$ for the derivatives $l+1, \dots, n$. These are given by

$$Z_t^i = E^*(X^i | \mathfrak{F}_{t \wedge T}) \ \text{ for any } t \in \mathbb{R}_+, \tag{3.12}$$

where E^* denotes expectation with respect to P^*.

Remark. $\mathbb{N}^*$-discrete markets can be expressed in terms of the transition probabilities $P(A|\mathfrak{F}_{t-1})$ for any $t \in \mathbb{N}^*$, $A \in \mathfrak{F}_t$. Relative to the pricing measure, the corresponding transition probabilities are given by

$$P^*(A|\mathfrak{F}_{t-1}) = E\bigg(\frac{u'_\kappa(\varphi_t \cdot \Delta Z_t)}{E(u'_\kappa(\varphi_t \cdot \Delta Z_t)|\mathfrak{F}_{t-1})} 1_A \bigg| \mathfrak{F}_{t-1}\bigg),$$

where the $\mathcal{F}_{t-1}$-measurable $\mathbb{R}^{l+1}$-valued random vector φ_t is chosen such that

$$\int x u'_\kappa(\varphi_t \cdot x)\, P^{\Delta Z_t|\mathcal{F}_{t-1}}(dx) = 0.$$

Observe that, in order to compute the transition probabilities from $\mathcal{F}_{t-1}$ to $\mathcal{F}_t$ relative to P^* (for fixed $t \in \mathbb{N}$), all we have to know are the corresponding probabilities relative to P. The past up to time $t-1$ or the future beyond t of the model are irrelevant. Derivative prices can now be recursively obtained from

$$Z^i_{t-1} = E^*(Z^i_t|\mathcal{F}_{t-1}).$$

Often one is not really interested in the Radon-Nikodým density of the pricing measure P^*, but rather in the dynamic of the price processes relative to P^*.

Corollary 3.38 *Suppose that $V = (V^1, \ldots, V^k)$ is an $\mathbb{R}^k$-valued stochastic process such that $(Z, V) = (Z^0, \ldots, Z^l, V^1, \ldots, V^k)$ is an extended Grigelionis process with extended characteristics $(\Theta, P^{(Z,V)_0}, \overline{b}, \overline{c}, \overline{F}, \overline{K})^E$. Assume that Conditions 1–3 in Theorem 3.36 or Condition 1 in Corollary 3.37 hold. If $V^1, \ldots, V^k$ are P^*-special semimartingales, then (Z, V) is an extended Grigelionis process relative to P^*, and its P^*-characteristics $(\Theta, P^{(Z,V)_0}, \widetilde{b}, c, \widetilde{F}, \widetilde{K})^E$ are given by*

$$\widetilde{b}^i_t = 0 \ \textit{ for } i = 0, \ldots, l,$$

$$\widetilde{b}^i_t = \overline{b}^i_t - \kappa \sum_{\alpha=0}^{l} \overline{c}^{i\alpha}_t \varphi^\alpha_t + \int x^i \Big(u'_\kappa\Big(\sum_{\alpha=0}^{l} \varphi^\alpha_t x^\alpha \Big) - 1 \Big)\, \overline{F}_t(dx) \textit{ for } i = l+1, \ldots, l+k,$$

$$\widetilde{F}_t(G) = \int 1_G(x) \Big(u'_\kappa\Big(\sum_{\alpha=0}^{l} \varphi^\alpha_t x^\alpha \Big) \Big)\, \overline{F}_t(dx) \textit{ for any } G \in \mathcal{B}^{l+1+k},$$

$$\widetilde{K}_t(G) = \int 1_G(x) \frac{u'_\kappa(\sum_{\alpha=0}^{l} \varphi^\alpha_t x^\alpha)}{\int u'_\kappa(\sum_{\alpha=0}^{l} \varphi^\alpha_t \widetilde{x}^\alpha)\, \overline{K}_t(d\widetilde{x})}\, \overline{K}_t(dx) \textit{ for any } G \in \mathcal{B}^{l+1+k}$$

for any $t \leq T$, where in $\widetilde{b} = (\widetilde{b}^0, \ldots, \widetilde{b}^{l+k})$ etc. the indices $0, \ldots, l$ correspond to Z and the components $l+1, \ldots, l+k$ to V. For any $t > T$, we have $(\widetilde{b}_t, \widetilde{c}_t, \widetilde{F}_t, \widetilde{K}_t) = (b_t, c_t, F_t, K_t)$.

Remarks.

1. The statements and proofs in this section still hold if one substitutes an arbitrary utility function u with risk aversion κ for the standard utility u_κ. If the processes $Z^0, \ldots, Z^l$ are continuous, then the resulting pricing measure P^* and hence the neutral derivative prices are independent of the choice of the utility function u (and of κ). In the jump case this is generally not true. It is an open question how strongly the prices are affected by the particular shape of u. We hope that this dependence is not very large.

2. We are interested in easily checkable sufficient conditions for Condition 4 in Theorem 3.36 and for integrability condition (RC 1) for the enlarged market $\overline{Z} = (Z^0, \ldots, Z^n)$.

Proofs

Proposition 3.39 *Let V^1, V^2 be special semimartingales. If X is a semimartingale with $V^1 \leq X \leq V^2$, then X is a special semimartingale as well.*

PROOF. Let $B := \sum_{t \leq \cdot} \Delta X_t 1_{\{|\Delta X_t|>1\}}$ and $\widetilde{X} := X - B$. By JS, I.4.24, $\widetilde{X}$ is a special semimartingale. Moreover, B has pathwise only finitely many jumps on any finite interval. Since $V^2 - V^1$ is a special semimartingale, we have $\sup_{t\leq\cdot} |(V_t^2 - V_t^1) - (V_0^2 - V_0^1)| \in \mathscr{A}_{\mathrm{loc}}^+$ (cf. JS, I.4.23). By Jacod (1979), Exercise 2.8 we have $\sqrt{[V^1, V^1]} \in \mathscr{A}_{\mathrm{loc}}^+$ and hence $\sqrt{\sum_{t\leq\cdot}(\Delta V_t^1)^2} \in \mathscr{A}_{\mathrm{loc}}^+$. Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of stopping times with $T_n \uparrow \infty$ P-almost surely and such that $\{t \leq T_n : |\Delta X_t| > 1\} \leq n$, $E(\sup_{t\leq T_n} |(V_t^2 - V_t^1) - (V_0^2 - V_0^1)|) < \infty$, $E\left(\sqrt{\sum_{t\leq T_n}(\Delta V_t^1)^2}\right) < \infty$ and $|V_0^2 - V_0^1| \leq n$ on $\{T_n > 0\}$. From $|\Delta X| \leq (V^2 - V^1) + |\Delta V^1|$ we can now conclude $\mathrm{Va}(B)_{T_n} = \sum_{t\leq T_n} |\Delta X_t| 1_{\{|\Delta X_t|>1\}} \leq n \sup_{t\leq T_n} |(V_t^2 - V_t^1) - (V_0^2 - V_0^1)| + n + \sum_{t\leq T_n} |\Delta V_t^1| 1_{\{|\Delta X_t|>1\}}$ for any $n \in \mathbb{N}$. Since $\frac{1}{n}\sum_{t\leq T_n} |\Delta V_t^1| 1_{\{|\Delta X_t|>1\}} \leq \sqrt{\frac{1}{n}\sum_{t\leq T_n} |\Delta V_t^1|^2}$, we obtain that $E(\mathrm{Va}(B)_{T_n}) < \infty$ as well. Therefore, $B \in \mathscr{A}_{\mathrm{loc}}$ and hence it is a special semimartingale (cf. JS, I.4.23). This proves the claim. □

PROOF OF THEOREM 3.36. *First step:* We show that N is well-defined and strictly positive on $\mathbb{R}_+$. Firstly, note that $\nu([0,t] \times G) = \int_0^t F_s(G)\,ds + \sum_{s\in\Theta\cap[0,t]} K_s(G)$ for any $G \in \mathcal{B}^{n+1}$, $t \in \mathbb{R}_+$. Since $|u_\kappa'(\varphi_s \cdot x) - 1| \leq \sup_{y\in\mathbb{R}} |u_\kappa''(y)||\varphi_s||x|$ and since φ is locally bounded and $\int_0^t \int (|x|^2 \wedge |x|)\, F_s(dx)\, ds < \infty$ P-almost surely for any t, we have that $\int_{[0,\cdot]\times\mathbb{R}^{n+1}} 1_{\Theta^C}(s)(|u_\kappa'(\varphi_s \cdot x) - 1|^2 \wedge |u_\kappa'(\varphi_s \cdot x) - 1|)\, \nu(ds, dx) \in \mathscr{V}^+$ and hence $\in \mathscr{A}_{\mathrm{loc}}^+$ by JS, I.3.10. Therefore, $1_{\Theta^C}(s)(u_\kappa'(\varphi_s \cdot x) - 1)$ is in $G_{\mathrm{loc}}(\mu)$ (cf. JS, II.1.33c). Moreover, one easily verifies that $|W| * \nu_t = \sum_{s\leq t} 1_\Theta(s) = |\Theta \cap [0,t]|$ for $W(\omega, s, x) := 1_{\Theta^C}(s) \frac{u_\kappa'(\varphi_s \cdot x)}{\int u_\kappa'(\varphi_s \cdot \widetilde{x})\, K_s(d\widetilde{x})}$. By JS, II.1.28, this implies that $W \in G_{\mathrm{loc}}(\mu)$. Together, we obtain that the integrand in the stochastic integral with respect to $\mu - \nu$ is in $G_{\mathrm{loc}}(\mu)$ and hence N is well-defined. Observe that, by definition of the integral with respect to $\mu - \nu$, the jumps of N are given by

$$\Delta N_t = \begin{cases} u_\kappa'(\varphi_t \cdot \Delta Z_t) - 1 & \text{if } t \notin \Theta, \\ \frac{u_\kappa'(\varphi_s \cdot \Delta Z_t)}{\int u_\kappa'(\varphi_s \cdot \widetilde{x})\, K_s(d\widetilde{x})} - 1 & \text{if } t \in \Theta. \end{cases} \tag{3.13}$$

Since $u_\kappa' > 0$, we have that $\Delta N_t + 1$ is positive, which, by Jacod (1979), (6.5), implies that N is positive as well.

Second step: Since L_T is a positive random variable with $E(L_T) = 1$, we have that P^* is a well-defined probability measure equivalent to P. From the boundedness of u_κ as well as Lemma 2.27 and Condition 3 in the remark following Theorem 2.26, it follows that $Z^0, \ldots, Z^l$ are P^*-special semimartingales. Assume that $V^1, \ldots, V^k$ are arbitrary processes as in Corollary 3.38. By Theorem 2.26, (Z, V) is an extended Grigelionis process relative to P^*. Corollary 3.23 yields that, outside some $(P \otimes \lambda)$-null set, we have

$$0 = \overline{b}^i_t - \kappa \sum_{\alpha=0}^{l} \overline{c}^{i\alpha}_t \varphi^\alpha_t + \int x^i \Big(u'_\kappa \Big(\sum_{\alpha=0}^{l} \varphi^\alpha_t x^\alpha \Big) - 1 \Big) \overline{F}_t(dx)$$

for $i = 0, \dots, l$ and $t \leq T$, since the first $l+1$ components of (Z, V) are $(Z^0, \dots, Z^l)$. Therefore, the shape of the P^*-characteristics in Corollary 3.38 follows immediately from Lemma 2.27, Theorem 2.26 and the fact that $1_\Theta * (\mu - \nu) = 0$. In particular, we have that

$$\Delta \widetilde{B}^i_t = \int x^i \widetilde{K}_t(dx) = \frac{1}{\int u'_\kappa(\sum_{\alpha=0}^{l} \varphi^\alpha_t \widetilde{x}^\alpha) \overline{K}_s(d\widetilde{x})} \int x^i u'_\kappa \Big(\sum_{\alpha=0}^{l} \varphi^\alpha_t x^\alpha \Big) \overline{K}_t(dx)$$

for $i = 0, \dots, l$ and $t \leq T$ (cf. Remark 2 in Section 2.4), where $\widetilde{B}$ denotes the predictable part of finite variation of (Z, V) relative to P^*. The last integral equals $\int x^i u'_\kappa(\sum_{\alpha=0}^{l} \varphi^\alpha_t x^\alpha)$ $K_t(dx)$, which is 0 by Corollary 3.23. Altogether, we obtain that $(\widetilde{B}^i)^T = 0$ for any $i \in \{0, \dots, l\}$, and hence that $(Z - Z_0)^T$ is a P^*-local martingale.

Third step: We will now prove that $X^{l+1}, \dots, X^n$ are P^*-integrable. Since $Z^T - Z_0$ is a P^*-uniformly integrable martingale and hence of class (D), it follows that $Z^T_\tau - Z_0$ is P^*-integrable for any stopping time τ. Since ξ, η in Condition 6 are in $\mathfrak{S}$, this implies that the X^i are P^*-integrable as well.

Fourth step: By Equation (3.11) we define P^*-martingales $Z^{l+1}, \dots, Z^n$. We will now show that $Z^{l+1}, \dots, Z^n$ are P-special semimartingales. Fix $i \in \{l+1, \dots, n\}$ and let M, ξ, η be as in Condition 6. By Lemma 2.22 we have that $-M + \int_0^\cdot \xi_s \cdot dZ^T_s$ and $M + \int_0^\cdot \eta_s \cdot dZ^T_s$ are extended Grigelionis processes and hence special semimartingales. Moreover, we have that

$$Z^i_t = E^*(X^i | \mathcal{F}_t) \leq M + E^* \Big(\int_0^T \eta_s \cdot dZ^T_s \Big| \mathcal{F}_t \Big) = M + \int_0^t \eta_s \cdot dZ^T_s$$

for any $t \in \mathbb{R}_+$, where the last equation follows from the martingale property of $\int_0^\cdot \eta_s \cdot dZ^T_s$. Similarly, one shows $Z^i \geq -M + \int_0^\cdot \xi_s \cdot dZ^T_s$. By Proposition 3.39 we have that Z^i is a special semimartingale.

Fifth step: We will now show that $\overline{Z} = (Z^0, \dots, Z^n)$ is an extended Grigelionis process relative to P. If the market is Θ-discrete, this follows immediately from Lemma 2.20 and the subsequent remark. Otherwise, let Y be as in Condition 5. W.l.o.g., we may assume that the set of fixed jump times in the extended characteristics of Y is also Θ. By $Y^{c,*}$, $\nu^{Y,*}$ we denote the continuous martingale part and the compensator of the jump measure μ^Y of Y relative to P^* instead of P. By Girsanov's theorem (cf. JS, III.3.24) we have that $\langle Y^{c,*}, Y^{c,*} \rangle$ and $\nu^{Y,*}([0, \cdot] \times G)$ are absolutely continuous with respect to $A_t = t + \sum_{s \leq t} 1_\Theta(s)$ in the sense of Lemma 2.10. In particular, $Y^{c,*}$ is an extended Grigelionis process relative to P^*. By JS, III.5.24 all P^*-local martingales have the representation property relative to Y. Therefore, $\overline{Z}^T$ can be written as $(Z^i)^T = Z^i_0 + \int_0^\cdot H^i_s \cdot dY^{c,*}_s + W^i * (\mu^Y - \nu^{Y,*})$ for some $H^i \in L^2_{\mathrm{loc}}(Y^{c,*})$, $W^i \in G_{\mathrm{loc}}(\mu^Y)$ (both relative to P^*) for $i = 0, \dots, n$. By JS, III.4.7 one obtains that the first integral is an extended Grigelionis process relative to P^*. The last term $W * (\mu^Y - \nu^{Y,*})$ and hence $\overline{Z}^T$ is, by Proposition 2.24 and the following remark, a P^*-extended Grigelionis process as well. Since $Z^0, \ldots, Z^n$ are P-special semimartingales, Theorem 2.26 yields that $\overline{Z}$ is an extended Grigelionis process relative to P.

Sixth step: We will show that $Z^{l+1}, \ldots, Z^n$ are neutral price processes for the derivatives $l+1, \ldots, n$ and that $\overline{Z} = (Z^0, \ldots, Z^n)$ allows no arbitrage on $[0,T]$. Denote by $(\Theta, P^{\overline{Z}_0}, \overline{b}, \overline{c}, \overline{F}, \overline{K})^E$ the extended characteristics of $\overline{Z}$ relative to P. Application of the second step (i.e. Corollary 3.38) to $Z^{l+1}, \ldots, Z^n$ yields for the extended characteristics $(\Theta, P^{*\overline{Z}_0}, \widetilde{b}, \overline{c}, \widetilde{F}, \widetilde{K})^E$ of $\overline{Z}$ relative to P^*:

$$\widetilde{b}^i_t = \overline{b}^i_t - \kappa \sum_{\alpha=0}^{l} \overline{c}^{i\alpha}_t \varphi^\alpha_t + \int x^i \Big(u'_\kappa \Big(\sum_{\alpha=0}^{l} \varphi^\alpha_t x^\alpha \Big) - 1 \Big) \overline{F}_t(dx) \tag{3.14}$$

$$\widetilde{K}_t(G) = \int 1_G(x) \frac{u'_\kappa(\sum_{\alpha=0}^{l} \varphi^\alpha_t x^\alpha)}{\int u'_\kappa(\sum_{\alpha=0}^{l} \varphi^\alpha_t \widetilde{x}^\alpha)\, \overline{K}_t(d\widetilde{x})} \overline{K}_t(dx)$$

for any $i \in \{0, \ldots, n\}$, $t \in [0,T]$. In particular,

$$\Delta \widetilde{B}^i_t = \int x^i \, \widetilde{K}_t(dx) = \frac{1}{\int u'_\kappa(\sum_{\alpha=0}^{l} \varphi^\alpha_t \widetilde{x}^\alpha)\, \overline{K}_t(d\widetilde{x})} \int x^i u'_\kappa \Big(\sum_{\alpha=0}^{l} \varphi^\alpha_t x^\alpha \Big) \overline{K}_t(dx) \tag{3.15}$$

for any $i \in \{0, \ldots, n\}$, $t \in \Theta \cap [0,T]$, where $\widetilde{B}$ denotes the predictable part of finite variation of the P^*-special semimartingale $\overline{Z}$. Since $\overline{Z}^T - \overline{Z}_0$ is a P^*-local martingale, we have $\widetilde{B}^T = 0$. Therefore, Equations (3.14), (3.15) and Corollary 3.23 yield that the strategy $\widetilde{\varphi}$, defined by $\widetilde{\varphi}^i := \varphi^i$ for $i = 0, \ldots, l$ and $\widetilde{\varphi}^i = 0$ for $i = l+1, \ldots, n$, is u_κ-optimal for $\mathfrak{A}$. Thus, $Z^{l+1}, \ldots, Z^n$ are neutral price processes for $l+1, \ldots, n$. Moreover, the market $\overline{Z}^T = (Z^0, \ldots, Z^n)^T$ allows no arbitrage (cf. Lemma 3.7).

Seventh step: We show that P^* does not depend on the choice of the optimal strategy φ. Let $\widetilde{\varphi} = (\widetilde{\varphi}^0, \ldots, \widetilde{\varphi}^l)$ be another u_κ-optimal strategy for the speculator in the market $(Z^0, \ldots, Z^l)$. In the proof of Lemma 3.24 we have shown that outside the usual null sets we have $c_t(\varphi_t - \widetilde{\varphi}_t) = 0$ and $\varphi_t \cdot x = \widetilde{\varphi}_t \cdot x$ for F_t- resp. K_t-almost all $x \in \mathbb{R}^{n+1}$. In particular, we have $W * \nu = 0$ for any $W \in G_{\mathrm{loc}}(\mu)$ of the form $W(\omega, t, x) = g(\omega, t, \varphi_t(\omega) \cdot x) - g(\omega, t, \widetilde{\varphi}_t(\omega) \cdot x)$, where $g : \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a $(\mathcal{P} \otimes \mathcal{B})$-measurable mapping. By JS, II.1.34 it follows that $\langle W*(\mu-\nu), W*(\mu-\nu) \rangle = 0$ and hence $W*(\mu-\nu) = 0$. This implies that the stochastic integrals relative to $\mu - \nu$ in the definition of N coincide regardless of whether we insert the strategy φ or $\widetilde{\varphi}$. Indeed, just observe that the integrand is of the form $g(\omega, t, \varphi_t(\omega) \cdot x)$ for some predictable g. Similarly, we have that $\int_0^\cdot \varphi_s \cdot dZ^c_s - \int_0^\cdot \widetilde{\varphi}_s \cdot dZ^c_s = \int_0^\cdot (\varphi_s - \widetilde{\varphi}_s) \cdot dZ^c_s = 0$ since $\langle \int_0^\cdot (\varphi_s - \widetilde{\varphi}_s) \cdot dZ^c_s, \int_0^\cdot (\varphi_s - \widetilde{\varphi}_s) \cdot dZ^c_s \rangle = \int_0^\cdot (\varphi_s - \widetilde{\varphi}_s)^\top c_s (\varphi_s - \widetilde{\varphi}_s)\, ds = 0$ by JS, II.4.7. Together, we obtain that the local martingale N is the same for φ and $\widetilde{\varphi}$.

Eighth step: We will show that, up to indistinguishability, there are no other neutral price processes for the derivatives such that the extended market allows no arbitrage. Otherwise, let $\widetilde{Z}^{l+1}, \ldots, \widetilde{Z}^n$ be another set of such processes. By definition, there is a strategy $\widetilde{\varphi} = (\widetilde{\varphi}^0, \ldots, \widetilde{\varphi}^l, 0, \ldots, 0)$ that is u_κ-optimal for the speculator in the market

$(Z^0, \dots, Z^l, \widetilde{Z}^{l+1}, \dots, \widetilde{Z}^n)$. Using Corollary 3.23, this implies that $(\widetilde{\varphi}^0, \dots, \widetilde{\varphi}^l)$ is u_κ-optimal for the speculator in the market $(Z^0, \dots, Z^l)$. By the seventh step, φ and $(\widetilde{\varphi}^0, \dots, \widetilde{\varphi}^l)$ lead to the same measure P^*. Fix $i \in \{l+1, \dots, n\}$. From Condition 6 and a simple arbitrage argument, it follows that

$$-M + \int_0^t \xi_s \cdot dZ_s \leq \widetilde{Z}_t^i \leq M + \int_0^t \eta_s \cdot dZ_s \tag{3.16}$$

for any $t \in [0, T]$. Therefore, $\widetilde{Z}^{l+1}, \dots, \widetilde{Z}^n$ are P^*-special semimartingales (cf. Proposition 3.39). Application of the second step (i.e. Corollary 3.38) to $\widetilde{Z}^{l+1}, \dots, \widetilde{Z}^n$ yields as before

$$\widetilde{b}_t^i = \overline{b}_t^i - \kappa \sum_{\alpha=0}^{l} \overline{c}_t^{i\alpha} \widetilde{\varphi}_t^\alpha + \int x^i \Big(u_\kappa' \Big(\sum_{\alpha=0}^{l} \widetilde{\varphi}_t^\alpha x^\alpha \Big) - 1 \Big) \, \overline{F}_t(dx), \tag{3.17}$$

$$\Delta \widetilde{B}_t^i = \int x^i \, \widetilde{K}_t(dx) = \frac{1}{\int u_\kappa'(\widetilde{\varphi}_t \cdot \widetilde{x}) \, \overline{K}_t(d\widetilde{x})} \int x^i u_\kappa' \Big(\sum_{\alpha=0}^{l} \widetilde{\varphi}_t^\alpha x^\alpha \Big) \, \overline{K}_t(dx)$$

for any $i \in \{0, \dots, n\}$, $t \in [0, T]$, where now $\overline{b}, \overline{B}$ etc. correspond to the P-characteristics of $(Z^0, \dots, Z^l, \widetilde{Z}^{l+1}, \dots, \widetilde{Z}^n)$ and $\widetilde{b}, \widetilde{B}$ etc. to the P^*-characteristics of the same process. By Corollary 3.23 and the u_κ-optimality of $(\widetilde{\varphi}^0, \dots, \widetilde{\varphi}^l, 0, \dots, 0)$ it follows that $\widetilde{B}^T = 0$, which in turn implies that $\widetilde{Z}^{l+1} - \widetilde{Z}_0^{l+1}, \dots, \widetilde{Z}^n - \widetilde{Z}_0^n$ are P^*-local martingales. Since they are bounded from below and above by uniformly integrable P^*-martingales (cf. Equation 3.16), it follows that they are of class (D) and therefore P^*-martingales with the same terminal values $X^{l+1}, \dots, X^n$ as $Z^{l+1}, \dots, Z^n$. This implies $\widetilde{Z}^i = Z^i$ for $i = l+1, \dots, n$. □

PROOF OF THE REMARKS. *2.* We have already shown in the seventh step of the preceding proof that P^* and hence $E^*(X^i | \mathcal{F}_{t \wedge T})$ does not depend on the choice of φ. Moreover, observe that for different choices of T the corresponding measures P^* coincide on the σ-field $\mathcal{F}_T$ with the smallest index T. The independence of κ follows from Statement 1 in Lemma 3.29, from $u_\kappa'(y) = u_1'(\kappa y)$ and the definition of N in Theorem 3.36.

3. Since the definition of φ on the open intervals beween neighbouring points of Θ does not affect its local utility, we may choose φ Θ-discrete with $\varphi_0 = 0$. By Lemma A.1, φ is locally bounded. Let us assume $\Theta = \mathbb{N}^*$ for ease of notation. Firstly, observe that N is a discrete local martingale. Therefore, $\mathscr{E}(N)$ is a discrete local martingale as well. Since it is non-negative, it is a supermartingale (cf. Jacod (1979), (5.17)) and in particular integrable. By JS, p.15 this implies that $\mathscr{E}(N)$ is a martingale. Fix $i \in \{0, \dots, l\}$. If $Z^i - Z_0^i$ is bounded from below by $D \in \mathbb{R}$, then $(Z^i)^T - Z_0^i - D$ is a non-negative local P^*-martingale and therefore a P^*-supermartingale (Jacod (1979), (5.17)), in particular all $(Z^i)^T - Z_0^i$ are integrable. This implies that $(Z^i)^T - Z_0^i$ is a P^*-martingale (cf. JS, p.15).

4. Observe that $\nu(\{t\} \times \mathbb{R}^{n+1}) = 1_\Theta(t)$ for any $t \in \mathbb{R}_+$. Moreover, we have for any predictable mapping $W : \Omega \times \mathbb{R}_+ \times \mathbb{R}^{n+1} \to \mathbb{R}$ that $\widehat{W}_t := \int W(t,x) \, \nu(\{t\} \times dx) = 1_\Theta(t) \int W(t,x) \, K_t(dx)$ in the sense of JS, II.1.24. In the case $W(s,x) = 1_{\Theta^C}(s)(u_\kappa'(\varphi_s \cdot x) - 1) + 1_\Theta(s) \frac{u_\kappa'(\varphi_s \cdot x)}{\int u_\kappa'(\varphi_s \cdot \widetilde{x}) \, K_s(d\widetilde{x})}$, we thus have $\widehat{W}_t = 1_\Theta(t)$. The claim now follows directly from application of Jacod (1979), (8.44) to $\mathscr{E}(N)$.

5. Relative to P^*, the local martingale $Z - Z_0$ can be decomposed as $Z^T - Z_0 = Z^{c,*} + \int x(\mu - \nu^*)(ds, dx)$, where $Z^{c,*}$ denotes the P^*-continuous martingale part of Z and ν^* the P^*-compensator of μ. From Girsanov's theorem (cf. Theorem 2.26), we conclude that $\langle Z^{c,*,i}, Z^{c,*,i}\rangle_t = \int_0^t c_s^{ii}\, ds$ for $i = 0, \dots, l$. The integrability condition in Remark 5 implies that $Z^{c,*,i}$ are square-integrable P^*-martingales on $[0, T]$. Also by Girsanov's theorem we have that

$$\begin{aligned}\int_{[0,T]\times\mathbb{R}^{n+1}} (|x|^2 \wedge |x|)\, \nu^*(dt, dx) &= \int_0^T \int (|x|^2 \wedge |x|) u'_\kappa(\varphi_t \cdot x)\, F_t(dx)\, dt \\ &\leq \sup_{y\in\mathbb{R}} |u'_\kappa(y)| \int_0^T \int (|x|^2 \wedge |x|)\, F_t(dx)\, dt.\end{aligned}$$

The second condition in Remark 5 and Proposition 2.8 yield that $x^i * (\mu - \nu^*)$ is a uniformly integrable P^*-martingale on $[0, T]$ for $i = 0, \dots, l$.

6. This follows immediately by skipping the third, fourth and eighth step in the proof of Theorem 3.36. □

PROOF OF COROLLARY 3.37. Condition 1 in Theorem 3.36 holds by Theorem 3.28. For Conditions 2–4, cf. Remark 3. The shape of $\frac{dP^*}{dP}$ follows from JS, I.4.63. □

PROOF OF THE REMARK. Firstly, observe that $K_t = P^{\Delta Z_t|\mathcal{F}_{t-1}}$ by Lemma 2.20. The expression for $P^*(A|\mathcal{F}_{t-1})$ follows as Statement 3 in Proposition 1.6. □

PROOF OF COROLLARY 3.38. This has already been shown in the second step of the proof of Theorem 3.36. □

3.5 Price Regions

As in Subsections 1.2.5 and 1.2.6, we now want to relax the assumption that non-speculators are not present in the derivative market in order to obtain reasonable price regions and improved derivative models. As in Chapter 1, we will introduce two notions of supply-consistent prices. The first one leads to price regions and models that are intuitive from an economic point of view but lead to difficulties on the mathematical side. These problems will be relaxed by defining approximate prices, although at the expense of a weaker theoretical foundation. In the following two sections we will work exclusively with standard utility functions (but cf. Remark 4 in Section 3.6). The general setting is as in the previous section, i.e. we are given underlyings $0, \dots, l$ and derivatives $l + 1, \dots, n$ at maturity. As before, we assume that there are many speculators in the market who all trade with u_κ-optimal strategies for different values of κ. By Lemma 3.29 the union of all these portfolios is again a u_κ-optimal strategy for some $\kappa > 0$. Contrary to the previous section, we do not assume that this union portfolio contains no derivatives. Instead, we suppose that it contains constantly $\rho^i \in \mathbb{R}$ shares of derivative i for $i = l + 1, \dots, n$. This implies that their price processes have to be $(\kappa, \rho^{l+1}, \dots, \rho^n)$-consistent in the sense of the following

Definition 3.40 Let $\kappa > 0$, $\rho^{l+1}, \dots, \rho^n \in \mathbb{R}$. We call stochastic processes $Z^{l+1}, \dots, Z^n$ $(\kappa, \rho^{l+1}, \dots, \rho^n)$*-consistent* for the derivatives $l+1, \dots, n$ if the following conditions hold.

1. $\overline{Z} = (Z^0, \dots, Z^n)$ is an extended Grigelionis process.
2. $Z^i_t = X^i$ P-almost surely for any $t \geq t_i$ and any $i \in \{l+1, \dots, n\}$.
3. The convexly restricted set $\mathfrak{M} := \{\varphi \in \mathfrak{A} : \varphi^i = \rho^i \text{ for any } i \in \{l+1, \dots, n\}\}$ contains a u_κ-optimal strategy φ for $\mathfrak{A}$.

Remarks.

1. Since $(\kappa, \rho^{l+1}, \dots, \rho^n)$-consistent prices are identical to $(1, \kappa\rho^{l+1}, \dots, \kappa\rho^n)$-consistent processes, it usually suffices to consider the case $\kappa = 1$.
2. Neutral price processes are $(\kappa, 0, \dots, 0)$-consistent price processes (for any $\kappa > 0$) and vice versa.

The following lemma means that $(\kappa, \rho^{l+1}, \dots, \rho^n)$-consistent prices are "usually" conditional expectations under some equivalent martingale measure that is given in terms of a u_κ-optimal strategy for the speculator.

Lemma 3.41 *Let $T := \sup\{t_{l+1}, \dots, t_n\}$ and fix $\kappa > 0$, $\rho^{l+1}, \dots, \rho^n \in \mathbb{R}$. Assume that $Z^{l+1}, \dots, Z^n$ are $(\kappa, \rho^{l+1}, \dots, \rho^n)$-consistent price processes for the derivatives $l+1, \dots, n$ and that the strategy φ in Definition 3.40 can be chosen P-almost surely pathwise bounded on $[0, T]$. Define the local martingale $L := \mathscr{E}(N)$ by*

$$
\begin{aligned}
N_t = &-\kappa \int_0^{t \wedge T} \varphi_s \cdot d\overline{Z}^c_s \\
&+ \int_{[0, t\wedge T] \times \mathbb{R}^{n+1}} \left(1_{\Theta^c}(s)(u'_\kappa(\varphi_s \cdot x) - 1) + 1_\Theta(s) \frac{u'_\kappa(\varphi_s \cdot x)}{\int u'_\kappa(\varphi_s \cdot \widetilde{x})\, \overline{K}_s(d\widetilde{x})} \right) (\overline{\mu} - \overline{\nu})(ds, dx),
\end{aligned}
$$

where $\overline{Z} = (Z^0, \dots, Z^n)$ and the random measures $\overline{\mu}, \overline{\nu}$ are defined as in Theorem 3.36, but relative to $\overline{Z}$ instead of Z, and $\overline{K}$ denotes the last component in the extended characteristics of $\overline{Z}$. Suppose that L^T is a martingale so that we can define a probability measure P^* by $\frac{dP^*}{dP} := L_T$. Then $\overline{Z} - \overline{Z}_0$ is a P^*-local martingale. If $\overline{Z}$ is a P^*-martingale, then obviously

$$
Z^i_t = E^*(X^i | \mathcal{F}_{t \wedge T}) \quad \textit{for any } t \in \mathbb{R}_+ \tag{3.18}
$$

and any $i \in \{l+1, \dots, n\}$, where E^* denotes conditional expectation with respect to P^*.

In Subsection 1.2.5 we define price regions as the set of all price processes that correspond to moderate values of external supply $\rho^{l+1}, \dots, \rho^n$. This is repeated here.

Definition 3.42 Fix $\kappa > 0$ and a *supply bound* $r \geq 0$. We say that processes $Z^{l+1}, \dots, Z^n$ belong to the κr*-price region* if they are $(\kappa, \rho^{l+1}, \dots, \rho^n)$-consistent derivative price processes for some $\rho^{l+1}, \dots, \rho^n$ satisfying $|\rho^i| \leq r$ for $i = l+1, \dots, n$.

Remark. By Remark 1 the κr-price region only depends on the product κr.

Although the previous lemma looks similar to Theorem 3.36 formally, the analogy is very limited. In Lemma 3.41 we do not show that $(\kappa, \rho^{l+1}, \dots, \rho^n)$-consistent price processes really exist. Moreover, the density process L does not help in computing the derivative prices, since it depends on the unknown processes $Z^{l+1}, \dots Z^n$. We will use the latter fact to define an approximate notion of consistency as in Subsection 1.2.5. For both the computation of a u_κ-optimal strategy for $\mathfrak{M}$, as well as for the definition of L, we substitute the known neutral derivative prices as an approximation for the unknown consistent processes. More precisely, we proceed as follows.

1. Let $T := \sup\{t_{l+1}, \dots, t_n\}$ and fix $\kappa > 0$, $\rho^{l+1}, \dots, \rho^n \in \mathbb{R}$.

2. Assume that the conditions in Theorem 3.36 or Corollary 3.37 hold. There are then unique neutral price processes $Z^{l+1}, \dots, Z^n$.

3. Let $\varphi \in \mathfrak{M} := \{\varphi \in \mathfrak{A} : \varphi^i = \rho^i$ for any $i \in \{l+1, \dots, n\}\}$ be a u_κ-optimal strategy for $\mathfrak{M}$ in the market $\overline{Z} = (Z^0, \dots, Z^n)$. Such a strategy exists by Theorem 3.26 and can be computed using Corollary 3.23. W.l.o.g., let $\varphi_0 := (0, \dots, 0, \rho^{l+1}, \dots, \rho^n)$.

4. Assume that φ is P-almost surely pathwise bounded on $[0, T]$. This is no restriction if the market is Θ-discrete (cf. Remark 3 following Theorem 3.36).

5. Now define the local martingale $\widetilde{L} := \mathscr{E}(\widetilde{N})$ by

$$\begin{aligned} \widetilde{N}_t &= -\kappa \int_0^{t\wedge T} \varphi_s \cdot d\overline{Z}^c_s + \int_{[0,t\wedge T]\times\mathbb{R}^{n+1}} \Bigg(1_{\Theta^c}(s)(u'_\kappa(\varphi_s \cdot x) - 1) \\ &\quad + 1_\Theta(s) \frac{u'_\kappa(\varphi_s \cdot x)}{\int u'_\kappa(\varphi_s \cdot \widetilde{x})\, \overline{K}_s(d\widetilde{x})} \Bigg) (\overline{\mu} - \overline{\nu})(ds, dx), \end{aligned}$$

   where $\overline{Z} = (Z^0, \dots, Z^n)$ and the random measures $\overline{\mu}, \overline{\nu}$ are defined as in Theorem 3.36 but relative to $\overline{Z}$ instead of Z, and $\overline{K}$ denotes the last component in the extended characteristics of $\overline{Z}$.

6. Assume that $\widetilde{L}^T$ is a martingale (This holds automatically if the market is Θ-discrete, cf. Remark 3 following Theorem 3.36). Define the probability measure $\widetilde{P}$ by $\frac{d\widetilde{P}}{dP} := \widetilde{L}_T$.

7. Assume that the $\widetilde{P}$-local martingale $Z^T - Z_0$ is a $\widetilde{P}$-martingale, where $Z = (Z^0, \dots, Z^l)$. This holds automatically in Θ-discrete markets, if $Z^i - Z^i_0$ is bounded from below by a constant $D \in \mathbb{R}$ for $i = 0, \dots, l$ (cf. Remark 3 following Theorem 3.36).

8. Define processes $\widetilde{Z}^{l+1}, \dots, \widetilde{Z}^n$ by taking a càdlàg version of $\widetilde{Z}^i_t := \widetilde{E}(X^i | \mathfrak{F}_{t\wedge T})$ for any $t \in \mathbb{R}_+$, where $\widetilde{E}$ denotes expectation with respect to $\widetilde{P}$.

Definition 3.43 $\widetilde{Z}^{l+1},\dots,\widetilde{Z}^n$ will be called $(\kappa,\rho^{l+1},\dots,\rho^n)$-*approximate price processes* for the derivatives $l+1,\dots,n$.

Lemma 3.44 *Let $\widetilde{Z}^{l+1},\dots,\widetilde{Z}^n$ be $(\kappa,\rho^{l+1},\dots,\rho^n)$-approximate price processes for the derivatives $l+1,\dots,n$. Then the market $\widetilde{Z}=(Z^0,\dots,Z^l,\widetilde{Z}^{l+1},\dots,\widetilde{Z}^n)$ is an extended Grigelionis process and allows no arbitrage on $[0,T]$. Moreover, we have $Z_t^i = X^i$ P-almost surely for any $t\geq t_i$ and any $i\in\{l+1,\dots,n\}$.*

Remarks.

1. As for consistent prices we have that $(\kappa,\rho^{l+1},\dots,\rho^n)$-approximate prices are identical to $(1,\kappa\rho^{l+1},\dots,\kappa\rho^n)$-approximate prices. Moreover, neutral price processes coincide with $(\kappa,0,\dots,0)$-approximate processes (for any $\kappa>0$).

2. The key idea in Definition 3.43 is to use neutral price processes as a zeroth approximation to consistent prices in steps 3 and 5. This suggests repeating steps 3 to 8, but this time using the (presumably) better first approximation $\widetilde{Z}^{l+1},\dots,\widetilde{Z}^n$ instead of $Z^{l+1},\dots,Z^n$. It would be interesting to know whether an iteration of this procedure leads, under suitable conditions, to $(\kappa,\rho^{l+1},\dots,\rho^n)$-consistent prices in the limit.

We now define approximate price regions as in Subsection 1.2.5.

Definition 3.45 Fix $\kappa>0$ and $r\geq 0$. We say that processes $Z^{l+1},\dots,Z^n$ belong to the *approximate κr-price region* if they are $(\kappa,\rho^{l+1},\dots,\rho^n)$-approximate derivative price processes for some $\rho^{l+1},\dots,\rho^n$ satisfying $|\rho^i|\leq r$ for $i=l+1,\dots,n$.

Let us make a final remark concerning $(\kappa,\rho^{l+1},\dots,\rho^n)$-consistent and -approximate price processes in $\mathbb{N}^*$-discrete markets.

Remark. In the remark following Theorem 3.37 we observe that the transition probabilities for the pricing measure P^* are given by

$$P^*(A|\mathcal{F}_{t-1}) = E\Big(\frac{u'_\kappa(\varphi_t\cdot\Delta Z_t)}{E(u'_\kappa(\varphi_t\cdot\Delta Z_t)|\mathcal{F}_{t-1})}1_A\Big|\mathcal{F}_{t-1}\Big) \tag{3.19}$$

for any $t\in\mathbb{N}^*$, $A\in\mathcal{F}_t$, where φ_t satisfies

$$\int x^i u'_\kappa(\varphi_t\cdot x)\,P^{\Delta Z_t|\mathcal{F}_{t-1}}(dx)=0 \text{ for } i=0,\dots,l. \tag{3.20}$$

Similar equations hold for the pricing measures in this section. Recall that one obtains the equivalent martingale measure leading to neutral derivative prices if Z in the Equations (3.19) and (3.20) denotes the underlyings' price process $Z=(Z^0,\dots,Z^l)$.

Now let Z instead denote the joint process $Z=(Z^0,\dots,Z^l,Z^{l+1},\dots,Z^n)$, where $Z^{l+1},\dots,Z^n$ are the neutral price processes from Section 3.4. Moreover, fix $\varphi_t^i=\rho^i$ for any $i\in\{l+1,\dots,n\}$, $t\in\mathbb{N}^*$. Then Equations (3.19) and (3.20) yield the probability measure $\widetilde{P}$ leading to $(\kappa,\rho^{l+1},\dots,\rho^n)$-approximate price processes in Definition 3.43.

Finally, one may replace Z again with a joint process $Z = (Z^0, \dots, Z^l, Z^{l+1}, \dots, Z^n)$, but this time with $(\kappa, \rho^{l+1}, \dots, \rho^n)$-consistent price processes $Z^{l+1}, \dots, Z^n$. We fix again $\varphi^i_t = \rho^i$ for any $i \in \{l+1, \dots, n\}$, $t \in \mathbb{N}^*$. With this choice, the Equations (3.19) and (3.20) yield the pricing measure P^* in Lemma 3.41, leading to $(\kappa, \rho^{l+1}, \dots, \rho^n)$-consistent price processes.

Proofs

PROOF OF LEMMA 3.41. Firstly, note that (RC 2) holds due to Corollary 3.27. Moreover, one may choose $\varphi_0 = (0, \dots, 0, \rho^{l+1}, \dots, \rho^n)$ w.l.o.g. With that choice, the pathwise boundedness of φ^T implies that it is locally bounded (cf. Lemma A.1). In order to establish the claim, just apply the first two steps of the proof of Theorem 3.36 to $\overline{Z}$ instead of Z. We obtain that P^* is a well-defined probability measure equivalent to P and that $\overline{Z} - \overline{Z}_0$ is a P^*-local martingale. If $\overline{Z}$ is a P^*-martingale, Equation (3.18) clearly holds. □

PROOF OF LEMMA 3.44. Firstly, note that $\widetilde{N}$ and $\widetilde{P}$ are well-defined (cf. the first two steps of the proof of Theorem 3.36). From the boundedness of u'_κ as well as Lemma 2.27 and Condition 3 in the remark following Theorem 2.26, it follows that $Z^0, \dots, Z^n$ are $\widetilde{P}$-special semimartingales. Application of Theorem 2.26 to $\overline{Z} = (Z^0, \dots, Z^n)$ yields that

$$\widetilde{b}^i_t = \overline{b}^i_t - \kappa \overline{c}^{i\cdot}_t \cdot \varphi_t + \int x^i (u'_\kappa(\varphi_t \cdot x) - 1)\, \overline{F}_t(dx),$$

$$\widetilde{K}_t(G) = \int 1_G(x) \frac{u'_\kappa(\varphi_t \cdot x)}{\int u'_\kappa(\varphi_t \cdot \widetilde{x})\, \overline{K}_t(d\widetilde{x})}\, \overline{K}_t(dx)$$

for any $i \in \{0, \dots, n\}$, $t \in [0, T]$ and hence

$$\Delta \widetilde{B}^i_t = \int x^i\, \widetilde{K}_t(dx) = \frac{1}{\int u'_\kappa(\varphi_t \cdot \widetilde{x})\, \overline{K}_t(d\widetilde{x})} \int x^i u'_\kappa(\varphi_t \cdot x)\, \overline{K}_t(dx)$$

for any $i \in \{0, \dots, n\}$, $t \in \Theta \cap [0, T]$, where $(\Theta, P^{\overline{Z}_0}, \overline{b}, \overline{c}, \overline{F}, \overline{K})^E$ and $(\Theta, P^{\overline{Z}_0}, \widetilde{b}, \overline{c}, \widetilde{F}, \widetilde{K})^E$ denote the P- resp. $\widetilde{P}$-characteristics of $\overline{Z}$ and $\widetilde{B}$ is the predictable part of finite variation of the $\widetilde{P}$-special semimartingale $\overline{Z}$. From the u_κ-optimality of φ and Corollary 3.23, we conclude that $(\widetilde{B}^i)^T = 0$ for $i = 0, \dots, l$ (but not necessarily for $i = l+1, \dots, n$). This implies that $Z^T - Z_0$ is a $\widetilde{P}$-local martingale. The $\widetilde{P}$-integrability of the random variables X^i follows as in the third step of the proof of Theorem 3.36. By definition $\widetilde{Z}^{l+1}, \dots, \widetilde{Z}^n$ are $\widetilde{P}$-martingales. As in the fourth and the fifth step of the proof of Theorem 3.36 we have that $\widetilde{Z} = (Z^0, \dots, Z^l, \widetilde{Z}^{l+1}, \dots, \widetilde{Z}^n)$ is an extended Grigelionis process relative to $\widetilde{P}$. From Lemma 3.7 we conclude that $\widetilde{Z}^T$ allows no arbitrage. □

PROOF OF THE REMARK. This follows as before from Corollary 3.23 and from the proof of Statement 3 in Proposition 1.6. □

3.6 Improved Derivative Models

As in Subsection 1.2.6, we now want to use consistent and approximate prices to construct models that are consistent with initially observed market prices. The general setting is as in the previous two sections, i.e. we are given underlyings $0, \ldots, l$ and derivatives $l+1, \ldots, n$ at maturity. Moreover, we assume that $\mathcal{F}_0$ is the trivial σ-field $\{\varnothing, \Omega\}$ or its P-completion and that the initial prices $p^{l+1}, \ldots, p^n$ of the derivatives are known. As before, we work with standard utility functions u_κ.

Definition 3.46 Stochastic processes $Z^{l+1}, \ldots, Z^n$ are called $(p^{l+1}, \ldots, p^n)$-*consistent* for the derivatives $l+1, \ldots, n$ if the following conditions hold.

1. $Z^{l+1}, \ldots, Z^n$ are $(\kappa, \rho^{l+1}, \ldots, \rho^n)$-consistent price processes for the derivatives $l+1, \ldots, n$ for some $\kappa > 0$, $\rho^{l+1}, \ldots, \rho^n \in \mathbb{R}$. (By Remark 1 following Definition 3.40 one can in fact choose any κ, e.g. $\kappa = 1$.)

2. $Z_0^i = p^i$ P-almost surely for $i = l+1, \ldots, n$.

As indicated in the previous section, we are usually unable to compute consistent prices except in very simple models. Since in practice one may still prefer to work with a model that does not contradict the initially observed derivative prices, we use approximately consistent prices instead.

Definition 3.47 We call stochastic processes $Z^{l+1}, \ldots, Z^n$ *approximately* $(p^{l+1}, \ldots, p^n)$-*consistent* for the derivatives $l+1, \ldots, n$ if the following conditions hold.

1. $Z^{l+1}, \ldots, Z^n$ are $(\kappa, \rho^{l+1}, \ldots, \rho^n)$-approximate price processes for the derivatives $l+1, \ldots, n$ for some $\kappa > 0$, $\rho^{l+1}, \ldots, \rho^n \in \mathbb{R}$. (By Remark 1 following Lemma 3.44 one can in fact choose any κ, e.g. $\kappa = 1$.)

2. $Z_0^i = p^i$ P-almost surely for $i = l+1, \ldots, n$.

Remarks.

1. Note that, in general, (approximately) $(p^{l+1}, \ldots, p^n)$-consistent prices do not exist if the initial prices $(p^{l+1}, \ldots, p^n)$ are not consistent with the absence of arbitrage.

2. As any reader may observe, the previous two sections leave many open questions, e.g. concerning existence, uniqueness and numerical computation of consistent prices (for details cf. Subsections 1.2.5 and 1.2.6).

3. In Definitions 3.40 and 3.43 we may replace the constant supply $\rho^{l+1}, \ldots, \rho^n$ with a predictable supply process $(\rho_t^{l+1}, \ldots, \rho_t^n)_{t \in \mathbb{R}_+}$. In this way we can obtain settings with *stochastic external supply*. We refer to Subsection 1.2.6 for a discussion of how models of this type can be used in practice.

4. As before, one may replace the standard functions u_κ in the previous two sections with some other utility function.

3.7 American Options

Our aim is to extend the approach in Section 3.4 to markets where some of the derivatives are American options. The setting is as follows. The underlyings $0, \ldots, l$ are given by their discounted price processes $Z = (Z^0, \ldots, Z^l)$ on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$. Z is assumed to be an extended Grigelionis process that meets integrability condition (RC 1'). As in Section 3.4, derivatives $l+1, \ldots, k$ are given by their $\mathcal{F}_{t_i}$-measurable discounted terminal price X^i at time t_i for $i = l+1, \ldots, k$. Moreover, the market contains securities $k+1, \ldots, n$ representing *American options*. These are characterized by their respective discounted exercise price processes Y^i, where for any $i \in \{k+1, \ldots, n\}$, Y^i is assumed to be a $\mathbb{R}_+$-valued, càdlàg, adapted process with $Y^i = (Y^i)^{t_i}$ (i.e. staying constant after t_i) for some $t_i \in \mathbb{R}_+$. This is to say that, at any time $t \in \mathbb{R}_+$, one may return the option i and get Y^i_t in exchange. As in Subsection 1.2.7, we have to make some assumptions to be able to price American options.

1. We base our derivation once more on the condition that the typical speculator has a zero position in the securities $l+1, \ldots, n$. But note that a speculator no longer corresponds to an investor who can choose his portfolio freely in the set of all strategies $\mathfrak{A}$. When the market price of an American option meets its exercise price, then it is likely that any trader with a long position in this security will return it. So, speculators may not be able to maintain a short position in the option beyond this first reasonable exercise time. As long as the market price is still higher than the exercise price, however, we may safely assume that no trader returns the option, which implies that its short sale is not yet restricted.

2. An American option i can be exercised at any stopping time σ. Hence, an investor may use it as a substitute for a usual contingent claim with terminal discounted value Y^i_σ at maturity. In Section 3.4 we define and derive unique neutral prices for this kind of derivative. For the definition of a corresponding notion for American options, we assume that the price of these securities is at least as high as the neutral price of the various implied options of European style.

The following definition takes these aspects into account. As in Section 3.4 we work with standard utility functions u_κ.

Definition 3.48 We call the stochastic processes $Z^{l+1}, \ldots, Z^n$ *neutral price processes* for the derivatives $l+1, \ldots, n$ if the following conditions hold.

1. $Z^{l+1}, \ldots, Z^n$ are adapted processes whose paths are P-almost surely càdlàg.

2. $Z^i_t = X^i$ P-almost surely for any $t \geq t_i$ and any $i \in \{l+1, \ldots, k\}$.

3. If $\sigma_{k+1}, \ldots, \sigma_n$ are bounded stopping times and if $(\widetilde{Z}^{k+1}, \ldots, \widetilde{Z}^n)$ are neutral price processes (in the sense of Definition 3.35) for derivatives $k+1, \ldots, n$ with discounted terminal value $Y^i_{\sigma_i}$ for $i = k+1, \ldots, n$, then $(\widetilde{Z}^i)^{\sigma_i} \leq (Z^i)^{\sigma_i}$ up to an evanescent set for $i = k+1, \ldots, n$.

4. For any $t_0 \in \mathbb{R}_+$, $i \in \{k+1, \dots, n\}$, define the *next reasonable exercise time* $\tau_{t_0} := \tau^1_{t_0,i} \wedge \tau^2_{t_0,i}$, where $\tau^1_{t_0,i} := \inf\{t > t_0 : Z^i_{t-} = Y^i_{t-}\}$ and $\tau^2_{t_0,i} := \inf\{t \geq t_0 : Z^i_t = Y^i_t\}$. Moreover, define for any $i \in \{k+1, \dots, n\}$ and any $t \in [t_0, \infty)$
$$\overline{Z}^i_t := \begin{cases} Z^i_t & \text{if } t < \tau_{t_0,i} \text{ or } t = t_0 \\ Z^i_{\tau_{t_0,i}-} & \text{if } t_0 \neq t \geq \tau_{t_0,i} \text{ and } Z^i_{\tau_{t_0,i}-} = Y^i_{\tau_{t_0,i}-} \\ Z^i_{\tau_{t_0,i}} & \text{if } t_0 \neq t \geq \tau_{t_0,i} \text{ and } Z^i_{\tau_{t_0,i}-} \neq Y^i_{\tau_{t_0,i}-} \text{ (and hence } Z^i_{\tau_{t_0,i}} = Y^i_{\tau_{t_0,i}}). \end{cases}$$
Then we have
   (a) $\tau_{t_0,i} \leq t_0 \vee t_i$ for any $t_0 \in \mathbb{R}_+$ and any $i \in \{k+1, \dots, n\}$.
   (b) The market $(\overline{Z}_t)_{t \in [t_0,\infty)} = (Z^0_t, \dots, Z^k_t, \overline{Z}^{k+1}_t, \dots, \overline{Z}^n_t)_{t \in [t_0,\infty)}$ is an extended Grigelionis process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [t_0,\infty)}, P)$ that allows no arbitrage, and the convexly restricted set $\mathfrak{M} := \{\varphi \in \mathfrak{A} : \varphi^i = 0 \text{ for any } i \in \{l+1, \dots, n\}\}$ contains a u_κ-optimal strategy for $\mathfrak{A}$.

Remarks.

1. Strictly speaking, we have defined extended Grigelionis processes, strategies etc. only on a stochastic basis with index set $\mathbb{R}_+$. Nevertheless, it should be clear what we mean by the corresponding notions on $[t_0, \infty)$ in Condition 4 of the previous definition.

2. The convention $Y^i_t = Y^i_{t_i}$ (instead of, e.g. $Y^i_t = 0$) for any $t > t_i$ is made for mathematical ease. Economically it means that, at the last possible exercise time t_i, the option is automatically converted into $Y^i_{t_i}$ units of the numeraire and hence practically vanishes from the market.

3. At first glance it may seem more intuitive to define the next reasonable exercise time as $\inf\{t \geq t_0 : Z^i_t = Y^i_t\}$. However, in continuous-time one can easily construct examples where the market price comes arbitrarily close to the exercise price (i.e. $Z^i_{t-} = Y^i_{t-}$ for some t), but does not really reach it before expiration.

4. Condition 4a in Definition 3.48 means that the option reaches or has already reached its terminal value $Y^i_{t_i}$ at time t_i.

5. $\overline{Z}^i$ stands for the option that is exercised at the next reasonable time and converted into shares of the numeraire. In contrast to Z^i, this security can be held short even when the option has vanished from the market by early exercise. Therefore the condition 4b, which corresponds to a zero position in the derivatives for a speculator facing no short sale restrictions, makes sense for the securities $\overline{Z}^i$, $i = k+1, \dots, n$.

The following two theorems correspond to Theorem 3.36 and Corollary 3.37.

Theorem 3.49 *Let* $T := \sup\{t_{l+1} \dots, t_n\}$ *and fix* $\kappa > 0$ *(e.g.* $\kappa = 1$*). Assume that*

1. *Conditions 1–3 and 5 in Theorem 3.36 hold.*

2. *$E^*(\sup_{t\in[0,T]}|Z_t - Z_0|) < \infty$, where P^* is defined in Theorem 3.36.*

3. *For any $i \in \{l+1,\dots,k\}$ there exist $\xi, \eta \in \mathfrak{S}$, $M \in \mathbb{R}$ such that*

$$-M + \int_0^T \xi_s \cdot dZ_s \leq X^i \leq M + \int_0^T \eta_s \cdot dZ_s.$$

For any $i \in \{k+1,\dots,n\}$ there exist $\xi \in \mathfrak{S}$, $M \in \mathbb{R}$ such that for any $t \in [0,T]$ we have

$$0 \leq Y_t^i \leq M + \int_0^t \xi_s \cdot dZ_s.$$

Then there exist up to indistinguishability unique neutral price processes for the derivatives $l+1,\dots,n$. For $i \in \{l+1,\dots,k\}$ these are the processes in Theorem 3.36. For $i \in \{k+1,\dots,n\}$ we have

1. *Z^i is the smallest P^*-supermartingale dominating Y^i (i.e. Z^i is a P^*-supermartingale, we have $Z_t^i \geq Y_t^i$ P^*-almost surely for any $t \in \mathbb{R}_+$, and if $\widetilde{Z}^i$ is another such process, then we have $\widetilde{Z}_t^i \geq Z_t^i$ P^*-almost surely for any $t \in \mathbb{R}_+$).*

2. *$Z_t^i = \operatorname{ess\,sup}\{E_T^*(Y_\sigma^i|\mathcal{F}_t) : \sigma\ [t,\infty)$-valued stopping time$\}$ P-almost surely for any $t \in \mathbb{R}_+$.*

Corollary 3.50 *Let $T := \sup\{t_{l+1},\dots,t_n\}$ and fix $\kappa > 0$ (e.g. $\kappa = 1$). Assume that*

1. *Condition 1 in Corollary 3.37 holds.*

2. *For any $i \in \{l+1,\dots,k\}$ there exist $\xi, \eta \in \mathfrak{S}$, $M \in \mathbb{R}$ such that*

$$-M + \int_0^T \xi_s \cdot dZ_s \leq X^i \leq M + \int_0^T \eta_s \cdot dZ_s.$$

For any $i \in \{k+1,\dots,n\}$ there exist $\xi \in \mathfrak{S}$, $M \in \mathbb{R}$ such that for any $t \in [0,T]$ we have

$$0 \leq Y_t^i \leq M + \int_0^t \xi_s \cdot dZ_s.$$

Define P^* as in Corollary 3.37. Then the assertion in Theorem 3.49 holds. Moreover, we have

$$Z_t^i = \begin{cases} Y_t^i & \text{for } t \geq T \\ \max\{Y_{s_{j-1}}^i, E^*(Z_{s_j}^i|\mathcal{F}_{s_{j-1}})\} & \text{for } s_{j-1} \leq t < s_j \end{cases}$$

for any $i \in \{k+1,\dots,n\}$, where $0 = s_0 \leq s_1 \leq \ldots \leq s_m = T$ with $\{s_0,\dots,s_m\} = [0,T] \cap (\Theta \cup \{0,T\})$.

Remarks.

1. The processes Z^i in Theorem 3.49 are called the *Snell envelope* of Y^i. It is well-known that in complete models the unique fair price processes are also given in terms of a Snell envelope under the equivalent martingale measure (cf. Lamberton & Lapeyre (1996)). In view of Statement 2 in Theorem 3.49 and the complicated Definition 3.48, one may wonder why we have not *defined* neutral American option prices to be the supremum of all neutral European option price processes with terminal value Y^i_σ for arbitrary stopping times σ. However, this easier concept already implies by definition that an American option is not worth more than the best of its implied European style derivatives (i.e. the *possibility* to choose the exercise time has no value in itself). We are interested in whether this fact can be deduced from weaker assumptions. Therefore, we prefer the seemingly more awkward Definition 3.48.

2. Most remarks in Section 3.4 carry over to this slightly more general setting.

3. It is an open question whether one can also define price regions and improved derivative models for American options in the spirit of Sections 3.5 and 3.6.

Proofs

Proposition 3.51 *Let X be a càdlàg adapted process, $t_0 \in \mathbb{R}_+$, $\tau := \inf\{t > t_0 : X_{t-} = 0\}$, $A := \{X_{\tau-} = 0$ and $X_{t_0} \neq 0\}$. Then $\tau_A = \tau 1_A + \infty 1_{A^C}$ is a predictable stopping time. (More precisely, τ_A is indistinguishable from a predictable stopping time.)*

PROOF. By Jacod (1979), (1.1) we may assume that the filtration is complete. Note that $\tau_A = \inf\{t \in \mathbb{R}_+ : 1_{\{X_{t_0}\neq 0\}} 1_{(t_0,\infty)} \neq 0$ and $X^{\tau-}_{t-} = 0\}$. Moreover, $(\omega, t) \in [\tau_A]$ implies $X^{\tau-}_{t-} = X_{\tau-} = 0$ and $t > t_0$. Therefore, $(\omega,t) \in \{1_{\{X_{t_0}\neq 0\}} 1_{(t_0,\infty)} \neq 0$ and $X^{\tau-}_{t-} = 0\} \in \mathcal{P}$. By JS, I.2.27 and JS, I.2.13 we have that τ_A is a predictable stopping time. □

PROOF OF THEOREM 3.49. Here and also occasionally in other proofs, we apply results stated only for P-complete filtered probability spaces. Following Jacod (1979), (1.1), the statements nevertheless hold in incomplete spaces as well.

First step: Define $Z^{l+1}, \dots, Z^k$ by Equation (3.11) and $Z^{k+1}, \dots, Z^n$ by Statement 1. Their existence follows from Fakeev (1970), Theorem 2. (Strictly speaking, Fakeev (1970) yields only the existence of minimal right-continuous supermartingales, whereas in JS (and thus for us) supermartingales are supposed to be càdlàg. In the second step we show that $Z^{k+1}, \dots, Z^n$ have left-hand limits as well.) Since by assumption $Y^i_t = Y^i_T$ for any $i \in \{k+1, \dots, n\}$, $t \in [T, \infty)$, Fakeev (1970), Theorem 1 yields that

$$\begin{aligned} Z^i_t &= \operatorname{ess\,sup}\{E^*(Y^i_\sigma|\mathcal{F}_t) : \sigma\ [t,\infty)\text{-valued stopping time}\} \\ &= \operatorname{ess\,sup}\{E^*(Y^i_\sigma|\mathcal{F}_t) : \sigma\ [t,T]\text{-valued stopping time}\} \end{aligned}$$

P-almost surely for any $t \in [0,T]$.

Second step: By definition $Z^{l+1},\dots,Z^k$ are adapted and càdlàg. In addition, $Z^{k+1},\dots,$ Z^n are also adapted and right-continuous. From the proof of Theorem 2 in Fakeev (1970), we conclude that the mapping $t\mapsto E^*(Z^i_t)$ is right-continuous for $i=k+1,\dots,n$. By Métivier (1982), Corollary 10.10 and the right-continuity of the Z^i, this implies that P^*-almost all paths of $Z^{k+1},\dots,Z^n$ are càdlàg. The second condition in Definition 3.48 follows from Theorem 3.36.

Third step: Since $Y^i_{\sigma_i}$ meets Condition 6 in Theorem 3.36, we have that $(\widetilde{Z}^{k+1},\dots,\widetilde{Z}^n)$ in Condition 3 of Definition 3.48 is uniquely given by $\widetilde{Z}^i_t=E^*(Y^i_{\sigma_i}|\mathcal{F}_{T\wedge t})$ for any $t\in\mathbb{R}_+$, $i\in\{k+1,\dots,n\}$. (W.l.o.g., we have assumed that $\sigma_i\le T$ for $i=k+1,\dots,n$.) Fix $i\in\{k+1,\dots,n\}$. From the definition and Doob's stopping theorem (cf. JS, I.1.39) we have that $\widetilde{Z}^i_{\sigma_i}=Y^i_{\sigma_i}\le Z^i_{\sigma_i}$ and hence $(\widetilde{Z}^i)^{\sigma_i}_t=E^*(\widetilde{Z}^i_{\sigma_i}|\mathcal{F}_{T\wedge t})\le E^*(Z^i_{\sigma_i}|\mathcal{F}_{T\wedge t})\le (Z^i)^{\sigma_i}_t$ P^*-almost surely for any $t\in\mathbb{R}_+$, where the latter inequality follows again from Doob's stopping theorem. Thus we have shown Condition 3 in Definition 3.48.

Fourth step: We will now show that $(\overline{Z}^i_t)_{t\in[t_0,\infty)}$ is a P^*-martingale for any $i\in\{k+1,\dots,n\}$, $t_0\in\mathbb{R}_+$. Fix i,t_0. For any $m\in\mathbb{N}$, let us define the stopping time $\sigma_m:=\inf\{t>t_0:Z^i_t\le Y^i_t+1/m\}$. For $m\to\infty$ we have $\sigma_m\uparrow\tau_{t_0,i}$ P-almost surely. Using Condition 2 in Theorem 3.49, one easily shows that $E^*(\sup_{t\in[0,T]}|M+\int_0^t\xi_s\cdot dZ_s|)<\infty$ for any $M\in\mathbb{R}$, $\xi\in\mathfrak{S}$. Since the process Y^i is dominated by a process of the form $M+\int_0^\cdot\xi_s\cdot dZ^T_s$, it follows that $E^*(\sup_{t\in[0,T]}|Y^i_t|)<\infty$. From Fakeev (1970), Equation (24) it follows that $((Z^i)^{\sigma_m}_t)_{t\in[t_0,\infty)}$ is a P^*-martingale for any $m\in\mathbb{N}$. Using Doob's stopping theorem (cf. JS, I.1.39), we obtain that $(Z^i_{\sigma_m})_{m\in\mathbb{N}}$ is a uniformly integrable $(\mathcal{F}_{\sigma_m})_{m\in\mathbb{N}}$-martingale relative to P^*. From the martingale convergence theorem (cf. Bauer (1978), Korollar 60.3), we can now conclude that $(Z^i_{\sigma_m})_{m\in\mathbb{N}}$ converges P^*-almost surely and in $L^1(P^*)$ to a $\sigma(\cup_{m\in\mathbb{N}}\mathcal{F}_{\sigma_m})$-measurable random variable R. Let $A:=\{Z^i_{\tau_{t_0,i}-}=Y^i_{\tau_{t_0,i}-}\}\cap\{Z^i_{t_0}\ne Y^i_{t_0}\}$. Since Z^i,Y^i are càdlàg, we have $R=Z^i_{\tau_{t_0,i}-}1_A+Z^i_{\tau_{t_0,i}}1_{A^C}=\overline{Z}^i_{\tau_{t_0,i}}$. For any $t\in[t_0,\infty)$ this implies that (with the convention $]\sigma_{-1},\sigma_0]:=[\sigma_0]$)

$$\begin{aligned}\overline{Z}^i_t &= \sum_{m\in\mathbb{N}}(Z^i)^{\sigma_m}_t 1_{]\sigma_{m-1},\sigma_m]}(t)+R1_{(\cup_{m\in\mathbb{N}}[0,\sigma_m])^C}(t)\\ &= \sum_{m\in\mathbb{N}}E^*(E^*(R|\mathcal{F}_{\sigma_m})|\mathcal{F}_t)1_{]\sigma_{m-1},\sigma_m]}(t)+E^*(R|\mathcal{F}_t)1_{(\cup_{m\in\mathbb{N}}[0,\sigma_m])^C}(t).\end{aligned}$$

Doob's stopping theorem (cf. JS, I.1.39) yields that $E^*(E^*(R|\mathcal{F}_{\sigma_m})|\mathcal{F}_t)1_{]\sigma_{m-1},\sigma_m]}(t)=E^*(R$ $|\mathcal{F}_t)\ 1_{]\sigma_{m-1},\sigma_m]}(t)$ P-almost surely. Therefore $\overline{Z}^i_t=E(R|\mathcal{F}_t)$, which implies that $(\overline{Z}^i_t)_{t\in[t_0,\infty)}$ is a P^*-martingale.

Fifth step: Let M,ξ be as in Condition 3 in Theorem 3.49, chosen relative to Y^i. Denote $\tau_A:=(\tau_{t_0,i})_A$. Then $\tau_A:=(\tau^1_{t_0,i})_{A_1\cap A_2}$ for $A_1:=\{Z^i_{\tau^1_{t_0,i}-}=Y^i_{\tau^1_{t_0,i}-}\}\cap\{Z^i_{t_0}\ne Y^i_{t_0}\}$ and $A_2:=\{\tau^1_{t_0,i}\le\tau^2_{t_0,i}\}\in\mathcal{F}_{\tau^1_{t_0,i-}}$. By Proposition 3.51 and JS, I.2.10 it follows that τ_A is a predictable stopping time. We may therefore define the predictable process $\widetilde{\xi}$ and the local martingale U by $\widetilde{\xi}:=\xi(1_{[0,\tau_{t_0,i}]}-1_{[\tau_A]})$ and $U:=\int_0^\cdot\widetilde{\xi}_s\cdot dZ^T_s$. Note that $|\int_0^\cdot(\widetilde{\xi}1_{[\tau_A]})\cdot dZ^T_s|\le|\widetilde{\xi}_{\tau_A}||Z^T_{\tau_A}-Z^T_{\tau_A-}|$. From $E^*(\sup_{t\in\mathbb{R}_+}|Z^T-Z_0|)<\infty$ it easily follows that $E^*(\sup_{t\in\mathbb{R}_+}|U_t|<\infty)$, which implies that U is a P^*-martingale (cf. JS, I.1.47). Note that $\overline{Z}^i_T = Y^i_{\tau_{t_0,i}-}1_A + Y^i_{\tau_{t_0,i}}1_{A^C} \leq M + \int_0^{\tau_{t_0,i}} \xi_s \cdot dZ^T_s - 1_A\Delta(\int_0^{\cdot} \xi_s \cdot dZ^T_s)_{\tau_{t_0,i}} = M + U_T$. Since $(\overline{Z}^i_t)_{t\in[t_0,\infty)}$ and U are both P^*-martingales, it follows that $0 \leq \overline{Z}^i_t \leq M + U_t$ for any $t \geq t_0$. From Lemma 2.22 and Proposition 3.39, we conclude that U and hence $(\overline{Z}^i_t)_{t\in[t_0,\infty)}$ is a P-special semimartingale.

Sixth step: As in the fifth step of the proof of Theorem 3.36, it follows that $(\overline{Z}^i_t)_{t\in[t_0,\infty)} = (Z^0_t, \dots, Z^k_t, \overline{Z}^{k+1}_t, \dots, \overline{Z}^n_t)_{t\in[t_0,\infty)}$ is an extended Grigelionis process relative to P. Condition 4b in Definition 3.48 now follows exactly as the sixth step of the proof of Theorem 3.36.

Seventh step: Let $\widetilde{Z}^{l+1}, \dots, \widetilde{Z}^n$ be another set of neutral price processes for the derivatives $l+1, \dots, n$. Fix $t_0 \in \mathbb{R}_+$. Moreover, fix $i \in \{k+1, \dots, n\}$ for the moment and let σ be a $[t_0, T]$-valued stopping time. Since $0 \leq Y^i_\sigma \leq M + \int_0^T \xi_s 1_{[0,\sigma]}(s) \cdot dZ_s$, Theorem 3.36 yields that $(E^*(Y^i_\sigma|\mathfrak{F}_t))_{t\in\mathbb{R}_+}$ is a neutral price process for a derivative with terminal value Y^i_σ. By Condition 2 we have $\widetilde{Z}^i_{t_0} \geq E^*(Y^i_\sigma|\mathfrak{F}_{t_0})$ P-almost surely. Since $Z^i_{t_0}$ is the essential upper bound of random variables of the type $E^*(Y^i_\sigma|\mathfrak{F}_{t_0})$, it follows that $\widetilde{Z}^i_{t_0} \geq Z^i_{t_0}$ P-almost surely.

Eighth step: Fix $t_0 \in \mathbb{R}_+$. By Condition 4b there is a u_κ-optimal strategy for $\mathfrak{A}$ of the form $\widetilde{\varphi} = (\widetilde{\varphi}^0, \dots, \widetilde{\varphi}^l, 0, \dots, 0)$ for the market $(\widetilde{Z}_t)_{t\in[t_0,\infty)} = (Z^0_t, \dots, Z^l_t, \widetilde{Z}^{l+1}_t, \dots, \widetilde{Z}^n_t)_{t\in[t_0,\infty)}$. Since the local utility of $\widetilde{\varphi}$ in the market $(\widetilde{Z}_t)_{t\in[t_0,\infty)}$ is the same as the local utility of $(\widetilde{\varphi}^0, \dots, \widetilde{\varphi}^l)$ in the market $(Z_t)_{t\in[t_0,\infty)}$ and furthermore the local utility of all optimal strategies coincides, we may assume that $(\widetilde{\varphi}^0, \dots, \widetilde{\varphi}^l) = (\varphi^0, \dots, \varphi^l)$, where $(\varphi^0, \dots, \varphi^l)$ is the strategy used for the definition of P^* in Theorem 3.36. Applying Corollary 3.23, it is easy to show that $(\varphi^0, \dots, \varphi^l, 0, \dots, 0)$ is also a u_κ-optimal strategy for $\mathfrak{A}$ in the market $(Z^0_t, \dots, Z^l_t, \widetilde{Z}^{l+1}_t, \dots, \widetilde{Z}^k_t, (\widetilde{Z}^{k+1})^{\sigma_{k+1}}_t, \dots, (\widetilde{Z}^n)^{\sigma_n}_t)_{t\in[t_0,\infty)}$, where $\sigma_{k+1}, \dots, \sigma_n$ are arbitrary stopping times. Fix $\varepsilon > 0$. For any $i \in \{k+1, \dots, n\}$ define the stopping time $\sigma_i := \inf\{t \geq t_0 : \widetilde{Z}^i_t \leq Y^i_t + \frac{1}{\varepsilon}\}$. The third condition in Theorem 3.49 yields that $0 \leq (\widetilde{Z}^i)^{\sigma_i}_T \leq (Y^i)^{\sigma_i}_T + \frac{1}{\varepsilon} \leq \frac{1}{\varepsilon} + M + \int_0^T 1_{[0,\sigma_i]}(s)\xi_s \cdot dZ_s$ for some $M \in \mathbb{R}$, $\xi \in \mathfrak{S}$. A simple arbitrage argument shows that $0 \leq (\widetilde{Z}^i)^{\sigma_i}_t \leq \frac{1}{\varepsilon} + M + \int_0^t 1_{[0,\sigma_i]}(s)\xi_s \cdot dZ_s$ for any $t \in [t_0, T]$. Since the right-hand side is a P-special semimartingale, Proposition 3.39 yields that $((\widetilde{Z}^i)^{\sigma_i}_t)_{t\in[t_0,\infty)}$ is a P-special semimartingale as well for $i \in \{k+1, \dots, n\}$. By basically the same argumentation as in the eighth step of Theorem 3.36, we conclude that $((\widetilde{Z}^i)^{\sigma_i}_t)_{t\in[t_0,\infty)}$ is a P^*-martingale. It follows that $\widetilde{Z}^i_{t_0} = E^*((\widetilde{Z}^i)^{\sigma_i}_T|\mathfrak{F}_{t_0}) \leq E^*((Y^i)^{\sigma_i}_T|\mathfrak{F}_{t_0}) + \frac{1}{\varepsilon} \leq Z^i_{t_0} + \frac{1}{\varepsilon}$ for $i \in \{k+1, \dots, n\}$ and $\widetilde{Z}^i_{t_0} = E^*(\widetilde{Z}^i_T|\mathfrak{F}_{t_0}) = E^*(X^i_T|\mathfrak{F}_{t_0}) = Z^i_{t_0}$ for $i \in \{l+1, \dots, k\}$. In view of the previous step, the proof is complete. □

PROOF OF COROLLARY 3.50. As in Corollary 3.37, Conditions 1–5 in Theorem 3.36 are met. Furthermore, we have $E^*(\sup_{t\in[0,T]} |Z^i_t - Z^i_0|) \leq \sum_{t\in[0,T]\cap\Theta} E^*(|Z^i_t - Z^i_0|) < \infty$, which yields the assumptions in Theorem 3.49. The recursion formula for the Snell envelope can be found in Gihman & Skorohod (1979), Theorem 1.8 (for $\Theta = \mathbb{N}^*$). □

3.8 Continuous Time Limits of Discrete Time Models

Any continuous-time market can be converted to discrete-time by restricting the index set $\mathbb{R}_+$ to $\varepsilon\mathbb{N}$ for some $\varepsilon > 0$. By letting $\varepsilon \to 0$ the model can be viewed as the limit of a sequence of discrete-time markets (which are treated in a continuous-time frame by the embedding in the appendix). In this section we show that under suitable conditions the local utility and u-optimal strategies also converge to the respective notions in continuous time. This is satisfactory for two reasons. Firstly, it supports the interpretation of local utility as a conditional expected utility for an infinitesimal time interval (cf. Lemma 3.16 and Theorem 3.54). Secondly, for numerical computations we may approximate continuous-time markets by neighbouring discrete models and vice versa. As in Section 3.1 we consider a market $Z = (Z^0, \dots, Z^n)$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$. We denote the extended characteristics of Z by $(\Theta, P^{Z_0}, b, c, F, K)^E$. u denotes a utility function.

Definition 3.52 Let Σ be a discrete subset of $\mathbb{R}_+$. For any $t \in \mathbb{R}_+$ we denote

$$\begin{aligned}
t^{\Sigma-} &:= \sup\{s \in \Sigma \cup \{0,\infty\} : s \leq t\} \\
t^{\Sigma+} &:= \inf\{s \in \Sigma \cup \{0,\infty\} : s \geq t\} \\
t^{\Sigma--} &:= \sup\{s \in \Sigma \cup \{0,\infty\} : s < t\} \\
t^{\Sigma++} &:= \inf\{s \in \Sigma \cup \{0,\infty\} : s > t\} \\
\Delta t^{\Sigma+} &:= t^{\Sigma++} - t^{\Sigma-}.
\end{aligned}$$

We define the *mesh-size* of Σ as $\|\Sigma\| := \sup_{t\in\mathbb{R}_+} |t - t^{\Sigma-}|$.

Definition 3.53 Let $\Sigma \subset \mathbb{R}_+$ be a discrete set. We call Σ*-discretized market* the stochastic process $(Z_t^\Sigma)_{t\in\mathbb{R}_+}$ on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t^\Sigma)_{t\in\mathbb{R}_+}, P)$, where $\mathcal{F}_t^\Sigma := \mathcal{F}_{t^{\Sigma-}}$ and $Z_t^\Sigma := Z_{t^{\Sigma-}}$ for any $t \in \mathbb{R}_+$. If Z^Σ is an extended Grigelionis process, then we denote by $(\Gamma^\Sigma, \gamma^\Sigma) = (\Gamma^\Sigma, 0)$ its local utility.

Remark. If Z meets regularity condition (RC 1), then Z^Σ is an extended Grigelionis process. If, in addition, the market Z allows no arbitrage, then the same is true for Z^Σ.

Theorem 3.54 *Let $(\Sigma_m)_{m\in\mathbb{N}}$ be a sequence of discrete subsets of $\mathbb{R}_+$ such that $\|\Sigma_m\| \to 0$ for $m \to \infty$. Assume that the market Z meets integrability condition (RC 1). Then we have for any compact set $A \subset \mathbb{R}^{n+1}$:*

$$\sup_{\psi\in A} |\Gamma^{\Sigma_m}_{t^{\Sigma_m+}}(\psi) - \Gamma_t(\psi)| \stackrel{m\to\infty}{\longrightarrow} 0 \quad \textit{in } L^1 \textit{ for any } t \in \mathbb{R}_+,$$

$$\sup_{\psi\in A} \left| \frac{1}{\Delta t^{\Sigma_m+}} \Gamma^{\Sigma_m}_{t^{\Sigma_m++}}(\psi) - \gamma_t(\psi) \right| \stackrel{m\to\infty}{\longrightarrow} 0 \quad \textit{in } L^1 \textit{ for } \lambda\textit{-almost all } t \in \mathbb{R}_+,$$

where the λ-null set in the second statement does not depend on the chosen sequence $(\Sigma_m)_{m\in\mathbb{N}}$.

Remark. In particular, we have that for any $\psi \in \mathbb{R}^{n+1}$,

$$\Gamma^{\Sigma_m}_{t^{\Sigma_m}+}(\psi) \overset{m\to\infty}{\longrightarrow} \Gamma_t(\psi) \quad \text{in probability for any } t \in \mathbb{R}_+,$$

$$\frac{1}{\Delta t^{\Sigma_m+}}\Gamma^{\Sigma_m}_{t^{\Sigma_m}++}(\psi) \overset{m\to\infty}{\longrightarrow} \gamma_t(\psi) \quad \text{in probability for } \lambda\text{-almost all } t \in \mathbb{R}_+.$$

Theorem 3.55 *Let $(\Sigma_m)_{m\in\mathbb{N}}$ be a sequence of discrete subsets of $\mathbb{R}_+$ such that $\|\Sigma_m\| \to 0$ for $m \to \infty$. Let $M \subset \mathbb{R}^n$ be such that $\mathfrak{M} = \{\varphi \in \mathfrak{A} : \varphi_t \in \mathbb{R} \times M \text{ for any } t \in \mathbb{R}_+\}$ is a set of strategies as in Theorem 3.22. We make the following assumptions.*

1. *The market Z meets the regularity conditions (RC 1), (RC 2).*

2. *There exists a u-optimal strategy for $\mathfrak{M}$ in the market Z and in any of the markets Z^{Σ_m} (e.g. by Theorem 3.26). We denote these with $\varphi \in \mathfrak{M}$ resp. $\varphi^{\Sigma_m} \in \mathfrak{M}$. W.l.o.g., we assume $\varphi^0 = 0$ and $(\varphi^{\Sigma_m})^0 = 0$ for any $m \in \mathbb{N}$ (cf. Remark 2 following Definition 3.11).*

3. *The u-optimal strategy φ is* strictly optimal *in the following sense.*

   (a) *P-almost surely and for any $t \in \Theta$ we have*

   $$\Gamma_t(\varphi_t) > \Gamma_t(\psi) \text{ for any } \psi \in (\{0\} \times M) \setminus \{\varphi_t\}.$$

   (b) *Outside some $(P \otimes \lambda)$-null set we have*

   $$\gamma_t(\varphi_t) > \gamma_t(\psi) \text{ for any } \psi \in (\{0\} \times M) \setminus \{\varphi_t\}.$$

Then we have

1. $\varphi^{\Sigma_m}_{t^{\Sigma_m}+} \overset{m\to\infty}{\longrightarrow} \varphi_t$ *in probability for any $t \in \Theta$*

2. $\varphi^{\Sigma_m}_{t^{\Sigma_m}++} \overset{m\to\infty}{\longrightarrow} \varphi_t$ *in probability for any $t \in \mathbb{R}_+$ outside some λ-null set that does not depend on the chosen sequence $(\Sigma_m)_{m\in\mathbb{N}}$.*

Remarks.

1. Strict optimality is needed to ensure that the u-optimal strategy φ is unique. Otherwise, we cannot hope for convergence.

2. It would be nice to have convergence results for pricing measures (e.g. relative to the total variation distance, cf. JS Subsection V.4a), neutral prices, consistent prices, approximate prices, price regions etc. as well. These questions should be addressed in future research.

Proofs

PROOF OF THE REMARK. Z^Σ is an adapted Σ-discrete process. Using Remark 2 in Section 2.4, we may decompose Z as $Z_t - Z_0 = \int_0^t b_s\,ds + \sum_{s\in\Theta\cap[0,t]}\int x\,K_s(dx) + \int_0^t 1\,dZ_s^c + x * (\mu-\nu)_t$ for any $t \in \mathbb{R}_+$. Using regularity condition (RC 1), JS, III.4.5d and Proposition 2.8, we conclude that $Z - Z_0$ is uniformly integrable on any interval $[0,T]$. In particular, $E(|Z_t - Z_s||\mathfrak{F}_s) < \infty$ P-almost surely for any $s,t \in \mathbb{R}_+$ with $s \le t$. By Lemma 2.20 and the following remark this implies that Z^Σ is an extended Grigelionis process. One easily sees that any arbitrage in the market Z^Σ is also an arbitrage in the market Z. Hence, the second claim follows. □

PROOF OF THEOREM 3.54. *First step:* Fix $t \in \mathbb{R}_+$. By the remark following Definition 3.11 we have that for any $\psi \in \mathbb{R}_+^{n+1}$

$$\begin{aligned}
|\Gamma^{\Sigma_m}_{t^\Sigma_m+}(\psi) - \Gamma_t(\psi)| &= |E(u(\psi\cdot(Z_{t^{\Sigma_m}+} - Z_{t^{\Sigma_m}--}))|\mathfrak{F}_{t^{\Sigma_m}--}) - E(\psi\cdot\Delta Z_t|\mathfrak{F}_{t-})| \\
&\le E(|u(\psi\cdot(Z_{t^{\Sigma_m}+} - Z_{t^{\Sigma_m}--})) - u(\psi\cdot\Delta Z_t)||\mathfrak{F}_{t^{\Sigma_m}--}) \qquad (3.21)\\
&\quad + |E(u(\psi\cdot\Delta Z_t)|\mathfrak{F}_{t^{\Sigma_m}--}) - E(\psi\cdot\Delta Z_t|\mathfrak{F}_{t-})| \qquad (3.22)
\end{aligned}$$

In the next two steps, we consider the terms (3.21) and (3.22) seperately.

Second step: By the mean value theorem and the Cauchy-Schwarz inequality we have that $|u(\psi\cdot(Z_{t^{\Sigma_m}+} - Z_{t^{\Sigma_m}--})) - u(\psi\cdot\Delta Z_t)| \le \sup_{y\in\mathbb{R}}|u'(y)||\psi||Z_{t^{\Sigma_m}+} - Z_{t^{\Sigma_m}--} - Z_t + Z_{t-}|$. Since Z is càdlàg, the right-hand side converges to 0 P-almost surely for $m \to \infty$. As $Z - Z_0$ is uniformly integrable on any bounded interval (cf. the preceding proof), we also have convergence in L^1. This implies that $\sup_{\psi\in A} E(|u(\psi\cdot(Z_{t^{\Sigma_m}+} - Z_{t^{\Sigma_m}--})) - u(\psi\cdot\Delta Z_t)||\mathfrak{F}_{t^{\Sigma_m}--})$ also converges in L^1 to 0 for $m \to \infty$.

Third step: Let $\varepsilon > 0$ and $M := E(\Delta Z_t)$. Choose $\psi_1, \dots, \psi_r \in A$ such that for any $\psi \in A$, there is a $\psi_{i(\psi)} \in \{\psi_1,\dots,\psi_r\}$ at most $\widetilde{\varepsilon} := \varepsilon/(3M\sup_{y\in\mathbb{R}}|u'(y)|)$ away. By the martingale convergence theorem (cf. JS, I.1.42) we have that, for any $\psi \in A$, $|E(u(\psi\cdot\Delta Z_t)|\mathfrak{F}_{t-}) - E(u(\psi\cdot\Delta Z_t)|\mathfrak{F}_{t^{\Sigma_m}--})| \overset{m\to\infty}{\longrightarrow} 0$ P-almost surely and in L^1. It follows that $E(|E(u(\psi_i\cdot\Delta Z_t)|\mathfrak{F}_{t-}) - E(u(\psi_i\cdot\Delta Z_t)|\mathfrak{F}_{t^{\Sigma_m}--})|) < \frac{\varepsilon}{3r}$ for any $i \in \{1,\dots,r\}$ and any m large enough. By the mean value theorem we have that $|u(\psi\cdot\Delta Z_t) - u(\psi_{i(\psi)}\cdot\Delta Z_t)| \le \widetilde{\varepsilon}|\Delta Z_t|\sup_{y\in\mathbb{R}}|u'(y)|$ for any $\psi \in A$. Using the triangular inequality, we obtain for sufficiently large m that

$$\begin{aligned}
& E\Big(\sup_{\psi\in A}|E(u(\psi\cdot\Delta Z_t)|\mathfrak{F}_{t^{\Sigma_m}--}) - E(u(\psi\cdot\Delta Z_t)|\mathfrak{F}_{t-})|\Big) \\
\le\ & E\Big(\sup_{\psi\in A} E(|u(\psi\cdot\Delta Z_t) - u(\psi_{i(\psi)}\cdot\Delta Z_t)||\mathfrak{F}_{t^{\Sigma_m}--})\Big) \\
& + \sum_{i=1}^r E\Big(|E(u(\psi_i\cdot\Delta Z_t)|\mathfrak{F}_{t-}) - E(u(\psi_i\cdot\Delta Z_t)|\mathfrak{F}_{t^{\Sigma_m}--})|\Big) \\
& + E\Big(\sup_{\psi\in A} E(|u(\psi\cdot\Delta Z_t) - u(\psi_{i(\psi)}\cdot\Delta Z_t)||\mathfrak{F}_{t-})\Big) \\
\le\ & \widetilde{\varepsilon}M\sup_{y\in\mathbb{R}}|u'(y)| + r\frac{\varepsilon}{3r} + \widetilde{\varepsilon}M\sup_{y\in\mathbb{R}}|u'(y)| = \varepsilon.
\end{aligned}$$

Therefore, $\sup_{\psi \in A} |E(u(\psi \cdot \Delta Z_t)|\mathcal{F}_{t^{\Sigma_m}--}) - E(u(\psi \cdot \Delta Z_t)|\mathcal{F}_{t-})| \to 0$ in L^1 for $m \to \infty$. Recalling the first and the second step, this yields the first statement in Theorem 3.54.

Fourth step: Let $p := 1 + \varepsilon$, $q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$, where ε is chosen as in regularity condition (RC 1). Define the increasing functions $V^1, V^2, V^3(\psi) : \mathbb{R}_+ \to \mathbb{R}$ and the real valued processes $V^4(\psi), V^5(\psi)$ (for any $\psi \in \mathbb{R}^{n+1}$) by

$$
\begin{aligned}
V_t^1 &:= \int_0^t E\Big(|b_s|^p + \sum_{i,j=0}^n |c_s^{ij}|^p + \Big(\int (|x|^2 \wedge |x|)\, F_s(dx)\Big)^p\Big)\, ds, \\
V_t^2 &:= \int_0^t E\Big(|b_s| + \sum_{i,j=0}^n |c_s^{ij}| + \int (|x|^2 \wedge |x|)\, F_s(dx)\Big)\, ds, \\
V^3(\psi)_t &:= \int_0^t E(|\gamma_s(\psi)|)\, ds, \\
V^4(\psi)_t &:= \int_0^t |\gamma_s(\psi)|\, ds, \\
V^5(\psi)_t &:= \int_0^t \gamma_s(\psi)\, ds.
\end{aligned}
$$

Firstly note that the finiteness of V^1, V^2, $V^3(\psi)$ follows from the integrability conditions (RC 1) (for $V^3(\psi)$ cf. the proof of Theorem 3.14). Since these functions and processes are absolutely continuous, there is a λ-null set $N \subset \mathbb{R}_+$ such that V^1, V^2, $V^3(\psi)$ are differentiable for any $\psi \in \mathbb{Q}^{n+1}$ (cf. Elstrodt (1996), VII.4.12, VII.4.14). Moreover, N can be chosen such that $V^4(\psi)$, $V^5(\psi)$ are for any $\psi \in \mathbb{Q}^{n+1}$ P-almost surely differentiable in any $t \in N^C$. Moreover, the finite derivatives are given by the respective integrands evaluated in t. Fix $t \in (N \cup \Theta)^C$. Similarly to the first step, we have for any $\psi \in \mathbb{R}^{n+1}$ that

$$
\begin{aligned}
&\Big|\frac{1}{\Delta t^{\Sigma_m +}} \Gamma^{\Sigma_m}_{t^{\Sigma_m}++}(\psi) - \gamma_t(\psi)\Big| \\
&= \frac{1}{\Delta t^{\Sigma_m +}} |E(u(\psi \cdot (Z_{t^{\Sigma_m}++} - Z_{t^{\Sigma_m}-})) - \Delta t^{\Sigma_m +}\gamma_t(\psi)|\mathcal{F}_{t^{\Sigma_m}-})| \\
&\quad + |E(\gamma_t(\psi)|\mathcal{F}_{t^{\Sigma_m}-}) - \gamma_t(\psi)|.
\end{aligned} \tag{3.23}
$$

Fifth step: Let $\varepsilon > 0$,

$$
L := 1 + \kappa \sup_{\psi \in A} |\psi| + \sup_{y \in \mathbb{R}} |u'(y)| + 2 \sup_{\psi \in A} |\psi| \sup_{y \in \mathbb{R}} |u''(y)|,
$$

and $M := E(|b_t| + \sum_{i,j=0}^n |c_t^{ij}| + \int (|x|^2 \wedge |x|)\, F_t(dx)) < \infty$. Choose $\psi_1, \dots, \psi_r \in A$ such that for any $\psi \in A$, there is a $\psi_{i(\psi)} \in \{\psi_1, \dots, \psi_r\}$ at most $\widetilde{\varepsilon} := \varepsilon/(3ML)$ away. By the martingale convergence theorem (cf. JS, I.1.42) we have that $E(|\gamma_t(\psi_i) - E(\gamma_t(\psi_i)|\mathcal{F}_{t^{\Sigma_m}-})| < \frac{\varepsilon}{3r}$ for any $i \in \{1, \dots, r\}$ and any m large enough. By applying the mean value theorem to u and u', we obtain that

$$
\begin{aligned}
&|(u(\psi \cdot x) - \psi \cdot x) - (u(\psi_{i(\psi)} \cdot x) - \psi_{i(\psi)} \cdot x)| \\
&\le (|x|^2 \wedge |x|)\widetilde{\varepsilon}\Big(1 + \sup_{y \in \mathbb{R}} |u'(y)| + 2 \sup_{\widetilde{\psi} \in A} |\widetilde{\psi}| \sup_{y \in \mathbb{R}} |u''(y)|\Big)
\end{aligned}
$$

for any $\psi \in A$. Moreover, we have $|\psi^\top c_t\psi - \psi_{i(\psi)}^\top c_t\psi_{i(\psi)}| = |(\psi - \psi_{i(\psi)})^\top c_t(\psi + \psi_{i(\psi)})| \leq 2\widetilde{\varepsilon}\sum_{i,j=0}^n |c_t^{ij}|\sup_{\widetilde{\psi}\in A}|\widetilde{\psi}|$. Summing the various terms up, it follows that for large m

$$E\Big(\sup_{\psi\in A}|\gamma_t(\psi) - \gamma(\psi_{i(\psi)})|\Big) \leq \widetilde{\varepsilon}LM = \frac{\varepsilon}{3}. \tag{3.24}$$

As in the third step, we can now conclude that $\sup_{\psi\in A}|E(\gamma_t(\psi)|\mathcal{F}_{t^{\Sigma_m}-}) - \gamma_t(\psi)| \to 0$ in L^1 for $m \to \infty$.

Sixth step: Let $m \in \mathbb{N}$ be so large that $[t^{\Sigma_m -}, t^{\Sigma_m ++}] \cap \Theta = \varnothing$. Fix $\psi \in \mathbb{R}^{n+1}$ for the moment. Define the process $(Y_s)_{s\in\mathbb{R}_+}$ by $Y_s := \psi \cdot (Z_s - Z_{t^{\Sigma_m}-})1_{(t^{\Sigma_m -},\infty)}(s)$. By Itô's formula (cf. Jacod (1979), (3.89)), we obtain as in the proof of Theorem 3.14 that

$$\begin{aligned}
&u(\psi\cdot(Z_{t^{\Sigma_m ++}} - Z_{t^{\Sigma_m -}})) - \Delta t^{\Sigma_m +}\gamma_t(\psi)\\
&= \int_0^{t^{\Sigma_m ++}} 1_{[0,t^{\Sigma_m -}]^C}(s)u'(Y_s-)\psi\cdot dZ_s^C && (3.25)\\
&\quad + \int_{[0,t^{\Sigma_m ++}]\times\mathbb{R}^{n+1}} 1_{[0,t^{\Sigma_m -}]^C}(s)(u(Y_{s-}+\psi\cdot x) - u(Y_{s-}))\,(\mu^Z - \nu)(ds,dx) && (3.26)\\
&\quad + \int_{t^{\Sigma_m -}}^{t^{\Sigma_m ++}} (\gamma_s(\psi) - \gamma_t(\psi))\,ds && (3.27)\\
&\quad + \int_{t^{\Sigma_m -}}^{t^{\Sigma_m ++}} (u'(Y_{s-}) - 1)b_s\cdot\psi\,ds && (3.28)\\
&\quad + \frac{1}{2}\int_{t^{\Sigma_m -}}^{t^{\Sigma_m ++}} (u''(Y_{s-}) - \kappa)\psi^\top c_s\psi\,ds && (3.29)\\
&\quad + \int_{t^{\Sigma_m -}}^{t^{\Sigma_m ++}}\int \Big(u(Y_{s-}+\psi\cdot x) - u(Y_{s-}) - u(\psi\cdot x) - (u'(Y_{s-}) - 1)\psi\cdot x\Big)\,F_s(dx)\,ds. && (3.30)
\end{aligned}$$

As in the proof of Theorem 3.14, it follows that the terms (3.25), (3.26) are uniformly integrable martingales (as processes of the upper integration limit). Hence, their conditional expectation given $\mathcal{F}_{t^{\Sigma_m -}}$ equals 0.

Seventh step: Let $\varepsilon > 0$ and L as in the fifth step. Since V^2 is differentiable in t, there exixts a $M > 0$ such that $(V^2)'_t < M$ and $\frac{1}{\Delta t^{\Sigma_m +}}(V^2_{t^{\Sigma_m ++}} - V^2_{t^{\Sigma_m -}}) < M$ for m large enough. Choose $\psi_1,\dots,\psi_r \in A\cap\mathbb{Q}^{n+1}$ such that for any $\psi\in A$, there is a $\psi_{i(\psi)} \in \{\psi_1,\dots,\psi_r\}$ at most $\widetilde{\varepsilon} := \varepsilon/(3ML)$ away. The differentiability of $V^3(\psi), V^4(\psi), V^5(\psi)$ in t yields that we have $\frac{1}{\Delta t^{\Sigma_m +}}\int_{t^{\Sigma_m -}}^{t^{\Sigma_m ++}}\gamma_s(\psi)\,ds \stackrel{m\to\infty}{\longrightarrow} \gamma_t(\psi)$ P-almost surely, $\frac{1}{\Delta t^{\Sigma_m +}}\int_{t^{\Sigma_m -}}^{t^{\Sigma_m ++}}|\gamma_s(\psi)|\,ds \stackrel{m\to\infty}{\longrightarrow} |\gamma_t(\psi)|$ P-almost surely and $E(\frac{1}{\Delta t^{\Sigma_m +}}\int_{t^{\Sigma_m -}}^{t^{\Sigma_m ++}}|\gamma_s(\psi)|\,ds) \stackrel{m\to\infty}{\longrightarrow} E(|\gamma_t(\psi)|)$ for any $\psi \in \mathbb{Q}^{n+1}$. By Elstrodt (1996), Korollar VI.5.5 it follows that $\frac{1}{\Delta t^{\Sigma_m +}}\int_{t^{\Sigma_m -}}^{t^{\Sigma_m ++}}\gamma_s(\psi)\,ds \stackrel{m\to\infty}{\longrightarrow} \gamma_t(\psi)$ in L^1 for any $\psi\in\mathbb{Q}^{n+1}$. Therefore, we have that $E(|\frac{1}{\Delta t^{\Sigma_m +}}\int_{t^{\Sigma_m -}}^{t^{\Sigma_m ++}}\gamma_s(\psi_i)\,ds - \gamma_t(\psi_i)|) < \frac{\varepsilon}{3r}$ for any $i\in\{1,\dots,r\}$ and any m large enough. The estimate (3.24) implies that $E(\sup_{\psi\in A}|\frac{1}{\Delta t^{\Sigma_m +}}\int_{t^{\Sigma_m -}}^{t^{\Sigma_m ++}}(\gamma_s(\psi) - \gamma_s(\psi_{i(\psi)}))\,ds|) < \widetilde{\varepsilon}ML = \frac{\varepsilon}{3}$ and likewise $E(\sup_{\psi\in A}|\gamma_t(\psi) - \gamma_t(\psi_{i(\psi)})|) < \frac{\varepsilon}{3}$ for m large enough. Adding terms up as in the third and the fifth step, we can conclude that

$$\sup_{\psi\in A}\frac{1}{\Delta t^{\Sigma_m+}}\Big|E\Big(\int_{t^{\Sigma_m-}}^{t^{\Sigma_m++}}(\gamma_s(\psi)-\gamma_t(\psi))\,ds\Big|\mathcal{F}_{t^{\Sigma_m-}}\Big)\Big|\overset{m\to\infty}{\longrightarrow}0$$

in L^1.

Eighth step: Firstly, observe that $\sup_{s\in[t^{\Sigma_m-},t^{\Sigma_m++}]}|u'(Y_{s-})-1|$ is bounded and converges P-almost surely to 0 for $m\to\infty$. By domonated convergence, this implies that $\|\sup_{s\in[t^{\Sigma_m-},t^{\Sigma_m++}]}|u'(Y_{s-})-1|\|_{L^q}\to 0$ for $m\to\infty$. Moreover, Jensen's inequality and the differentiability of V^1 in t yield that $E((\frac{1}{\Delta t^{\Sigma_m+}}\int_{t^{\Sigma_m-}}^{t^{\Sigma_m++}}|b_s|\,ds)^p)\le\frac{1}{\Delta t^{\Sigma_m+}}\int_{t^{\Sigma_m-}}^{t^{\Sigma_m++}}E(|b_s|^p)\,ds=O(1)$ for $m\to\infty$. Together, we obtain with Hölder's inequality that

$$\begin{aligned}
&\sup_{\psi\in A}\frac{1}{\Delta t^{\Sigma_m+}}\Big|E\Big(\int_{t^{\Sigma_m-}}^{t^{\Sigma_m++}}(u'(Y_{s-})-1)b_s\cdot\psi\,ds\Big)\Big|\\
&\le\ \sup_{\psi\in A}|\psi|\Big\|\sup_{s\in[t^{\Sigma_m-},t^{\Sigma_m++}]}|u'(Y_{s-})-1|\Big\|_{L^q}\Big\|\frac{1}{\Delta t^{\Sigma_m+}}\int_{t^{\Sigma_m-}}^{t^{\Sigma_m++}}|b_s|\,ds\Big\|_{L^p}\to 0
\end{aligned}$$

for $m\to\infty$. This implies that $\sup_{\psi\in A}\frac{1}{\Delta t^{\Sigma_m+}}|E(\int_{t^{\Sigma_m-}}^{t^{\Sigma_m++}}(u'(Y_{s-})-1)b_s\cdot\psi\,ds|\mathcal{F}_{t^{\Sigma_m-}})|\to 0$ in L^1 for $m\to\infty$. With basically the same proof, we prove the convergence for the term (3.29) instead of (3.28).

Ninth step: As in the proof of Theorem 3.14, it follows that

$$\begin{aligned}
&\sup_{\psi\in A}\Big|\frac{1}{\Delta t^{\Sigma_m+}}\int_{t^{\Sigma_m-}}^{t^{\Sigma_m++}}\int\Big(u(Y_{s-}+\psi\cdot x)-u(Y_{s-})\\
&\qquad -u(\psi\cdot x)-(u'(Y_{s-})-1)\psi\cdot x\Big)F_s(dx)\,ds\Big|\\
&\le\ \Big(\sup_{\psi\in A}|\psi|^2\vee 1\Big)\frac{1}{\Delta t^{\Sigma_m+}}\int_{t^{\Sigma_m-}}^{t^{\Sigma_m++}}(|x|^2\wedge|x|)\,F_s(dx)\,ds\\
&\qquad\cdot\Big(\Big(3\sup_{y\in\mathbb{R}}|u'(y)|+1+2\sup_{y\in\mathbb{R}}|u''(y)|\Big)\\
&\qquad\wedge\sup_{s\in[t^{\Sigma_m-},t^{\Sigma_m++}]}|Y_{s-}|\Big(2\sup_{y\in\mathbb{R}}|u''(y)|+\sup_{y\in\mathbb{R}}|u'''(y)|\Big)\Big).
\end{aligned}$$

As in the eighth step one shows that the first factor is bounded, the second converges to 0 in L^q, and the L^p-norm of the third factor is $O(1)$ for $m\to\infty$. This yields the same kind of convergence for the term (3.30) as for (3.28) and (3.29).

Tenth step: Summarizing steps 6–9 shows L^1-convergence to 0 of Term (3.23), uniformly over all $\psi\in A$. In view of the fourth and the fifth step, this implies that $\sup_{\psi\in A}|\frac{1}{\Delta t^{\Sigma_m+}}\Gamma^{\Sigma_m}_{t^{\Sigma_m}++}(\psi)-\gamma_t(\psi)|\to 0$ in L^1 for $m\to\infty$, and hence the proof of Theorem 3.54 is complete. □

PROOF OF THEOREM 3.55. *First step:* Fix $t\in\Theta$. According to Bauer (1978), Satz 19.6, it suffices to show that for any subsequence $(\Sigma'_m)_{m\in\mathbb{N}}$ of $(\Sigma_m)_{m\in\mathbb{N}}$, there is a subsequence

$(\Sigma''_m)_{m\in\mathbb{N}}$ of $(\Sigma'_m)_{m\in\mathbb{N}}$ such that $\varphi^{\Sigma''_m}_{t^{\Sigma''_m}+} \to \varphi_t$ P-almost surely for $m \to \infty$. Let $(\Sigma'_m)_{m\in\mathbb{N}}$ be given. From Theorem 3.54, it follows that for any $N \in \mathbb{N}$ there is a subsequence $(\Sigma''_m)_{m\in\mathbb{N}}$ of $(\Sigma'_m)_{m\in\mathbb{N}}$ such that $\sup_{\psi\in\overline{K}_N(0)} |\Gamma^{\Sigma''_m}_{t^{\Sigma''_m}+}(\psi) - \Gamma_t(\psi)| \to 0$ P-almost surely for $m \to \infty$, where $\overline{K}_N(0)$ denotes the closed ball with radius N around 0. By a diagonal procedure we can even find a subsequence $(\Sigma''_m)_{m\in\mathbb{N}}$ of $(\Sigma'_m)_{m\in\mathbb{N}}$ and a P-null set $N_1 \in \mathcal{F}$ such that $\sup_{\psi\in A} |\Gamma^{\Sigma''_m}_{t^{\Sigma''_m}+}(\psi) - \Gamma_t(\psi)| \stackrel{m\to\infty}{\longrightarrow} 0$ pointwise outside N_1 for any compact set $A \subset \mathbb{R}^{n+1}$. Moreover, Conditions 3 and 2 imply that outside some P-null set N_2 we have $\Gamma_t(\varphi_t) > \Gamma_t(\psi)$ for any $\psi \in (\{0\} \times M) \setminus \{\varphi_t\}$. Fix $\omega \in (N_1 \cup N_2)^C$ and $\varepsilon > 0$.

Second step: We will now show that the existence of a $\delta > 0$ such that for any $\psi \in (\{0\} \times M) \setminus K_{\frac{\varepsilon}{2}}(\varphi_t)$ we have $\Gamma_t(\varphi_t) - \delta > \Gamma_t(\psi)$, where $K_{\frac{\varepsilon}{2}}(\varphi_t)$ denotes the ball with radius $\frac{\varepsilon}{2}$ around φ_t. Assume that there is no such δ. Then there exists for any $k \in \mathbb{N}^*$ some $\psi_k \in (\{0\} \times M) \setminus K_{\frac{\varepsilon}{2}}(\varphi_t)$ with $\Gamma_t(\psi_k) > \Gamma_t(\varphi_t) - \frac{1}{k}$. Since the mapping $\psi \to \Gamma_t(\psi)$ is concave (cf. the proof of Theorem 3.22), the same holds for any ψ on the straight line between ψ_k and φ_t. By convexity of $\{0\} \times M$ this implies that for any $k \in \mathbb{N}$ there is a $\widetilde{\psi} \in (\{0\} \times M) \cap \partial K_{\frac{\varepsilon}{2}}(\varphi_t)$ with $\Gamma_t(\widetilde{\psi}_k) \geq \Gamma_t(\varphi_t) - \frac{1}{k}$. Since the set $(\{0\} \times M) \cap \partial K_{\frac{\varepsilon}{2}}(\varphi_t)$ is compact, the sequence $(\widetilde{\psi}_k)_{k\in\mathbb{N}}$ has a cluster point $\widetilde{\psi} \in (\{0\} \times M) \cap \partial K_{\frac{\varepsilon}{2}}(\varphi_t)$. Moreover, by continuity of the mapping $\psi \to \Gamma_t(\psi)$ we have $\Gamma_t(\psi) \geq \Gamma_t(\varphi_t)$. This is a contradiction to the assumption that φ_t is the unique maximal point of the mapping $\{0\} \times M \to \mathbb{R}$, $\psi \mapsto \Gamma_t(\psi)$.

Third step: From the uniform convergence in step 1, it follows that $|\Gamma^{\Sigma''_m}_{t^{\Sigma''_m}+}(\psi) - \Gamma_t(\psi)| < \frac{\delta}{2}$ for any $\psi \in \overline{K}_\varepsilon(\varphi_t)$ and any $m \in \mathbb{N}$ large enough. In view of the second step, this implies $\Gamma^{\Sigma''_m}_{t^{\Sigma''_m}+}(\psi) < \Gamma^{\Sigma''_m}_{t^{\Sigma''_m}+}(\varphi_t)$ for any $\psi \in (\{0\} \times M) \cap \overline{K}_\varepsilon(\varphi_t) \setminus K_{\frac{\varepsilon}{2}}(\varphi_t)$ and any $m \in \mathbb{N}$ large enough. Since the mapping $(\{0\} \times M) \cap \overline{K}_\varepsilon(\varphi_t) \to \mathbb{R}$, $\psi \mapsto \Gamma^{\Sigma''_m}_{t^{\Sigma''_m}+}(\psi)$ is a continuous function defined on a compact set, it attains its supremum. Because of the above inequality, the maximal point, say ψ_m, is situated in the open ball $K_{\frac{\varepsilon}{2}}(\varphi_t)$. Therefore, the concave function $\{0\} \times M \to \mathbb{R}$, $\psi \mapsto \Gamma^{\Sigma''_m}_{t^{\Sigma''_m}+}(\psi)$ has a local maximum in ψ_m, which is, by concavity, also a global maximum (cf. Rockafellar (1970), p.264). Since the set of extremal points of a concave function defined on a convex set is convex and since there is no extremal point in $\psi \in (\{0\} \times M) \cap \overline{K}_\varepsilon(\varphi_t) \setminus K_{\frac{\varepsilon}{2}}(\varphi_t)$, any extremal point of the mapping $(\{0\} \times M) \cap \overline{K}_\varepsilon(\varphi_t) \to \mathbb{R}$, $\psi \mapsto \Gamma^{\Sigma''_m}_{t^{\Sigma''_m}+}(\psi)$ must lie in $M \cap \overline{K}_{\frac{\varepsilon}{2}}(\varphi_t)$. In particular, this is true for $\varphi^{\Sigma''_m}_{t^{\Sigma''_m}+}$, which yields $|\varphi_t - \varphi^{\Sigma''_m}_{t^{\Sigma''_m}+}| < \varepsilon$ for m large enough. This shows Statement 1 in Theorem 3.55. The second statement follows along the same lines. □

Chapter 4

Examples

In this chapter our goal is to illustrate our approach in concrete settings. We have chosen primarily classical examples of different kinds. We perform explicit numerical calculations only in those cases where we can do without sophisticated algorithms. The examples are not intended primarily for model comparison or testing, nor to give new insights into the implications of these settings. Rather, we want to suggest how our approach may be applied in practice. As in the previous chapters, proofs are to be found at the end of any subsection. For easier readability we do not note all of the regularity assumptions in the statements, e.g. if they depend on the choice of parameters. In these cases we comment at the beginning of the respective proof.

4.1 A Two-period Model

The following two-period model is one of the simplest market models altogether. Nevertheless, it should become evident how to pass from here to any multiperiod setting with a finite state space. To begin with, we consider a market consisting of two securities. The first one represents the bank account and serves as the numeraire (i.e. $Z^0 = 1$). We denote the second security by Z^1 and call it stock. Its dynamics is given in Figure 4.1. The

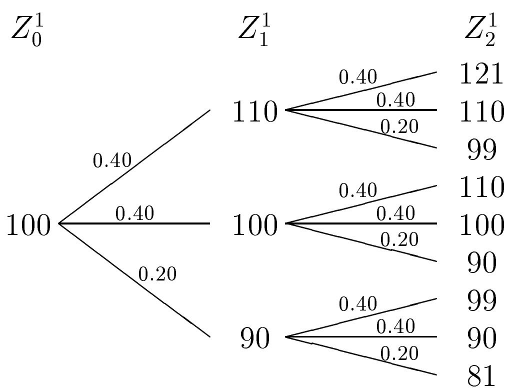

Figure 4.1: The market model

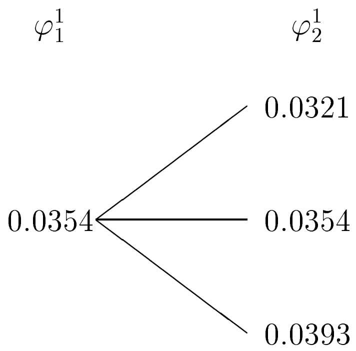

Figure 4.2: Strategy of the speculator for $\kappa = 1$

large numbers denote the possible prices at time $t = 0, 1, 2$, whereas the small ones indicate transition probabilities (e.g. we have $P(Z_2^1 = 99|Z_1^1 = 90) = 0.4$). Suppose that we are working with a filtered space whose filtration is generated by (Z^0, Z^1). In the language of the preceding two chapters the stochastic process (Z^0, Z^1) has the extended characteristics $(\{1, 2\}, \varepsilon_{(1,100)}, 0, 0, 0, K)^E$, where

$$K_t = \begin{cases} 0 & \text{for } t \notin \{1, 2\} \\ \varepsilon_0 \otimes (0.4\varepsilon_{0.1Z_{t-1}^1} + 0.4\varepsilon_0 + 0.2\varepsilon_{-0.1Z_{t-1}^1}) & \text{for } t \in \{1, 2\}. \end{cases}$$

4.1.1 Derivative Pricing

By means of Corollary 3.23 or the remark following Corollary 3.37 (or Lemma 1.2) we can now easily compute a u_1-optimal strategy φ for $\mathfrak{A}$ (i.e. for the speculator). Note that φ^0 as well as φ_t^1 for $t \notin \{1, 2\}$ can be freely chosen. For $t \in \{1, 2\}$, φ_t^1 is the uniquely determined $\mathfrak{F}_{t-1}$-measurable random variable represented in Figure 4.2. The upper, middle, and lower branch correspond to the respective transitions from Z_0^1 to Z_1^1 in Figure 4.1. With the help of Corollary 3.37 (or Equation (1.4)) we can determine the probabilities under the equivalent martingale measure P^* that is needed to obtain neutral derivative prices. These are given in Figure 4.3, where one can also find the corresponding probabilities relative to P. The small numbers on the branches indicate the transition probabilities with respect to P^*, which have been computed as in the remark following Corollary 3.37. Of course multiplication of these conditional probabilities also yields P^*. Now let us consider a European call option on the stock with discounted strike price 95, i.e. a derivative with terminal value $X^2 = (Z_2^1 - 95) \vee 0$. The corresponding neutral price process Z^2 is obtained by Equation (3.11) (or Lemma 1.7) and can be found in Figure 4.3 as well.

4.1.2 Hedging

Assume that you are a bank trading in the market with securities 0, 1, 2 as in Figure 4.3. You have sold 100 options to a customer and want to hedge your risk by investing in the stock. We suppose that you do this by choosing a u_{100}-optimal strategy for the set of all portfolios

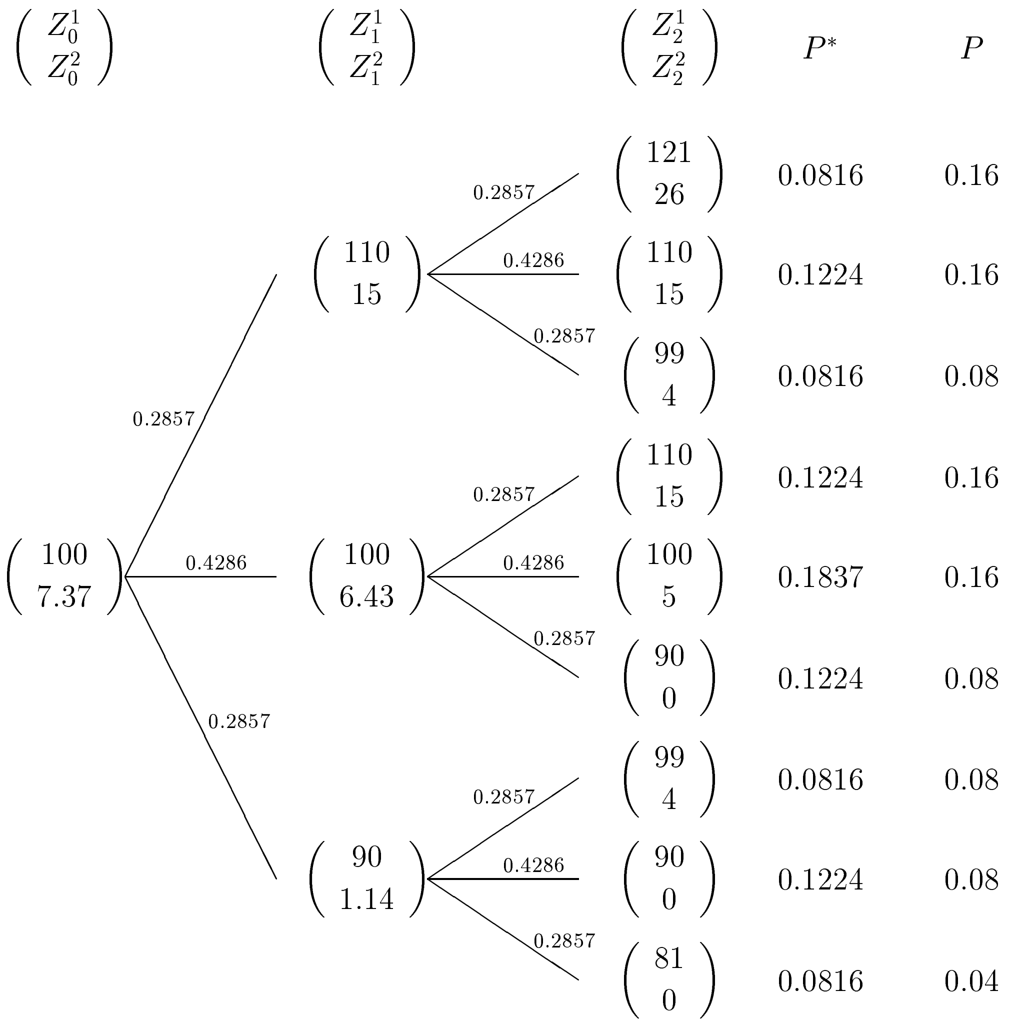

Figure 4.3: Neutral option prices and pricing measure

with fixed value $\varphi^2 = -100$, which corresponds to a very risk-averse attitude. In this case maximization of the expected utility is almost equivalent to minimizing the expected loss. In order to compute the optimal portfolio using Corollary 3.23, you need the extended characteristics of the process $\overline{Z} = (Z^0, Z^1, Z^2)$. Since this process is discrete, they are given by $(1, 2, \varepsilon_{(1,100,7.37)}, 0, 0, 0, \overline{K})^E$ (cf. Lemma 2.20 and the following remark), where $\overline{K}_t(\cdot) = E(\Delta\overline{Z}_t \in \cdot|\mathcal{F}_{t-1})$ for $i = 1, 2$ can be obtained from Figure 4.3 and the transition probabilities in Figure 4.1. The resulting u_{100}-optimal strategy φ is noted in Figure 4.4. As for the speculator in the previous subsection, φ^0 and φ_t for $t \notin \{1, 2\}$ can be arbitrarily chosen. Observe that using Lemma 1.2 instead of Corollary 3.23 yields the same results. In the same manner we can now compute the optimal hedge if you have bought 100 options (i.e. $\varphi^2 = 100$ instead of $\varphi^2 = -100$) (cf. Figure 4.5). Observe that the strategies in Figure 4.5 are not just the negative of those in Figure 4.4, as would be the case for the *perfect hedge* in a complete model. This is due to the fact that we are working with an asymmetric utility function that distinguishes possible losses and gains. Note also that the u_κ-optimal portfolios are not pure hedging strategies. This becomes apparent if we choose small values for the risk aversion κ and the fixed position φ^2 as we do in Figure 4.6 ($\kappa = 1, \varphi^2 = -1$). On the upper branch the optimal number of stocks is 1.032. This may be surprising, since $\varphi^1_2 = 1$ would be a perfect hedge for this part of the market (cf. Figure 4.3). This property of overhedging is due to the fact that u_κ-optimal trading is, by the shape of the utility function u_κ, a mixture between minimization of expected losses and maximization of expected gains. For small κ the expected gains are more important, whereas for large κ the losses become predominant so that the name *hedging strategy* is adequate.

4.1.3 Trading Corridors

Trading corridors allow you to choose a reasonable strategy without accepting too many or too large transactions. In Figures 4.7 and 4.8 we calculate (u_{100}, ε)-trading corridors for the hedging problems in Figure 4.4 resp. 4.5 and two values of ε (10 and 0.01, respectively). ε here corresponds to ε_1 in Definition 3.33 (or ε in Definition 1.4). ε_2 is irrelevant since we are working in a discrete-time model. The boundary points of the trading corridors can be numerically easily obtained by Lemma 3.34. In Figures 4.7 and 4.8 we only indicate the possible intervals for φ^1 since φ^2 is fixed. Since $\frac{1}{2}u_{100}(x) \approx x \wedge 0$, the utility bandwidth ε has an intuitive interpretation. Choosing $\varepsilon = 10$ (resp. 0.01) means sorting out the strategies whose expected loss does not exceed the optimal value by more than 20 (resp. 0.02). In Figure 4.8 one may make an interesting observation. Even for the very small value $\varepsilon = 0.01$ the allowed intervals are surprisingly large. This means that there is a comparatively broad range of portfolios with approximately the same expected utility. In particular, it shows that multiplying the strategies from Figure 4.4 by -1 produces an almost optimal portfolio for the problem in Figure 4.5. The converse, however, is not true.

$$
\begin{tikzcd}
\begin{pmatrix}\varphi_1^1\\ \varphi_1^2\end{pmatrix} & \begin{pmatrix}\varphi_2^1\\ \varphi_2^2\end{pmatrix} \\
 & \begin{pmatrix}100\\ -100\end{pmatrix} \\
\begin{pmatrix}76.3\\ -100\end{pmatrix} \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & \begin{pmatrix}85.7\\ -100\end{pmatrix} \\
 & \begin{pmatrix}31.7\\ -100\end{pmatrix}
\end{tikzcd}
$$

Figure 4.4: Hedging strategy for $\kappa = 100$, $\varphi^2 = -100$

$$
\begin{tikzcd}
\begin{pmatrix}\varphi_1^1\\ \varphi_1^2\end{pmatrix} & \begin{pmatrix}\varphi_2^1\\ \varphi_2^2\end{pmatrix} \\
 & \begin{pmatrix}-100\\ 100\end{pmatrix} \\
\begin{pmatrix}-68.1\\ 100\end{pmatrix} \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & \begin{pmatrix}-73.2\\ 100\end{pmatrix} \\
 & \begin{pmatrix}-20.6\\ 100\end{pmatrix}
\end{tikzcd}
$$

Figure 4.5: Hedging strategy for $\kappa = 100$, $\varphi^2 = 100$

$$
\begin{tikzcd}
\begin{pmatrix}\varphi_1^1\\ \varphi_1^2\end{pmatrix} & \begin{pmatrix}\varphi_2^1\\ \varphi_2^2\end{pmatrix} \\
 & \begin{pmatrix}1.032\\ -1\end{pmatrix} \\
\begin{pmatrix}0.772\\ -1\end{pmatrix} \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & \begin{pmatrix}0.862\\ -1\end{pmatrix} \\
 & \begin{pmatrix}0.325\\ -1\end{pmatrix}
\end{tikzcd}
$$

Figure 4.6: Hedging strategy for $\kappa = 1$, $\varphi^2 = -1$

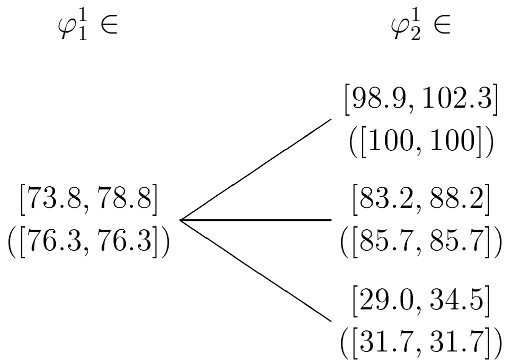

Figure 4.7: Trading corridor for $\varphi^2 = -100$, $\kappa = 100$, $\varepsilon_2 = 10$ (0.01)

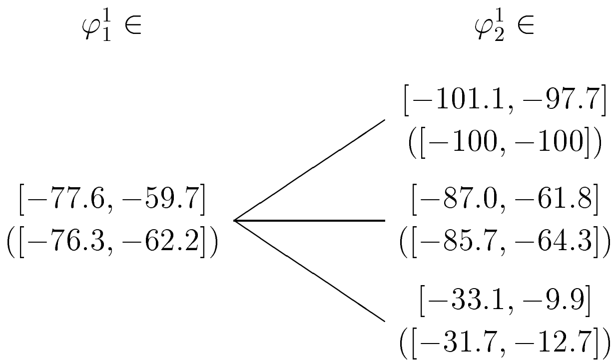

Figure 4.8: Trading corridor for $\varphi^2 = 100$, $\kappa = 100$, $\varepsilon_2 = 10$ (0.01)

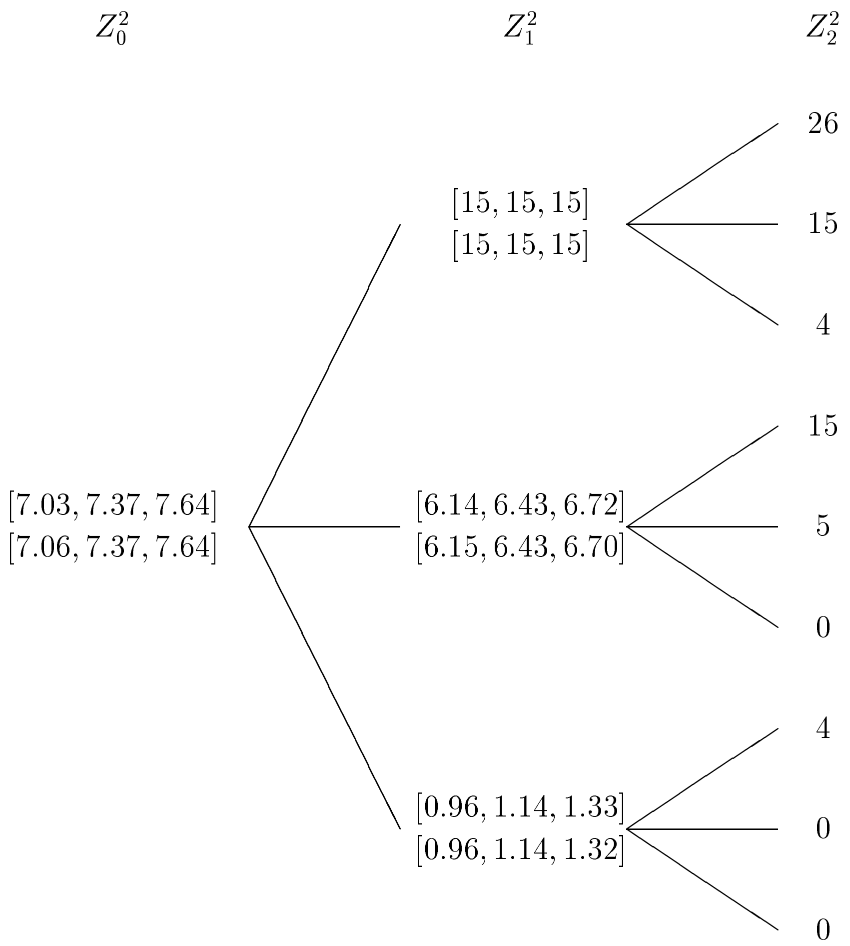

Figure 4.9: Lower, neutral, upper prices for $\kappa r = 0.2$

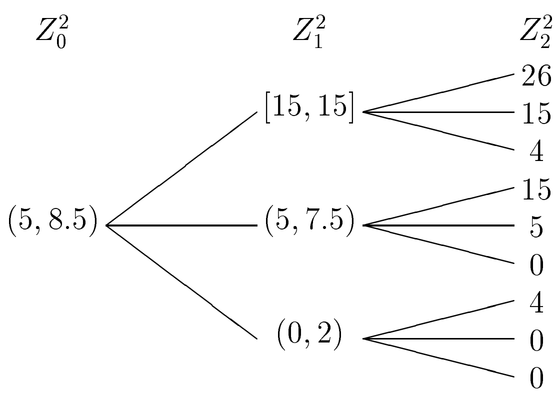

Figure 4.10: Arbitrage bounds

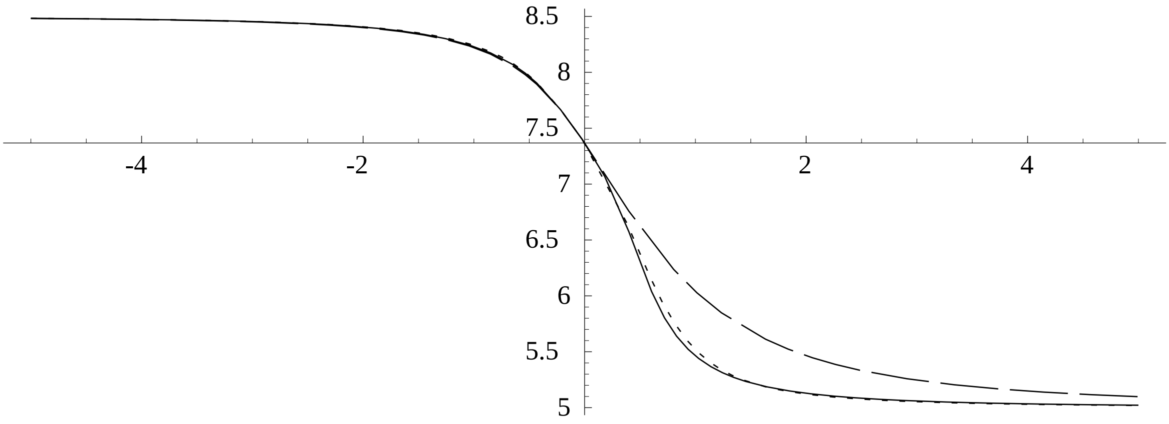

Figure 4.11: Consistent and approximate prices as a function of external supply

4.1.4 Price Regions

The concept of κr-price regions is based on (κ, ρ^2)-consistent (resp. (κ, ρ^2)-approximate) price processes. The computation of (κ, ρ^2)-approximate prices is relatively straightforward (cf. Section 3.5, in particular the remark following Definition 3.45, or alternatively Subsection 1.2.5). On the other hand, (κ, ρ^2)-consistent processes are usually hard to obtain, but since our model is a simple multiperiod market with a finite state space, we can apply the recursive algorithm sketched in Subsection 1.2.5. In Figure 4.9 we list the 0.2-price regions for the derivative $X^2 = (Z_2^1 - 95) \vee 0$ in the market from Figure 4.1. More precisely, the upper triplets contain the $(1, 0.2)$-, $(1, 0)$- and $(1, -0.2)$-consistent prices at each time. Below one can find the $(1, 0.2)$-, $(1, 0)$- and $(1, -0.2)$-approximate prices, respectively. Since the numbers in the middle correspond to zero supply, they equal the neutral option prices from Figure 4.3. To be very strict, we have not shown that the 0.2-price region actually consists of all prices between the upper and the lower value, which correspond to minimal and maximal external supply, respectively. But we believe that this holds in at least simple models of this kind. For a comparison, we note the arbitrage bounds for the option in Figure 4.10. In Figure 4.9 one can observe a certain difference between (κ, ρ^2)-consistent and -approximate prices, but it seems to be small from a numerical point of view. One may wonder whether this still holds for arbitrary values of ρ^2. In Figure 4.11 we plot the initial $(1, \rho^2)$-consistent (straight line) and $(1, \rho^2)$-approximate (dashed line) option price Z_0^2 as a function of the external supply ρ^2. As one may expect, the prices increase (resp. decrease) with growing demand (resp. supply) to the upper (resp. lower) arbitrage bound. The difference becomes the greatest for medium-sized positive values of external supply. In Subsection 1.2.5 and Section 3.5 we raise the question as to whether an iteration of the procedure leading to approximate prices yields a better agreement with consistent prices. In our simple example this is in fact true, as the dotted line in Figure 4.11 indicates. It corresponds to a single repetition of steps 3 to 5 on page 26 resp. steps 3 to 8 on page 123.

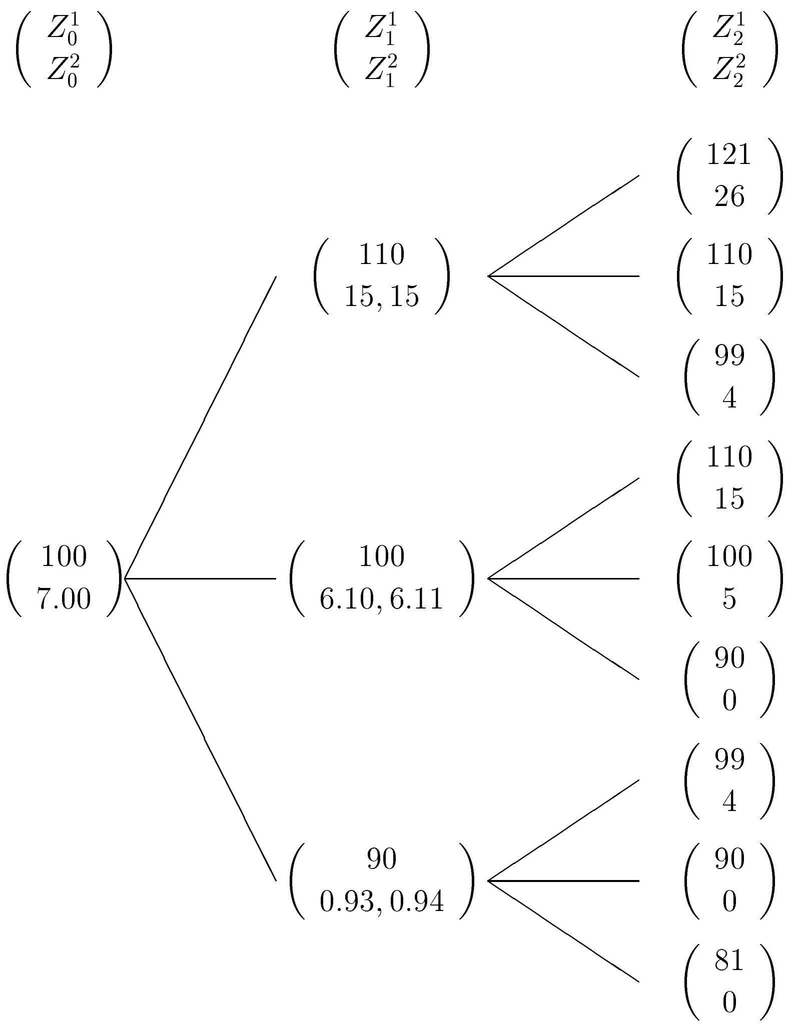

Figure 4.12: Improved model for $Z_0^2 = 7.00$

4.1.5 Improved Derivative Models

If the option price Z_0^2 on the market equals 7.00 instead of its neutral value 7.37, one may want to take this into account by working with 7.00-consistent price processes in the sense of Section 3.6 (or Subsection 1.2.6). The improved market models can be found in Figure 4.12, where the prices on the left (i.e. 15; 6.10; 0.93) correspond to the 7.00-consistent price process and the prices on the right (i.e. 15; 6.11; 0.94) to the approximate 7.00-consistent process, respectively. The value of the external supply ρ^2 leading to the initial price $Z_0^2 = 7.00$ must be determined numerically. For an initial value $Z_0^2 = 7.70$ we obtain the numbers in Figure 4.13. Based on the market model in Figure 4.13, we can now once more tackle the hedging problems from Subsection 4.1.2 that lead to Figures 4.4 and 4.5. The resulting strategies are listed in Figures 4.14 and 4.15. The numbers on the left (resp. right) again correspond to the numbers on the left (resp. right) in Figure 4.13, i.e. to consistent prices or approximately consistent prices, respectively.

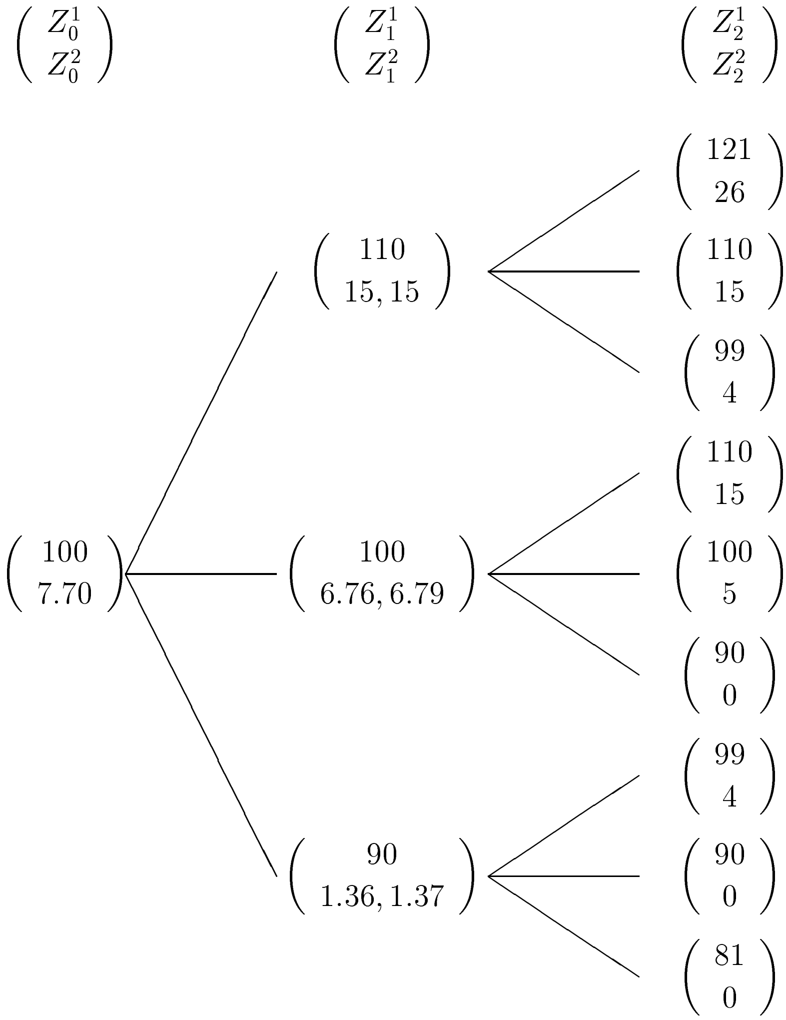

Figure 4.13: Improved model for $Z_0^2 = 7.70$

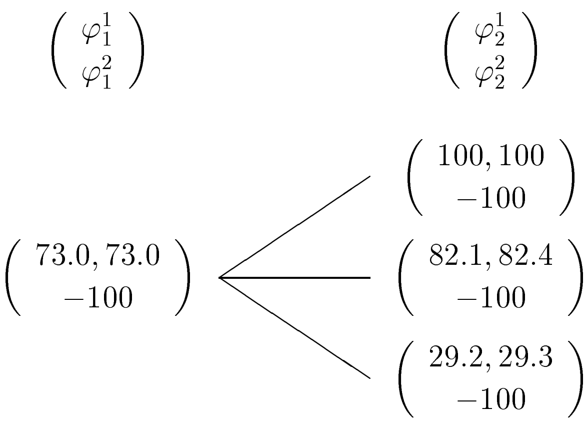

Figure 4.14: Hedging strategy for $Z_0^2 = 7.70$, $\kappa = 100$, $\varphi^2 = -100$

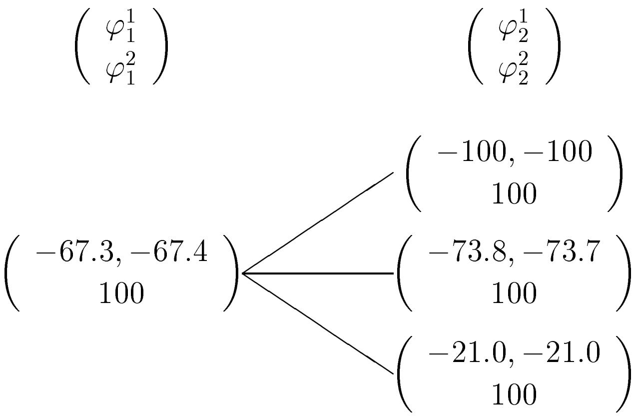

Figure 4.15: Hedging strategy for $Z_0^2 = 7.70$, $\kappa = 100$, $\varphi^2 = 100$

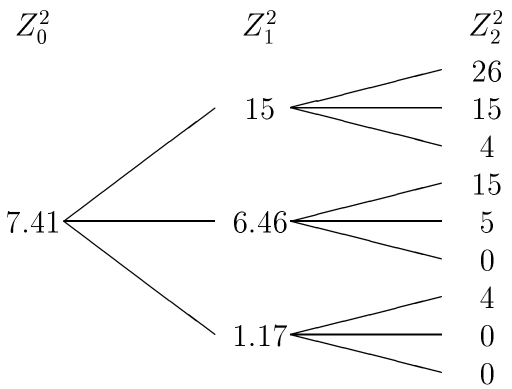

Figure 4.16: Neutral option prices if the stock is the numeraire

4.1.6 Change of Numeraire

In this final subsection we want to examine the effect of a numeraire change on the option prices. To this end, let us assume that the market considers the stock to be a riskless investment and the money market account as risky. We repeat the calculations from the beginning of the section for the market $\widehat{Z} = (\widehat{Z}^0, \widehat{Z}^1)$, where $\widehat{Z}^0 := Z^1/Z^1 = 1$ now is the discounted numeraire and $\widehat{Z}^0 := Z^0/Z^1 = 1/Z^1$ stands for the risky bank account. The discounted payout of the option is now given by $\widehat{X}^2 = X^2/Z_2^1 = (1 - 95\widehat{Z}^0) \vee 0$. If one computes the corresponding neutral derivative price process $\widehat{Z}^2$ and reconverts the values into multiples of the bank account by setting $Z^2 := \widehat{Z}^2 Z^1/Z^0 = \widehat{Z}^2 Z^1$, then one obtains the price process in Figure 4.16. A comparison with Figure 4.3 shows that the prices differ, though not greatly when compared to the unsuitable choice of the numeraire in this subsection. Note that the value $Z_1^2 = 15$ coincides in all option pricing models we have considered in this section, since it is the only value consistent with the absence of arbitrage.

4.2 Models with Continuous Paths

Since the formulas become much easier when jumps are absent, it is worthwhile to repeat some of the results from the previous chapter for models with continuous paths.

4.2.1 Hedging

We consider a market with three securities 0, 1, 2, where the first one denotes the numeraire. Since the corresponding discounted price processes are assumed to be continuous, their extended characteristics are of the form $(\varnothing, P^{(Z_0^0, Z_0^1, Z_0^2)}, b, c, 0, 0)^E$ for $\mathbb{R}^3$-valued resp. $\mathbb{R}^{3\times 3}$-valued processes b and c. Assume now that you have sold one share of Security 2 and you want to hedge the risk.

Lemma 4.1 *The u-optimal strategy for the hedging problem $\varphi_t^2 = -1$ is given by*

$$\varphi_t^1 = \frac{c_t^{12}}{c_t^{11}} + \frac{1}{\kappa}\frac{b_t^1}{c_t^{11}}$$

for any $t \in \mathbb{R}_+$, where $\kappa := -u''(0)$ denotes the risk aversion of the applied utility function. (As usual, φ^0 can be arbitrarily chosen.)

Remark. For large values of κ the u-optimal strategies deserve the name hedging strategy. One may call the limiting strategy $\varphi_t^1 = c_t^{12}/c_t^{11}$ for $\kappa \to \infty$ *pure hedge*. It coincides with the strategy derived by Föllmer & Schweizer (1991) (Theorem (3.14)), which is based on a different optimality criterion.

4.2.2 Trading Corridors

For the computation of $(u, \varepsilon_1, \varepsilon_2)$-trading corridors the parameter ε_1 is irrelevant, since the fixed jump part Γ_t of the local utility is 0. Trading corridors can be easily computed for the above hedging problem.

Lemma 4.2 *The $(u, \varepsilon_1, \varepsilon_2)$-trading corridor for the hedging problem $\varphi_t^2 = -1$ is given by*

$$J(\omega, t) = \mathbb{R} \times \left[\varphi_t^1 - \sqrt{\frac{2\varepsilon_1}{\kappa c_t^{11}}}, \varphi_t^1 + \sqrt{\frac{2\varepsilon_1}{\kappa c_t^{11}}}\right] \times \{-1\},$$

where φ^1 and κ are defined as in Lemma 4.1.

4.2.3 Derivative Pricing

Suppose you are in a market consisting of only one underlying besides the numeraire. Again, the extended characteristics of the corresponding discounted price process $Z = (Z^0, Z^1)$ is of the simple form $(\varnothing, P^{(Z_0^0, Z_0^1)}, b, c, 0, 0)^E$ for some $\mathbb{R}^2$-valued resp. $\mathbb{R}^{2\times 2}$-valued predictable processes b and c. The following lemma characterizes the equivalent probability measure P^* from Theorem 3.36 that allows computation of neutral derivative prices.

Lemma 4.3 *The density process L of P^* in Theorem 3.36 is of the form*

$$L_t = \exp\Big(- \int_0^{t\wedge T} \frac{b_s^1}{c_s^{11}}\, dZ_s^{1,c} - \frac{1}{2}\int_0^{t\wedge T} \frac{(b_s^1)^2}{c_s^{11}}\, ds\Big)$$

for any $t \in \mathbb{R}_+$. Moreover, the P^*-extended characteristics of (Z^0, Z^1) on $[0, T]$ are given by $(\varnothing, P^{(Z_0^0, Z_0^1)}, 0, c, 0, 0)^E$.

Remarks.

1. Note that P^* is independent of the applied utility function.

2. The measure P^* in Lemma 4.3 is called *minimal martingale measure* by Föllmer & Schweizer (1991) (Theorem 3.5). It is used to determine hedging strategies that are optimal in a locally quadratic sense. Note that this equality only holds in the case of continuous processes.

4.2.4 Price Regions and Improved Derivative Models

Since we have no result concerning the existence of consistent prices (cf. Section 3.5), we confine ourselves to indicate the density process of the probability measure $\widetilde{P}$ leading to (κ, ρ^2)-approximate prices for a derivative. The setting is as in the previous subsection. Denote by Z^2 the neutral price process of a derivative given by its terminal value X^2 at time $T > 0$. Denote the joint extended characteristics of $\overline{Z} = (Z^0, Z^1, Z^2)$ by $(\varnothing, P^{(Z_0^0, Z_0^1, Z_0^2)}, \overline{b}, \overline{c}, 0, 0)^E$.

Lemma 4.4 *Let $\kappa > 0$, $\rho^2 \in \mathbb{R}$ be given. The density process $\widetilde{L}$ of $\widetilde{P}$ in step 6 before Definition 3.43 is given by*

$$\widetilde{L}_t = L_t \cdot \exp\left(\kappa\rho^2\Big(\int_0^{t\wedge T} \frac{\overline{c}_s^{12}}{\overline{c}_s^{11}}\, dZ_s^{1,c} - Z_t^{2,c}\Big) + \frac{1}{2}(\kappa\rho^2)^2 \int_0^{t\wedge T} \Big(\frac{\overline{c}_s^{12}}{\overline{c}_s^{11}} - \overline{c}_s^{22}\Big)\, ds\right),$$

for any $t \in \mathbb{R}_+$, where L is the process in Lemma 4.3.

Proofs

PROOF OF LEMMA 4.1. Suppose that $c_t^{11} \neq 0$ outside some evanescent set. Otherwise an optimal strategy does not necessarily exist.

By Corollary 3.23 a strategy φ is u-optimal for the set of all strategies with $\varphi_t^2 = -1$ if and only if $b_t^1 - \kappa c_t^{11}\varphi_t^1 - \kappa c_t^{12}\varphi_t^2 = 0$, i.e. if $\varphi_t^1 = \frac{c_t^{12}}{c_t^{11}} + \frac{1}{\kappa}\frac{b_t^1}{c_t^{11}}$ for any $t \in \mathbb{R}_+$. □

PROOF OF LEMMA 4.2. We have to assume that $c_t^{11} \neq 0$ outside some evanescent set as in the previous proof.

Fix $(\omega, t) \in \Omega \times \mathbb{R}_+$. We have that $\psi \in J(\omega, t)$ if and only if $\psi^2 = -1$ and $\gamma_t(\psi) \geq \gamma(\varphi_t^1) - \varepsilon_1$. The latter condition is equivalent to

$$-\frac{1}{2}\kappa c_t^{11}\Big((\psi^1 - \varphi_t^1)^2 + (\psi^1 - \varphi_t^1)\Big(2\varphi_t^1 - 2\frac{b_t^1}{\kappa c_t^{11}} - 2\frac{c_t^{12}}{c_t^{11}}\Big) - 2\frac{\varepsilon_1}{\kappa c_t^{11}}\Big) \geq 0$$

which in turn is equivalent to $(\psi^1 - \varphi_t^1)^2 \leq \frac{2\varepsilon}{\kappa c_t^{11}}$. This implies the claim. □

PROOF OF LEMMA 4.3. Note that the conditions in Theorem 3.36 depend on the particular model and have to be checked.

By Corollary 3.23, a strategy $\varphi = (\varphi^0, \varphi^1)$ is u-optimal for $\mathfrak{A}$ if and only if $b_t^1 - \kappa c_t^{11}\varphi_t^1 = 0$, i.e. if $\varphi_t^1 = \frac{1}{\kappa}\frac{b_t^1}{c_t^{11}}$. The shape of the Radon-Nikodým density and of the P^*-extended characteristics of (Z^0, Z^1) then follows from Theorem 3.36 and Corollary 3.38. □

PROOF OF LEMMA 4.4. Note that the assumptions in the steps before Definition 3.43 depend on the particular model and still have to be checked.

As in the proof of Lemma 4.1, one verifies that the strategy φ in step 3 on page 123 is given by $\varphi_t^1 = \frac{1}{\kappa}\frac{\overline{b}_t^1}{\overline{c}_t^{11}} - \frac{\overline{c}_t^{12}}{\overline{c}_t^{11}}\rho^2$ for any $t \in \mathbb{R}_+$. Hence $\widetilde{L}$ in step 5 is of the form

$$\begin{aligned}\widetilde{L} = \exp\Big(&- \int_0^{T\wedge t} \kappa\varphi_s^1\, dZ_s^{1,c} - \kappa\rho^2 Z_t^{2,c} \\ &- \frac{1}{2}\int_0^{T\wedge t} (\kappa\varphi_s^1)^2 c_s^{11}\, ds - \int_0^{T\wedge t} \kappa^2\varphi_s^1\rho^2 c_s^{12}\, ds - \frac{1}{2}\int_0^{T\wedge t} (\kappa\rho_s^2)^2 c_s^{22}\, ds\Big).\end{aligned}$$

The claim now follows from a simple calculation. □

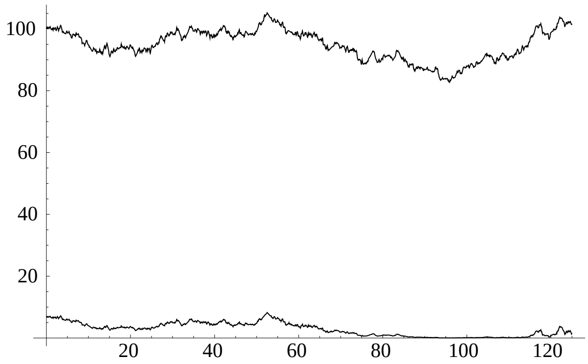

Figure 4.17: Stock price and European call price

4.3 The Black-Scholes Model

The aim of this section is to show that our approach is applicable to this now classical model and yields the same formulas. The setting is as follows. Consider a market with a money market account and a stock whose price processes S^0, S^1 solve the stochastic differential equations

$$\begin{aligned} dS_t^0 &= rS_t^0\,dt \\ dS_t^1 &= \mu S_t^1\,dt + \sigma S_t^1\,dW_t \end{aligned}$$

or, in discounted terms,

$$\begin{aligned} dZ_t^0 &= 0 \\ dZ_t^1 &= (\mu - r)Z_t^1\,dt + \sigma Z_t^1\,dW_t, \end{aligned} \tag{4.1}$$

where $S_0^0 := 1$, $S_0^1, \sigma \in \mathbb{R}_+^*$, $r, \mu \in \mathbb{R}$ are given and W denotes a standard Wiener process. Of course, the solution to these equations are $S_t^0 = e^{rt}$, $S_t^1 = S_0^1 \exp((\mu - \frac{\sigma^2}{2})t + \sigma W_t)$. By Lemma 2.22, $Z = (Z^0, Z^1)$ is an extended Grigelionis process with extended characteristics $(\varnothing, \varepsilon_{(1,S_0^1)}, b, c, 0, 0)^E$, where $b_t^0 = 0$, $b_t^1 = (\mu - r)Z_t^1$, $c_t^{00} = c_t^{01} = c_t^{10} = 0$, $c_t^{11} = (\sigma Z_t^1)^2$. Assume that the filtration is the canonical filtration of (S^0, S^1) (or equivalently, of Z^1 or of W) or its P-completion.

4.3.1 Derivative Pricing

Lemma 4.5 *Let $T \in \mathbb{R}_+$ be given. Then regularity condition (RC 1) and Conditions 1 to 5 in Theorem 3.36 are met. Relative to the pricing measure P^* (whose density may be found in Lemma 4.3), (Z^0, Z^1) has the extended characteristics $(\varnothing, \varepsilon_{(1,S_0^1)}, 0, c, 0, 0)^E$ on $[0, T]$.*

Remarks.

1. The P^*-dynamic of (Z^0, Z^1) is the same as the P-dynamic but with drift 0 instead of $\mu - r$ for Z^1.

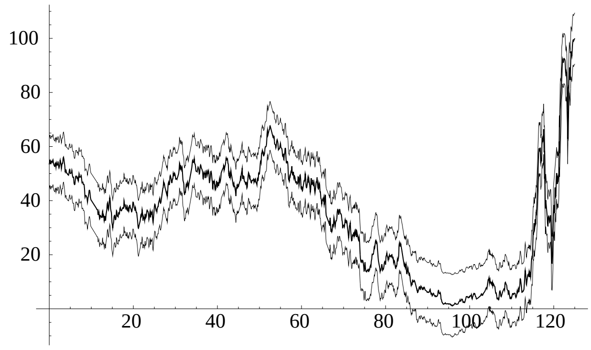

Figure 4.18: Optimal hedge and trading corridor

2. As is well-known, there exists but one equivalent martingale measure on $\mathcal{F}_T$ in this model (cf. Harrison & Pliska (1981), p.246). Hence, any approach to derivative pricing that is based on an EMM yields the same results in this setting. In particular, all consistent and approximate prices as in Section 3.5 coincide with the neutral price process. Moreover, the price regions consist only of a single process.

Now we turn to the explicit computation of derivative and especially European call option prices.

Lemma 4.6 *Let $T \in \mathbb{R}_+$ and $X^2 = g(Z_t^1)$ for a measurable random variable $g : \mathbb{R} \to \mathbb{R}$. Suppose that there are $M_1, M_2 \in \mathbb{R}$ with $|g(x)| \leq M_1 + M_2|x|$ for any $x \in \mathbb{R}$. Define a mapping $C_{BS} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ by $(y, v) \mapsto \int g(y \exp(\sqrt{v}x - \frac{v}{2}))\, \phi(x)dx$, where $\phi : \mathbb{R} \to \mathbb{R}$ denotes the density of the standard normal distribution. Then the unique neutral price process Z^2 for the derivative with terminal value X^2 at T is of the form $Z_t^2 = C_{BS}(Z_t^1, \sigma^2(T - t))$ for any $t \in [0, T]$.*

Corollary 4.7 *Let $K \in \mathbb{R}$. For the European call option $X^2 = ((S_T^1 - K) \vee 0)/S_t^0 = (Z_T^1 - e^{-rT}K) \vee 0$, the neutral price process Z^2 is of the form*

$$Z_t^2 = Z_t^1 \Phi\left(\frac{\log(Z_t^1 e^{rT}/K)}{\sigma\sqrt{T-t}} + \frac{\sigma}{2}\sqrt{T-t}\right) - Ke^{-rT}\Phi\left(\frac{\log(Z_t^1 e^{rT}/K)}{\sigma\sqrt{T-t}} - \frac{\sigma}{2}\sqrt{T-t}\right)$$

for any $t \in [0, T)$, where $\Phi : \mathbb{R} \to [0, 1]$ denotes the cumulative distribution function of the standard normal distribution.

Figure 4.17 shows a sample path of S^1 and the corresponding European call price $S^2 = Z^2 S^0$ for $S_0^1 = 100$, $r = \log(1.05)/250$ (i.e. 5%/year), $\mu = \log(1.09)/250$ (i.e. 9%/year), $\sigma = 0.2387/\sqrt{250}$ (i.e. an annual volatility of 23.87%), where time is measured in trading days ($:= 1/250$ year).

4.3.2 Hedging

Consider a market with three securities 0, 1, 2, where Z^0, Z^1 are as in the previous subsection and Z^2 denotes the neutral price process of a European call option with the strike price

$K \in \mathbb{R}$ and expiration date $T \in \mathbb{R}_+$. Assume that you have sold one option and you want to hedge your risk.

Lemma 4.8 *The u-optimal strategy for the hedging problem $\varphi_t^2 = -1$ is given by*

$$\varphi_t^1 = \Phi\left(\frac{\log(Z_t^1 e^{rT}/K)}{\sigma\sqrt{T-t}} + \sigma\sqrt{T-t}\right) + \frac{1}{\kappa Z_t^1}\frac{\mu - r}{\sigma^2}$$

for any $t \in [0,T)$, where $\kappa := -u''(0)$ is the risk aversion of u and Φ is defined as in Corollary 4.7. (As usual, φ_0 can be arbitrarily chosen.)

Remark. As noted previously, u-optimal strategies deserve the name hedging strategy only for large values of the risk aversion κ. Indeed, for $\kappa \to \infty$ the portfolio φ^1 in the previous lemma converges to the first term, which is the *perfect hedge* or *duplicating strategy* for X^2 in the Black-Scholes model. The second term equals the u-optimal strategy of a speculator.

4.3.3 Trading Corridors

We can easily obtain a $(u, \varepsilon_1, \varepsilon_2)$-trading corridor for the above hedging problem using Lemma 4.2.

Lemma 4.9 *The $(u, \varepsilon_1, \varepsilon_2)$-trading corridor for the hedging problem $\varphi_t^2 = -1$ is given by*

$$J(\omega, t) = \mathbb{R} \times \left[\varphi_t^1 - \sqrt{\frac{2\varepsilon_1}{\kappa}}\frac{1}{\sigma Z_t^1}, \varphi_t^1 + \sqrt{\frac{2\varepsilon_1}{\kappa}}\frac{1}{\sigma Z_t^1}\right] \times \{-1\},$$

where φ^1 and κ are defined as in Lemma 4.8.

In Figure 4.18 we plot the u-optimal strategy from Lemma 4.8 and the stock component of the trading corridor in Lemma 4.9 for the sample paths from Figure 4.17. The chosen values of the parameters are $\kappa = 100$, $K = 100$, $\varepsilon_1 = 10{,}000$. Note that the width $\sqrt{2\varepsilon_1\kappa^{-1}}/\sigma Z_t^1$ of the allowed interval for φ^1 hardly changes over time since Z^1 is approximately constant. Nevertheless, for options closing roughly at the money, one has to rebalance the hedging portfolio more and more often towards the end because the optimal value φ_t^1 changes more violently.

Proofs

PROOF OF LEMMA 4.5. Observe that $\varphi_t^1 = \frac{b_t^1}{\kappa c_t^{11}} = \frac{\mu - r}{\kappa\sigma^2 Z_t^1}$ is a well-defined locally bounded process such that $\varphi = (0, \varphi^1) \in \mathfrak{A}$ is u-optimal for $\mathfrak{A}$, e.g. by Corollary 3.23. Moreover, we have that $N_t = -\kappa\int_0^{T\wedge t}\varphi_s \cdot dZ_s^C = -\frac{\mu - r}{\sigma}W_{T\wedge t}$ in Theorem 3.36 which implies that Condition 3 is met. Note that $(LZ^1)^T$ is a stochastic exponential of a Wiener process without drift and hence a P-martingale. By JS, III.3.8 this implies that $(Z^1)^T$ and hence

Z^T is a P^*-martingale. Assumption 5 holds since the filtration is generated by the standard Wiener process W and any local martingale has the representation property relative to W (e.g. by Theorem 2.65). Since $Z_s^1 = Z_0^1 \exp((\mu - r - \frac{\sigma^2}{2})s + \sigma W_s)$, one easily shows that $\sup_{s\in[0,t]} E((Z_t^1)^4) < \infty$ for any $t \in \mathbb{R}_+$. This implies that $E(\int_0^t (b_s)^2\, ds) = (\mu - r)^2 \int_0^t E((Z_t^1)^2)\, ds < \infty$ and similarly $E(\int_0^t (c_s^{11})^2\, ds) < \infty$ for any $t \in \mathbb{R}_+$. Hence integrability condition (RC 1) holds. $\square$

PROOF OF LEMMA 4.6. By Theorem 3.36, we have that $Z_t^2 = E^*(g(Z_T^1)|\mathcal{F}_t) = \int g(\bar{\omega}_{T-t})\, P^{*(Z_{t+s}^1)_{s\geq 0}|\mathcal{F}_t}(d\bar{\omega})$ P-almost surely for any $t \in [0, T]$. By Lemma 2.33, $P^{*(Z_{t+s}^1)_{s\geq 0}|\mathcal{F}_t}(\omega)$ is for P-almost all $\omega \in \Omega$ a solution to the random martingale problem $(\varnothing, \varepsilon_{Z_t^1(\omega)}, \breve{b}, \breve{c}, 0, 0)^M$ in $\mathbb{R}$ with $\breve{b}_s = 0$ for $s \leq T - t$ and $\breve{c}_s(\bar{\omega}) = (\sigma\bar{\omega}_s)^2$. By Corollary 2.41 this martingale problem has a unique solution. If $\overline{W}$ denotes a standard Wiener process, then $Y_s = Z_t^1(\omega) \exp(-\frac{\sigma^2}{2}s + \sigma\overline{W}_s)$ is obviously a solution-process to this martingale problem on $[0, T - t]$. Therefore, we have $Z_t^2 = \int g(Z_t^1 \exp(-\frac{\sigma^2}{2}(T - t) + \sigma u))\, N(0, T - t)(du) = C_{BS}(Z_t^1, \sigma^2(T - t))$ P-almost surely for any $t \in [0, T]$. $\square$

PROOF OF COROLLARY 4.7. The well-known pricing formula is obtained by integration (cf. Lamberton & Lapeyre (1996), Remark 4.3.3). We will now show that integrability condition (RC 1) holds in the enlarged market. Denote by $(\varnothing, \varepsilon_{(1,Z_0^1,Z_0^2)}, \widetilde{b}, \widetilde{c}, 0, 0)^E$ the extended characteristics of $\widetilde{Z} = (Z^0, Z^1, Z^2)$. Since we have already shown that (RC 1) holds for $Z = (Z^0, Z^1)$, it suffices to show that $\int_0^t E(|\widetilde{b}_s^2|^2)\, ds < \infty$, $\int_0^t E(|\widetilde{c}_s^{12}|^2)\, ds < \infty$, and $\int_0^t E(|\widetilde{c}_s^{22}|^2)\, ds < \infty$ for any $t \in \mathbb{R}_+$. It is enough to consider $t = T$, because Z^2 is constant on $[T, \infty)$. The claim follows if we can show that $|\widetilde{b}_t^2| \leq |\widetilde{b}_t^1|$, $|\widetilde{c}_t^{12}| \leq |\widetilde{c}_t^{11}|$, and $|\widetilde{c}_t^{22}| \leq |\widetilde{c}_t^{11}|$ for any $t \in [0, T)$ since we have shown that integrability condition (RC 1) holds for Z and $\varepsilon = 1$. For fixed $m \in \mathbb{N}$, define the stopping time $T_m := \inf\{t \in \mathbb{R}_+ : t = T - \frac{1}{m} \text{ or } Z_t^1 < \frac{1}{m}\}$. Denote by $(\varnothing, \varepsilon_{(1,Z_0^1,Z_0^2)}, \widehat{b}, \widehat{c}, 0, 0)^E$ the extended characteristics of $\widetilde{Z}^{T_m}$, i.e. $\widehat{b} = \widetilde{b}^{T_m}$, $\widehat{c} = \widetilde{c}^{T_m}$. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a C^2-function with $f(t, z) = (1, z, z\Phi(\frac{\log(z/Ke^{-rT})}{\sigma\sqrt{T-t}} + \frac{\sigma}{2}\sqrt{T - t}) - Ke^{-rT}\Phi(\frac{\log(z/Ke^{-rT})}{\sigma\sqrt{T-t}} - \frac{\sigma}{2}\sqrt{T - t}))$ for any $(t, z) \in [0, T - \frac{1}{m}] \times [\frac{1}{m}, \infty)$. Since $\widetilde{Z}_t^{T_m} = f(t, Z_t^1)$ for any $t \in [0, T_m]$, application of Itô's formula (cf. Theorem 2.25) and the fact that f^3 is a solution to the partial differential equation $D_1 f^3(t, z) + \frac{1}{2}(\sigma z)^2 D_{22} f^3(t, z) = 0$ yields

$$\widetilde{b}_t = \begin{pmatrix} 0 \\ \widetilde{b}_t^1 \\ D_2 f^3(t, Z_t^1)\widetilde{b}_t^1 \end{pmatrix}$$

$$\widetilde{c}_t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_t^{11} & D_2 f^3(t, Z_t^1) c_t^{11} \\ 0 & D_2 f^3(t, Z_t^1) c_t^{11} & (D_2 f^3(t, Z_t^1))^2 c_t^{11} \end{pmatrix}$$

for any $t \in [0, T_m]$. One easily verifies that for any $t \in [0, T - \frac{1}{m}]$ the mapping $[\frac{1}{m}, \infty) \to \mathbb{R}$, $z \mapsto D_2 f^3(t, z) = \Phi(\frac{\log(z/Ke^{-rT})}{\sigma\sqrt{T-t}} + \sigma\sqrt{T - t})$ is positive, increasing, and converging to 1 for $z \to \infty$ (cf. Lamberton & Lapeyre (1996), Remark 4.3.6). Hence, we have $|D_2 f^3(t, Z_t^1)| \leq 1$ on $[0,T]$, which implies $|\widetilde{b}_t^2| = |\widehat{b}_t^2| \leq |\widetilde{b}_t^1|$, $|\widetilde{c}_t^{12}| = |\widehat{c}_t^{12}| \leq |\widetilde{c}_t^{11}|$, $|\widetilde{c}_t^{22}| = |\widehat{c}_t^{22}| \leq |\widetilde{c}_t^{11}|$ for any $t \in [0,T]$. By letting $m \to \infty$ it follows that this also holds on $[0,T]$. Hence, we are done. $\square$

PROOF OF LEMMA 4.8. By Lemma 4.1 we have that the optimal strategy is given by $\varphi_t^1 = \frac{\widetilde{c}_t^{12}}{\widetilde{c}_t^{11}} + \frac{1}{\kappa}\frac{\widetilde{b}_t^1}{\widetilde{c}_t^{11}}$ for any $t \in [0,T]$, where $\widetilde{b}$, $\widetilde{c}$ are as in the previous proof. From the proof of Corollary 4.7, it follows that $\frac{\widetilde{c}_t^{12}}{\widetilde{c}_t^{11}} = D_2 f^3(i, Z_t^1) = \Phi(\frac{\log(Z_t^1/Ke^{-rT})}{\sigma\sqrt{T-t}} + \sigma\sqrt{T-t})$ for any $t \in [0, T_m]$ and hence for any $t \in [0,T)$, because m can be chosen arbitrarily small. Since $\frac{\widetilde{b}_t^1}{\widetilde{c}_t^{11}} = \frac{\mu - r}{\sigma^2 Z_t^1}$, the claim follows. $\square$

PROOF OF LEMMA 4.9. This follows immediately from Lemma 4.2. $\square$

4.4 Models with Independent Discrete Returns

In this section we consider a discrete-time version of the model from the previous section. As before, the market consists of two assets 0, 1, namely the numeraire and a stock. We work on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$, where $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is a discrete filtration (cf. Definition A.4). We assume that the discounted price process Z^1 is discrete and, moreover, given by

$$Z_t^1 = Z_{t-1}^1(1+\varepsilon_t) \text{ for any } t \in \mathbb{N}^*,$$

where $Z_0^1 \in \mathbb{R}_+^*$ and $(\varepsilon_t)_{t\in\mathbb{N}^*}$ is a sequence of identically distributed random variables (with distribution Q on $(\mathbb{R}, \mathcal{B})$) such that ε_t is independent of $\mathcal{F}_{t-1}$ for any $t \in \mathbb{N}^*$. Assume that $\int |x|\, Q(dx) < \infty$ and moreover $Q((0,\infty)) > 0$, $Q((-\infty,0)) > 0$. By Lemma 2.20 this implies that $Z = (Z^0, Z^1)$ is an extended Grigelionis process with extended characteristics $(\mathbb{N}^*, \varepsilon_{(1,Z_0^1)}, 0, 0, 0, K)^E$, where $K_t(G) = \int 1_G(0, Z_{t-1}^1 x)\, Q(dx)$ for any $t \in \mathbb{N}^*$, $G \in \mathcal{B}^2$. From Theorem 3.28 one easily concludes that the market allows no arbitrage.

Remarks.

1. If Q is a lognormal distribution with parameters $-\mu + r + \sigma^2/2$, σ^{-1}, -1 (i.e. the law of $\log(1+\varepsilon_t)$ is $N(\mu - r - \sigma^2/2, \sigma^2)$, cf. Johnson & Kotz (1970a), Chapter 14), then the process $(Z_t^1)_{t\in\mathbb{N}}$ has the same distribution as in the model in Section 4.3. Or, to put it another way, the $\mathbb{N}$-discretized market of the Black-Scholes setting is a model of the type above.

2. The conditions on Q are met in particular for α-stable distributions $S_\alpha(\sigma, \beta, \mu)$ with $\alpha \in (1,2]$, $\sigma \in (0,\infty)$, $\beta \in [-1,1]$, $\mu \in \mathbb{R}$ (cf. Samorodnitsky & Taqqu (1994), Property 1.2.16 and p.16).

3. Under the above assumptions, Conditions 1-5 in Theorem 3.36 hold (cf. Corollary 3.37).

4.4.1 Derivative Pricing and Hedging

The following lemma yields the dynamic of (Z^0, Z^1) under the measure P^* in Corollary 3.37 leading to neutral derivative prices.

Lemma 4.10 *Let $T \in \mathbb{N}$. The extended characteristics $(\mathbb{N}^*, \varepsilon_{(1,Z_0^1)}, 0, 0, 0, \widetilde{K})^E$ of $Z = (Z^0, Z^1)$ relative to the measure P^* in Corollary 3.37 are given by*

$$\widetilde{K}_t(G) = \int 1_G(0, Z_{t-1}^1 x) \frac{u'_\kappa(\psi x)}{\int u'_\kappa(\psi\widetilde{x})\, Q(d\widetilde{x})}\, Q(dx)$$

for any $t \in \{1, 2, \ldots, T\}$, $G \in \mathcal{B}^2$, where $\psi \in \mathbb{R}$ solves $0 = \int x u'_\kappa(\psi x)\, Q(dx)$.

Remark. Relative to P^*, the dynamic of (Z^0, Z^1) is basically the same as with respect to P, but with $\widetilde{Q}$ instead of Q, where the probability measure $\widetilde{Q}$ is given by its Radon-Nikodým density

$$\frac{d\widetilde{Q}}{dQ}(x) := \frac{u'_\kappa(\psi x)}{\int u'_\kappa(\psi\widetilde{x})\, Q(d\widetilde{x})} \text{ for any } x \in \mathbb{R}.$$

In other words, $(\varepsilon_t)_{t\in\mathbb{N}^*}$ are independent $\widetilde{Q}$-distributed random variables under P^*.

The following corollary shows how to compute derivative prices explicitly.

Corollary 4.11 *Let $T \in \mathbb{N}$ and $X^2 = g(Z_T^1)$, where $g : \mathbb{R} \to \mathbb{R}$ is a measurable mapping. Assume that there are $M_1, M_2 \in \mathbb{R}$ with $|g(x)| \leq M_1 + M_2|x|$ for any $x \in \mathbb{R}$. Define a mapping*

$$\pi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}, \quad (y, t) \mapsto \int g\Big(y \prod_{i=1}^{T-[t]} (1 + x_i)\Big) \Big(\bigotimes_{i=1}^{T-[t]} \widetilde{Q}\Big)(d(x_1, \ldots, x_{T-[t]}))$$

(i.e. $\pi(y,t) = E(g(Y))$, where $Y = y \prod_{i=1}^{[T-t]}(1 + \widetilde{\varepsilon}_i)$ for independent $\widetilde{Q}$-distributed random variables $\widetilde{\varepsilon}_1, \widetilde{\varepsilon}_2, \ldots$), where $\widetilde{Q}$ is defined as in the preceding remark. Then the unique neutral price process Z^2 for the derivative with terminal value X^2 at T occurs in the form $Z_t^2 = \pi(t, Z_t^1)$ for any $t \in [0, T]$. Moreover, the extended characteristics $(\mathbb{N}, \varepsilon_{(1,Z_0^1,\pi(0,Z_0^1))}, 0, 0, 0, \overline{K})^E$ of $\overline{Z} = (Z^0, Z^1, Z^2)$ are given by

$$\overline{K}_t(G) = \int 1_G\Big(0, xZ_{t-1}^1, \pi(t, Z_{t-1}^1(1 + x)) - \pi(t - 1, Z_{t-1}^1)\Big)\, Q(dx)$$

for any $t \in \{1, 2, \ldots, T\}$, $G \in \mathcal{B}^3$.

The preceding corollary allows relatively straightforward numerical computation of derivative prices and hedging strategies. Observe that

$$\pi(y, t) = \int g(y \exp(x)) \Big(\ast_{i=1}^{T-[t]} R\Big)(dx),$$

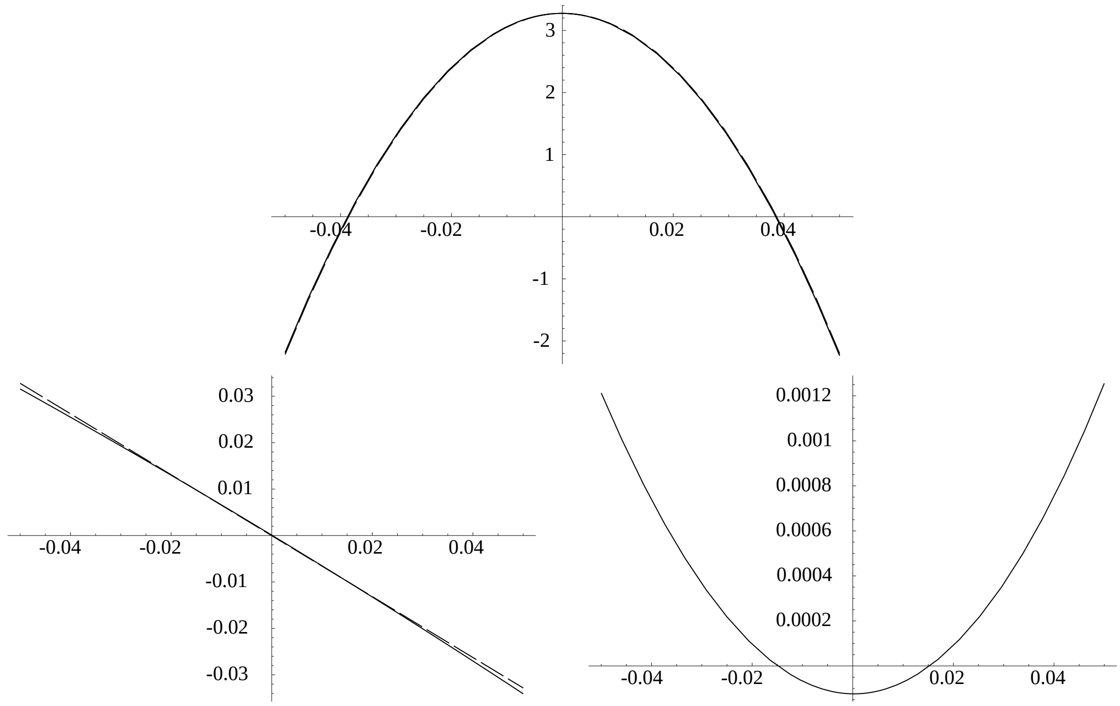

Figure 4.19: Logarithmic 1-day log-return densities

where the probability measure R on $(\mathbb{R}, \mathcal{B})$ is given by $R(G) = \int 1_G(\log(1+x))\, \widetilde{Q}(dx)$ for any $G \in \mathcal{B}$ and the asterisk denotes convolution. Hence in order to obtain prices, one may simply calculate the convolutions of R (by means of characteristic functions) and perform a numerical integration. Since the joint extended characteristics of $\overline{Z} = (Z^0, Z^1, Z^2)$ are known in terms of π and Q, one can now evaluate optimal hedging strategies numerically by means of Theorem 3.22 and Corollary 3.23.

4.4.2 Lognormal Returns

In this subsection we consider the $\mathbb{N}$-discretized version of the Black-Scholes setting as discussed in Remark 1. More specifically, we choose $S_t^0 = e^{r[t]}$, $S_0^1 = 100$, $r = \log(1.05)/250$, $\mu = \log(1.09)/250$, $\sigma = 0.2387/\sqrt{250}$ and Q as the lognormal distribution with parameters $-\mu + r + \sigma^2/2$, σ^{-1}, -1. As before, time is measured in trading days ($= 1/250$ year). Note that the distribution of the *log-return* $X_t = \log(Z_t^1) - \log(Z_{t-1}^1)$ (namely $N(\mu - r - \sigma^2/2, \sigma^2)$) is the same in this model and in the Black-Scholes setting. Moreover, it does not depend on t. One easily verifies that the laws of X_t under the pricing measures, either from the Black-Scholes model or from this section, are also independent of t. By $P, \widetilde{P}, P^*$, we denote the given probability measure, the pricing measure in the Black-Scholes model, and the pricing measure in this section, respectively. Accordingly, the densities of the law of the log-return X_t relative to $P, \widetilde{P}, P^*$ are called $f, \widetilde{f}, f^*$. Hence, $f, \widetilde{f}$ are the densities of a $N(\mu - r - \sigma^2/2, \sigma^2)$- and $N(-\sigma^2/2, \sigma^2)$-distribution, respectively. f^* does not correspond to a normal distribution. The logarithms of these densities (being parabolas for $f, \widetilde{f}$) are

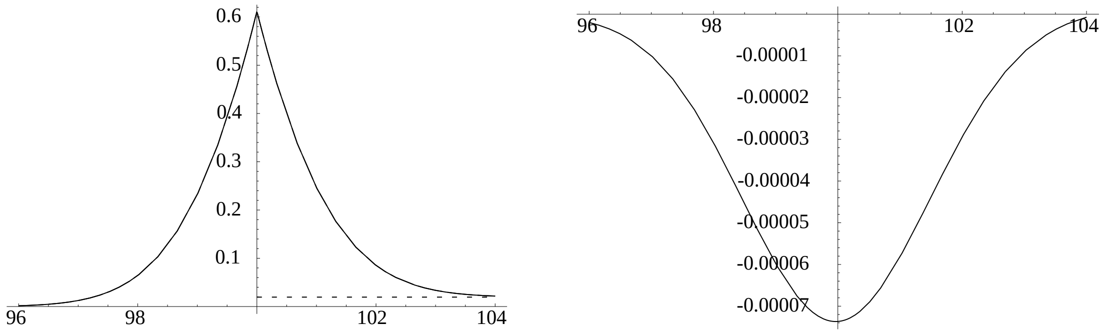

Figure 4.20: Time value and difference to Black-Scholes prices 1 day to maturity

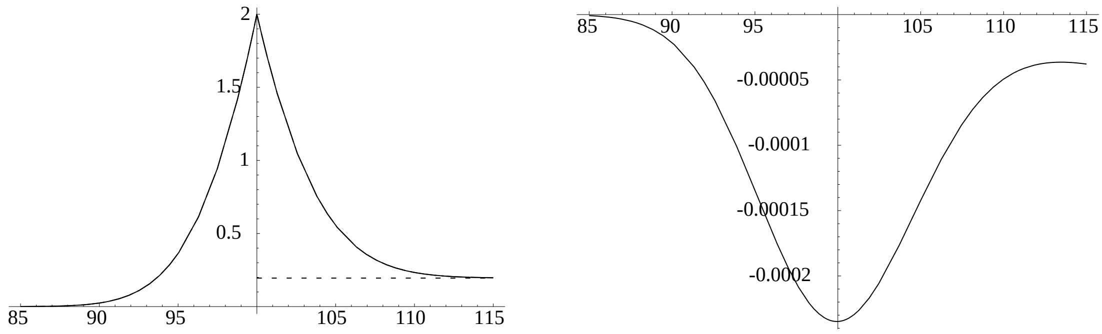

Figure 4.21: Time value and difference to Black-Scholes prices 10 days to maturity

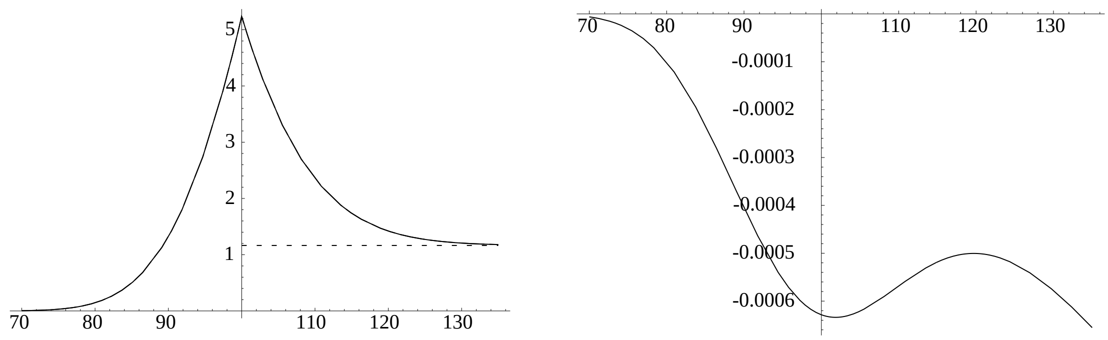

Figure 4.22: Time value and difference to Black-Scholes prices 60 days to maturity

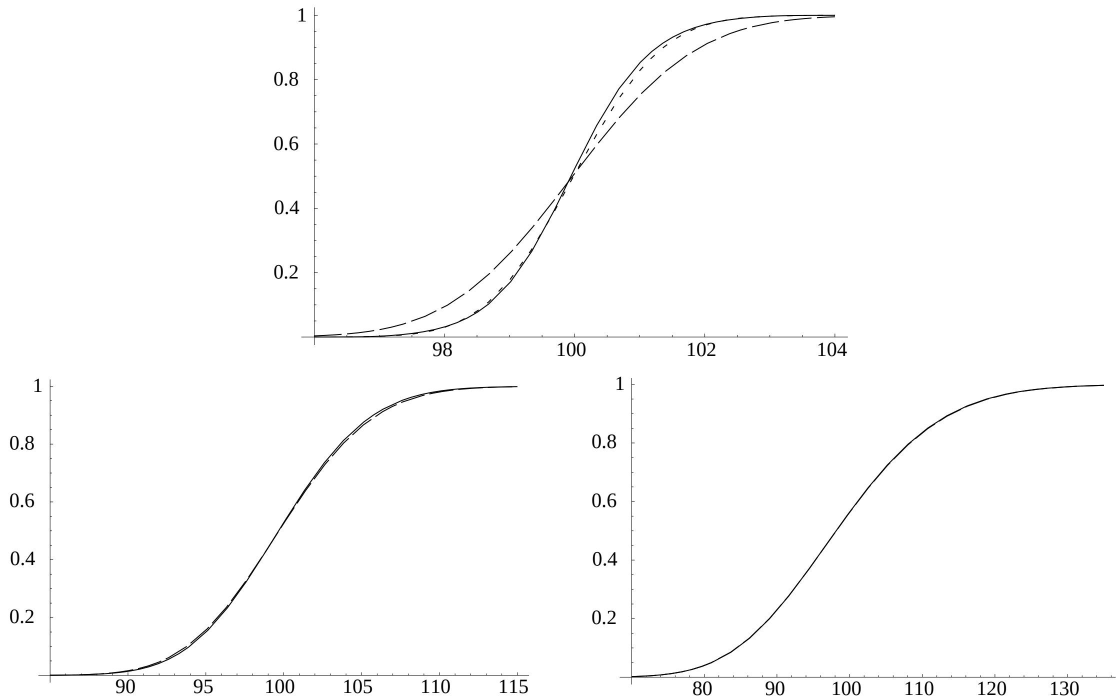

Figure 4.23: Hedging strategies 1, 10, 60 trading days to maturity

plotted in the upper diagram in Figure 4.19. The dashed line shows $\log(f)$ and is hardly visible since it is very close to a solid line representing $\log(\widetilde{f})$ and $\log(f^*)$, which tally even more. For better visibility we add two further graphs, indicating the differences between these functions. The dashed line on the left represents $\log(\widetilde{f}) - \log(f)$, whereas the solid line marks $\log(f^*) - \log(f)$. On the right-hand side in Figure 4.19, we plot $\log(\widetilde{f}) - \log(f^*)$. Observe that the distribution of the return under the continuous-time and the discrete-time pricing measure is very similar but not identical. Moreover, note that the discrete model is not complete and hence does not allow derivative pricing solely based on the absence of arbitrage.

One may wonder how strongly the discretization of the Black-Scholes model affects option prices. Consider a European call option with strike price $K = 100$ expiring in 1, 10, 60 trading days, respectively. We define the *time value* of the option as $S_t^2 - ((S_t^1 - K) \vee 0)$, where $S_t^2 = Z_t^2 S_t^0$ is the current price of the option in undiscounted terms and $((S_t^1 - K) \vee 0)$ its payoff if it were to expire immediately. Note that the time value of a European call option is non-negative since the even larger number $(S_t^1 - Ke^{-r(T-t)}) \vee 0$ is a lower arbitrage bound, as one may easily verify. The diagram on the left in Figure 4.20 shows the time value of our European call one day before expiration as a function of the current stock price S_0^1. The dotted horizontal line represents the lower arbitrage bound. In fact, the solid line in the left diagram consists of two curves, firstly the time value in the discrete-time setting and secondly in the Black-Scholes model from the previous section. The tiny difference between the two curves is plotted on the right, i.e. the Black-Scholes value is slightly greater than the price in the discrete model. In Figures 4.21 and 4.22 we repeat the calculations for an option ten and sixty trading days before expiration. Although we expect the difference between the prices to be small, we are surprised to note – especially in the case of the short-term option – that the relative deviation turns out to be so tiny.

Having seen that the effect of discretization to European call option prices is negligible, let us now turn to hedging strategies. Assume that you have sold one option and you want to hedge your risk by trading in the stock according to a u_κ-optimal strategy. We choose the relatively large value $\kappa = 100$ for the risk aversion. In Figure 4.23 we plot the number φ_t^1 of shares of stock in the u_{100}-optimal portfolio in terms of the current stock price S_0^1. The upper diagram corresponds to the option one trading day before expiration. The solid line shows the optimal portfolio in the discrete model, whereas the dashed line indicates the hedging strategy in the Black-Scholes setting. Observe that the strategies differ significantly. For larger time horizons the difference gets rapidly smaller as the second and third graph indicate. In the diagram on the left (10 trading days before expiration), one can still observe a small difference which seems to have vanished in the right picture (corresponding to 60 trading days before expiration).

From Figure 4.23 we may draw the following lesson. If you are functioning in a Black-Scholes market (or in its discretized form – a negligible difference as far as option prices are concerned), and can rebalance your portfolio only once a day, then the continuous-time hedging portfolio seems to be reasonable if the option is still valid for the near future. Just before expiration, however, one may do better. The dotted line in the upper diagram of Figure 4.23 shows the hedging strategy in the continuous-time Black-Scholes model 1/2 day before expiration. It coincides quite well with the optimal portfolio in our discrete-time model, where the last rebalance takes place one day before maturity.

It occasionally makes sense to mix continuous- and discrete-time models. If one believes that the market reacts very rapidly, one should use a continuous-time framework for the computation of derivative prices. On the other hand, if you can only afford to trade on a relatively coarse time grid, you are practically investing in a discrete-time market. Therefore, you may convert the model into a discretized market in the sense of Definition 3.53 before you actually compute optimal hedging strategies. The above example shows that even when you work with the option prices from the continuous-time model, the discretization has an effect on the optimal portfolio. However, for fine-meshed time grids the difference converges to 0 as shown in Theorem 3.55.

4.4.3 Stable Returns

As noted in Remark 2 above, α-stable distributions with stability index $\alpha > 1$ are a possible distribution Q for the return. Stable distributions for stock returns were proposed as early as 1964 (Mandelbrot (1963), Fama (1964)) for dealing with the observed heavy tails of market data. For our numerical computations we consider the example $Q = S_\alpha(\sigma, \beta, \mu)$ with $\alpha = 1.9$, $\sigma = 0.2387/(\sqrt{250}\sqrt{2})$, $\beta = 0$, $\mu = \log(1.09)/250$. As in the previous subsection the bank account $S_t^0 = e^{r[t]}$ with $r = \log(1.05)/250$ serves as a numeraire, where the time is measured in trading days. Moreover, we let $S_0^1 = 100$. From a theoretical point of view,

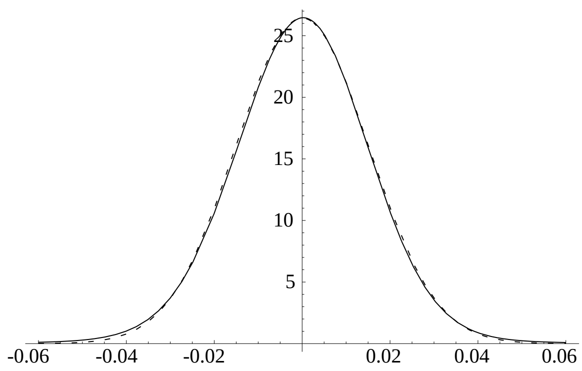

Figure 4.24: 1-day return densities

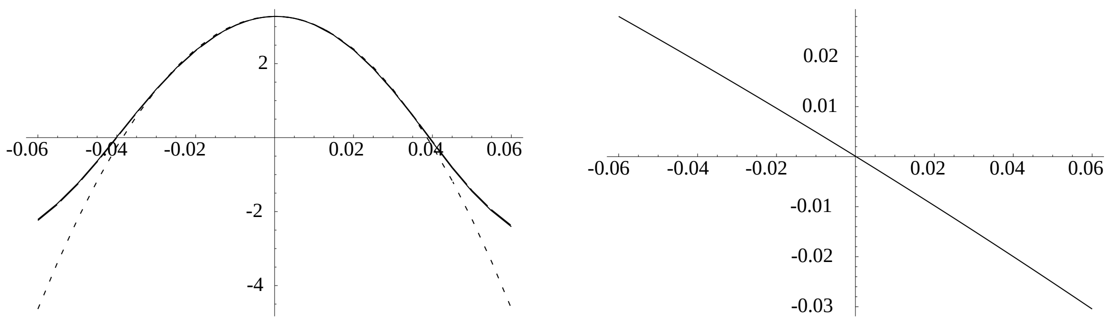

Figure 4.25: Logarithmic 1-day return densities

one may want to replace $S_\alpha(\sigma,\beta,\mu)$ with $S_\alpha(\sigma,\beta,\mu)|_{\mathbb{R}_+} + \varepsilon_0 S_\alpha(\sigma,\beta,\mu)((-\infty,0))$, since otherwise the stock price might jump to negative values. For the concrete numerical results in this subsection, however, this does not make any difference. In the following paragraphs we will compare the implications of this model to those of the Black-Scholes model with parameters r, μ, $\sigma\sqrt{2}$. Note that, similarly to the previous subsection, the distribution of the *daily return* $\varepsilon_t = (Z_t^1 - Z_{t-1}^1)/Z_{t-1}^1$ does not depend on t, neither under the given probability measure P in this or in the Black-Scholes model, nor under the pricing measure P^*. We denote by $f, \widetilde{f}, f^*$ the densities of the law of ε_t, relative firstly to the given probability measure in this stable increment model, secondly to the given probability measure in the Black-Scholes setting and thirdly the pricing measure P^* for the stable increment model. Hence, f is the density of Q, $\widetilde{f}$ of a lognormal distribution with parameters $-\mu+r+\sigma^2/2, \sigma^2$, σ^{-1}, -1 and f^* of Q^*. Figure 4.24 shows that these densities behave very similarly. The solid line marks both f and f^*, whereas the dotted line represents the lognormal density $\widetilde{f}$. Of course, the difference between the distributions should become visible in the tails. This is indeed so, as the left-hand diagram in Figure 4.25 illustrates. The solid line designating $\log(f^*)$ and the dashed line for $\log(f)$ are hardly distinguishable, whereas the dotted line for $\log(\widetilde{f})$ indicates that this distribution has thinner tails. The graph on the right in Figure 4.25 shows the difference $\log(f^*) - \log(f)$.

The fat tails of the stable distribution lead to increased option prices as we will now see. As in the previous subsection, we consider here a European call option with strike price

$K = 100$, expiring in 1, 10, 60 trading days, respectively. Figures 4.26 – 4.28 correspond to the Figures 4.20 – 4.22 in the lognormal case. The solid line in the diagrams on the left represents the time value of the European call for the stable model, the dashed line for the Black-Scholes case and the dotted horizontal line marks the lower arbitrage bound. On the right, we plot the difference between the neutral price in the stable and the Black-Scholes model.

Figure 4.29 illustrates the disagreement between the Black-Scholes model and the discrete stable case from another perspective. By inversion of the Black-Scholes formula, any European call price from the stable model can be converted into a theoretical implied Black-Scholes volatility. The graphs in Figure 4.29 show the implied annual volatility of our European call as a function of the current stock price, the upper diagram one day before expiration, and the lower left and right graph for a remaining life time of 10 and 60 trading days, respectively. The height of the abscissa indicates the annual volatility of the Black-Scholes model we used for comparisons (i.e. $\sqrt{250\sigma^2}$). What do the curves in Figure 4.29 mean? They indicate the kind of implied volatility smiles a Black-Scholes economist would observe if the real market followed a discrete stable return process with neutral option prices. Note that these smiles do not imply that either of the models is better from a statistical point of view. Judgements of that kind can only be based on the analysis of real market data.

Let us once more examine the u_{100}-optimal portfolio for the hedging problem $\varphi^2 = -1$ (i.e. one option has been sold short). Exactly as in Figure 4.23, the diagrams in Figure 4.30 show the number of shares of stock in the optimal hedging portfolio in terms of the current stock price S_0^1. The solid line marks the strategy in the stable case, the dashed line the Black-Scholes hedge and the dotted line in the upper diagram the Black-Scholes hedge 1/2 day before expiration. All in all, one may say that the optimal portfolios are quite similar to those in the case of lognormal returns (cf. Figure 4.23). This is surprising and reassuring at the same time. It indicates that the hedging strategies seem to be quite robust against variation of the underlying probabilistic model, even if the option *prices* are strongly affected. But one should be aware that this does not imply that the optimal portfolios perform equally well in the different models. In the continuous-time Black-Scholes setting the hedge is perfect, whereas in the discrete lognormal or even stable world there exists a significant chance of losing money.

Proofs

PROOF OF LEMMA 4.10. By Corollary 3.23, the u_κ-optimal strategy $\varphi = (\varphi^0, \varphi^1)$ for $\mathfrak{A}$ is any solution to $0 = \int x u'_\kappa(\varphi_t \cdot x)\, K_t(dx) = Z^1_{t-1} \int x u'_\kappa(Z^1_{t-1}\varphi^1_t x)\, Q(dx)$, i.e. of the form $(\varphi^0, \psi/Z^1_{t-1})$, where φ^0 is arbitrary and ψ is chosen as in Lemma 4.10. The form of the characteristics now follows easily from Corollary 3.38. □

PROOF OF THE REMARK. For any $t \in \mathbb{N}^*$, $G \in \mathcal{B}$, we have $P^{*\varepsilon_t|\mathcal{F}_{t-1}}(G) = \int 1_G(x/Z^1_{t-1})$ $P^{*\Delta Z^1_t|\mathcal{F}_{t-1}}(dx) = \int 1_G(x^2/Z^1_{t-1})\, \widetilde{K}_t(d(x^1, x^2)) = \widetilde{Q}(G)$. This shows that the ε_t are inde-

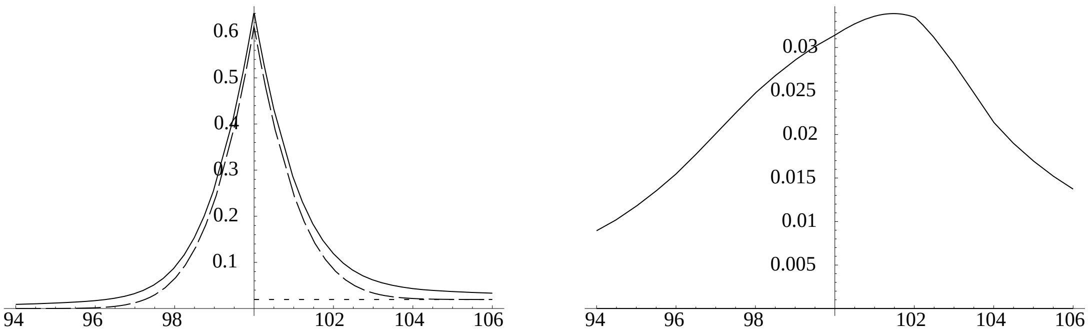

Figure 4.26: Time value and difference to Black-Scholes prices 1 day to maturity

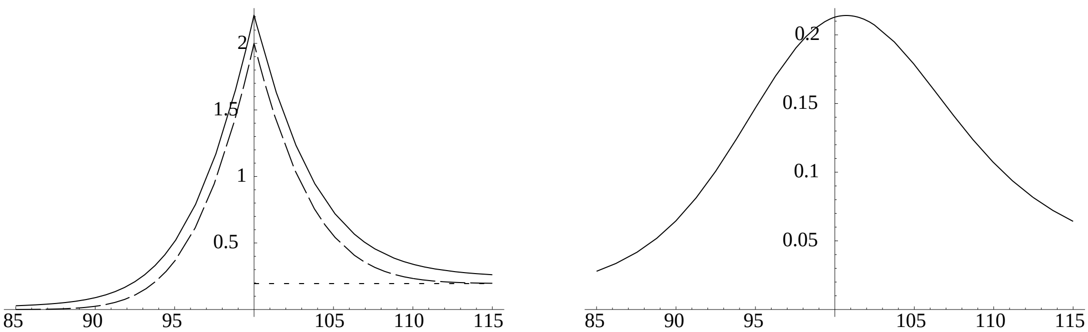

Figure 4.27: Time value and difference to Black-Scholes prices 10 days to maturity

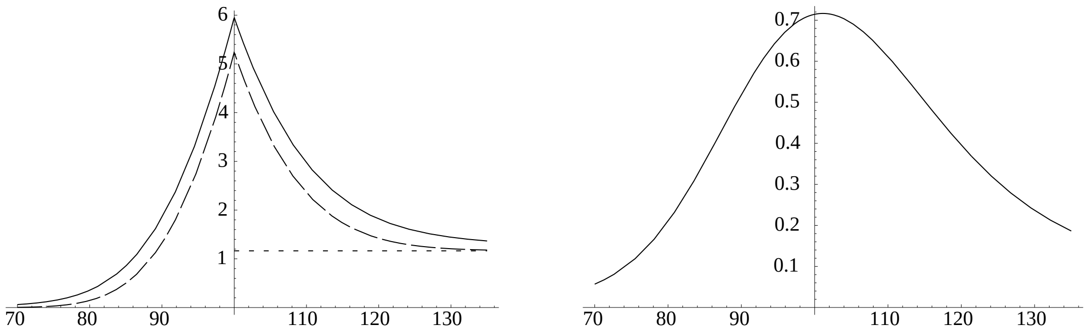

Figure 4.28: Time value and difference to Black-Scholes prices 60 days to maturity

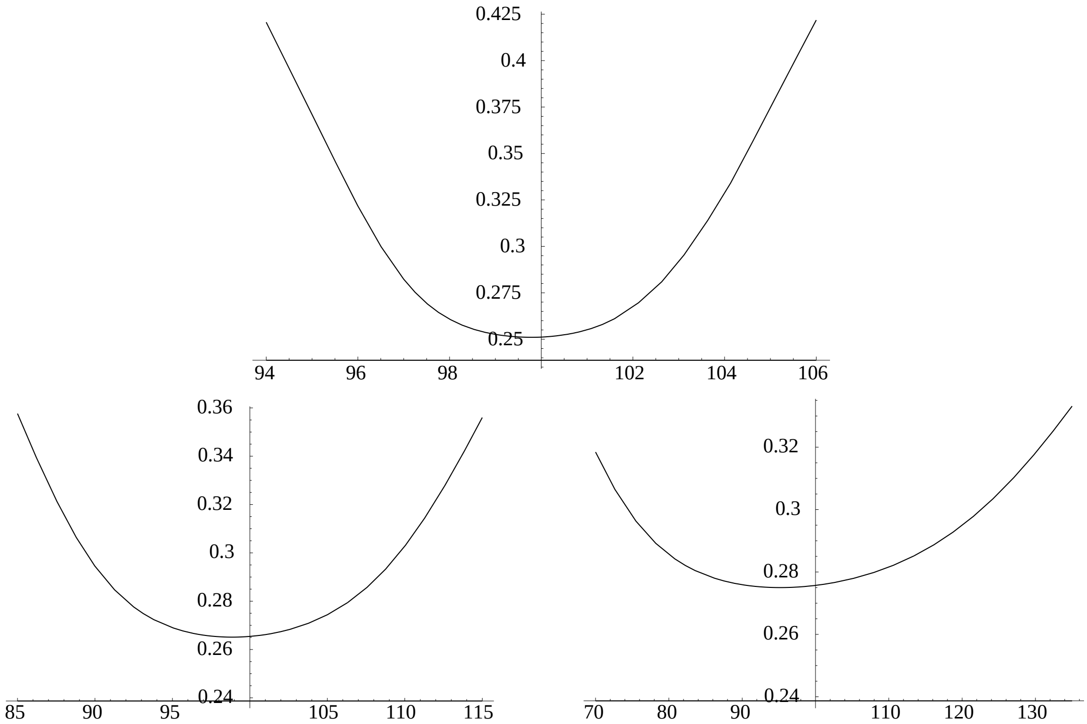

Figure 4.29: Implied Black-Scholes volatilities 1, 10, 60 trading days to maturity

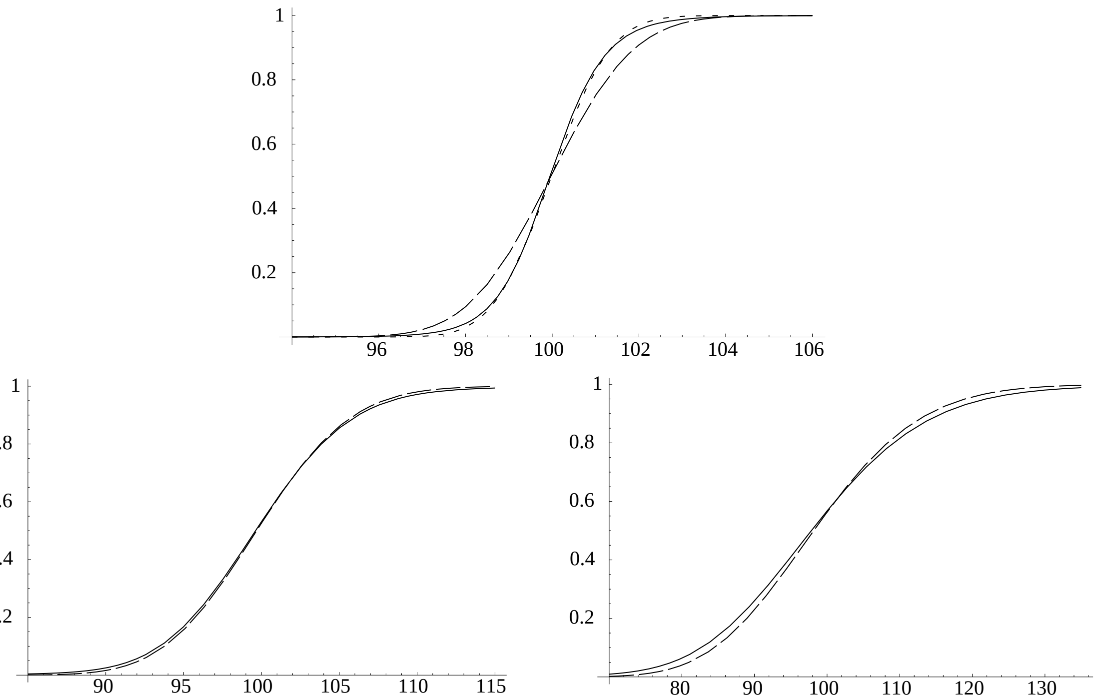

Figure 4.30: Hedging strategies 1, 10, 60 trading days to maturity

pendent and have distribution $\widetilde{Q}$ relative to P^*. □

PROOF OF COROLLARY 4.11. Since $(\pi(t, Z_t^1))_{t\in[0,T]}$ and the neutral price process from Corollary 3.37 are discrete, it suffices to show the claim for $t \in \{0, 1, \ldots, T\}$. We proceed by backward induction. For $t = T$, we have $Z_t^2 = X^2 = g(Z_t^1) = \pi(t, Z_t^1)$. Now, assume that equality holds for $t \in \{1, 2, \ldots, T\}$. By Corollary 3.37, the assumption, and the preceding proof, we have

$$
\begin{aligned}
Z_{t-1}^2 &= E^*(Z_t^2|\mathcal{F}_{t-1}) \\
&= \int \pi(t, Z_{t-1}^1(1+x))\, P^{*\varepsilon_t|\mathcal{F}_{t-1}}(dx) \\
&= \iint g\Big(Z_{t-1}^1(1+x)\prod_{i=1}^{T-[t]}(1+x_i)\Big)\Big(\bigotimes_{i=1}^{T-[t]}\widetilde{Q}\Big)(d(x_1,\ldots,x_{T-[t]}))\,\widetilde{Q}(dx) \\
&= \pi(t, Z_{t-1}^1).
\end{aligned}
$$

The shape of the characteristics follows from

$$
\begin{aligned}
\overline{K}_t(G) &= E(1_G(Z_t^0 - Z_{t-1}^0, Z_t^1 - Z_{t-1}^1, \pi(t, Z_t^1) - \pi(t-1, Z_{t-1}^1))|\mathcal{F}_{t-1}) \\
&= \int 1_G\Big(0, x, \pi(t, Z_{t-1}^1(1+x)) - \pi(t-1, Z_{t-1}^1)\Big)\, P^{\varepsilon_t|\mathcal{F}_{t-1}}(dx).
\end{aligned}
$$

□

4.5 ARCH-type Models

ARCH-models have become popular for modelling financial time series because they explain leptokurtosis and persistency of volatility clustering (cf. Bollerslev et al. (1992)). We focus here on a GARCH(1,1)-M stock price model for which option prices and hedging strategies have been derived by Duan (1995) and Kallsen & Taqqu (1998). We compare our formulas qualitatively to theirs. Our setting is similar to that of the previous section. We work on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$, where $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is a discrete filtration. The market consists of two assets 0, 1, namely the bank account $S_t^0 := e^{r[t]}$ and the stock price process S^1 which satisfies the recursive equation

$$
\log\left(\frac{S_t^1}{S_{t-1}^1}\right) = \mu(\sigma_t) - \frac{\sigma_t^2}{2} + \sigma_t\varepsilon_t \tag{4.2}
$$

for any $t \in \mathbb{N}^*$, where $r \in \mathbb{R}_+$, $S_0^1 \in \mathbb{R}_+^*$, $\mu : \mathbb{R}_+ \to \mathbb{R}$ are given and $(\varepsilon_t)_{t\in\mathbb{N}^*}$ is a sequence of identically distributed random variables such that ε_t is independent of $\mathcal{F}_{t-1}$ for any $t \in \mathbb{N}^*$. The $\mathbb{R}_+^*$-valued stochastic process $(\sigma_t)_{t\in\mathbb{N}}$ is given by the GARCH(1,1)-M equation

$$
\sigma_t^2 = \omega + \alpha(\sigma_{t-1}\varepsilon_{t-1})^2 + \beta\sigma_{t-1}^2
$$

for any $t \in \mathbb{N} \setminus \{0, 1\}$, with $\sigma_0, \sigma_1, \omega \in \mathbb{R}_+^*$, $\alpha, \beta \in \mathbb{R}_+$ being fixed constants. The distribution Q of ε_t is chosen as $N(0, 1)$ or, more generally, any distribution satisfying

$\int e^{\sigma|x|}\, Q(dx) < \infty$ for any $\sigma \in \mathbb{R}_+$. From Lemma 2.20 one easily concludes that $Z = (Z^0, Z^1)$ is an extended Grigelionis process with extended characteristics $(\mathbb{N}^*, \varepsilon_{(0,S_0^1)}, 0, 0, 0, K)^E$, where

$$K_t(G) = \int 1_G\Big(0, Z_{t-1}^1\Big(\exp\Big(\mu(\sigma_t) - \frac{\sigma_t^2}{2} - r + \sigma_t x\Big) - 1\Big)\Big)\, Q(dx)$$

for any $t \in \mathbb{N}^*$, $G \in \mathcal{B}^2$. If $Q = N(0,1)$, then it follows easily from Theorem 3.28 that the market allows no arbitrage.

Remarks.

1. Equation (4.2) can be rewritten as

$$Z_t^1 = Z_{t-1}^1 \cdot \exp\Big(\mu(\sigma_t) - \frac{\sigma_t^2}{2} - r + \sigma_t\varepsilon_t\Big) \text{ for any } t \in \mathbb{N}^*.$$

   The very similar model

$$Z_t^1 = Z_{t-1}^1\Big(1 + \mu(\sigma_t) - \frac{\sigma_t^2}{2} - r + \sigma_t\varepsilon_t\Big) \text{ for any } t \in \mathbb{N}^*.$$

   can be treated analogously.

2. If we let $Q := N(0,1)$ and $\mu(\sigma_t) := r + \lambda\sigma_t$ for some $\lambda \in \mathbb{R}_+$, then the above model is the same as in Duan (1995). Moreover, it coincides for integer times with the continuous-time ARCH-model in Kallsen & Taqqu (1998), Section 3. More precisely, it is the $\mathbb{N}$-discretized market (in the sense of Definition 3.53) of the model in Kallsen & Taqqu.

4.5.1 Derivative Pricing

The following lemma yields the dynamic of (Z^0, Z^1) under the equivalent measure P^* in Corollary 3.37, leading to neutral derivative prices.

Lemma 4.12 *Let $T \in \mathbb{N}$. The extended characteristics $(\mathbb{N}^*, \varepsilon_{(0,S_0^1)}, 0, 0, 0, \overline{K})^E$ of $Z = (Z^0, Z^1)$ relative to the measure P^* in Corollary 3.37 is given by*

$$\begin{aligned}\widetilde{K}_t(G) &= \int 1_G\Big(0, Z_{t-1}^1 \exp(\widetilde{\mu}(\sigma_t) + \sigma_t x) - 1\Big) \\ &\quad \cdot \frac{u_\kappa'(\psi(\sigma_t)(\exp(\widetilde{\mu}(\sigma_t) + \sigma_t x) - 1))}{\int u_\kappa'(\psi(\sigma_t)(\exp(\widetilde{\mu}(\sigma_t) + \sigma_t \widetilde{x}) - 1))\, Q(d\widetilde{x})}\, Q(dx)\end{aligned}$$

for any $t \in \{1, 2, \dots, T\}$, $G \in \mathcal{B}^2$, where $\widetilde{\mu}(\sigma_t) = \mu(\sigma_t) - \frac{\sigma_t^2}{2} - r$,

$$\sigma_t^2 = \omega + \alpha\Big(\log\Big(\frac{Z_{t-1}^1}{Z_{t-2}^1}\Big) - \widetilde{\mu}(\sigma_{t-1})\Big)^2 + \beta\sigma_{t-1}^2$$

for any $t \in \mathbb{N} \setminus \{0, 1\}$, and for any $\sigma \in \mathbb{R}$ the real number $\psi(\sigma)$ solves

$$0 = \int \Big(\exp(\widetilde{\mu}(\sigma) + \sigma x) - 1\Big) u_\kappa'\Big(\psi(\sigma)(\exp(\widetilde{\mu}(\sigma) + \sigma x) - 1)\Big)\, Q(dx).$$

Remarks.

1. Relative to P^*, the $(\varepsilon_t)_{t\in\mathbb{N}^*}$ are no longer i.i.d. random variables. Instead, we have
$$P^*(\varepsilon_t \in G|\mathcal{F}_{t-1}) = \int 1_G(x) \frac{u'_\kappa(\psi(\sigma_t)(\exp(\widetilde{\mu}(\sigma_t)+\sigma_t x)-1))}{\int u'_\kappa(\psi(\sigma_t)(\exp(\widetilde{\mu}(\sigma_t)+\sigma_t\widetilde{x})-1))\,Q(d\widetilde{x})}\,Q(dx) \tag{4.3}$$
P-almost surely for any $t\in\{1,2,\dots,T\}$ and any $G\in\mathcal{B}$.

2. The dynamic of $(Z_t^1)_{t\in\mathbb{N}}$ under P^* in Lemma 4.12 is not the same as with respect to the pricing measures of Duan and Kallsen & Taqqu, who obtain lognormal returns under the EMM as well (cf. Equations (3.10) – (3.12) in Kallsen & Taqqu (1998)). Kallsen & Taqqu consider a continuous-paths interpolation of the discrete GARCH(1,1)-M model. Hence, their setting fits into Section 4.2. Indeed, their density of the pricing measure is the same as in Lemma 4.3 (cf. Lemma 2.1 in Kallsen & Taqqu (1998)). This is not surprising since their model is complete and hence allows only one equivalent martingale measure. Similarly, one easily shows that the hedging strategy in Kallsen & Taqqu (1998), Theorem 3.6, is the limit of the portfolio in Lemma 4.1 for infinite risk aversion κ. In this section, however, we are dealing with a discrete stock price process. The relationship between both settings is essentially the same as between the Black-Scholes model and its discretized counterpart in Subsection 4.4.2. In fact, this is the special case if the GARCH parameters α,β are 0. Therefore, we conjecture that the option prices and long-term trading strategies behave numerically very similarly for the discrete GARCH-model and its continuous embedding.

   Finally, let us remark that Duan's derivative prices coincide with those in Kallsen & Taqqu, although he works in the same discrete-time framework as we do in this section.

Proofs

PROOF OF LEMMA 4.12. Note that Condition 1 in Corollary 3.37 (namely the absence of arbitrage) depends on the choice of the distribution Q and still has to be checked for any particular model.

By Corollary 3.23 a strategy $\varphi = (\varphi^0,\varphi^1)$ is u_κ-optimal for $\mathfrak{A}$ if and only if $0 = \int x u'_\kappa(\varphi\cdot x)\,K_t(dx) = Z_{t-1}^1\int(\exp(\widetilde{\mu}(\sigma_t)+\sigma_t x)-1)u'_\kappa(\varphi_t^1 Z_{t-1}^1(\exp(\widetilde{\mu}(\sigma_t)+\sigma_t x)-1))\,Q(dx)$ for any $t\in\mathbb{R}_+$. Hence, the optimal strategies are of the form $(\varphi_t^0,\psi(\sigma_t)/Z_{t-1}^1)$, where $\psi(\sigma_t)$ is as in Lemma 4.12. The form of the extended characteristics of (Z^0,Z^1) relative to P^* now follows easily from Corollary 3.38. □

PROOF OF REMARK 1. Observe that $\varepsilon_t = (\log(1+\Delta Z_t^1/Z_{t-1}^1)-\widetilde{\mu}(\sigma_t))/\sigma_t$ and $P^{*\Delta Z_t|\mathcal{F}_{t-1}} = \widetilde{K}_t$. Equation (4.3) now follows from a straightforward calculation using Equation (4.3). □

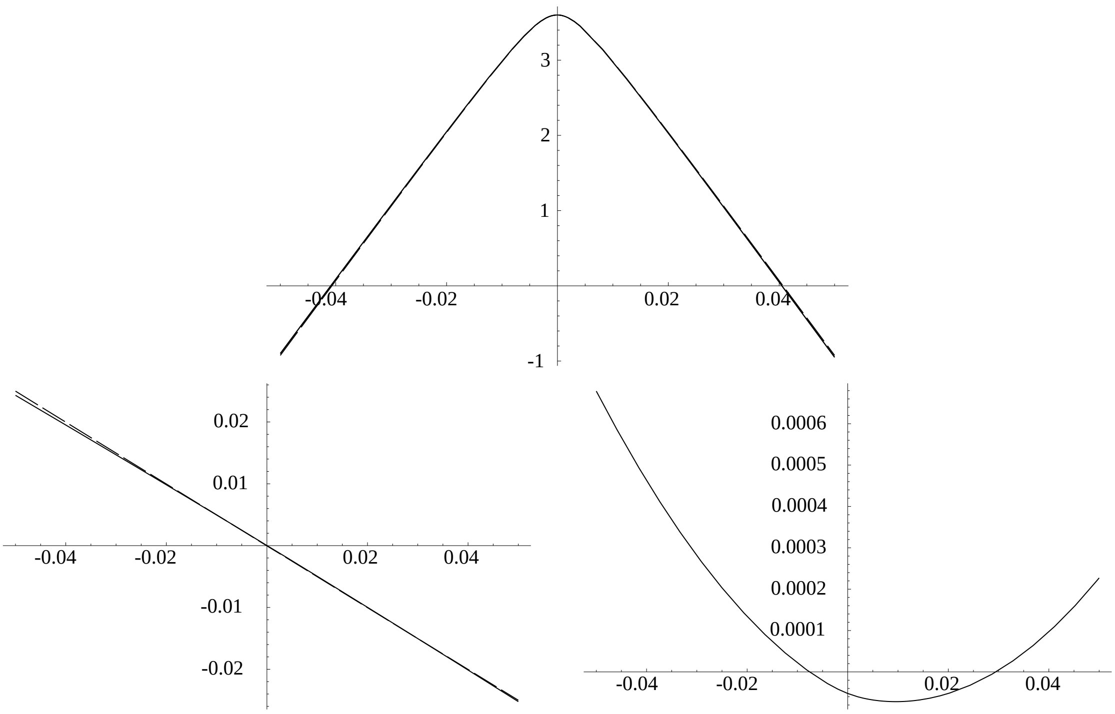

Figure 4.31: Logarithmic 1-day log-return densities

4.6 Exponential Lévy Processes

In this section we want to consider a class of models that generalizes the Black-Scholes setting by replacing the Wiener process in Equation (4.1) with a quite arbitrary Lévy process. Similarly to Section 4.3 we consider a market with a bank account $S_t^0 = e^{rt}$ for any $t \in \mathbb{R}_+$ and a stock whose discounted price process Z^1 satisfies the stochastic differential equation

$$dZ_t^1 = (\mu - r)Z_{t-}^1\,dt + \sigma Z_{t-}^1\,dW_t + \int xZ_{t-}^1\,(p-q)(dt,dx), \tag{4.4}$$

where $Z_0^1 \in \mathbb{R}_+^*$, $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$, W denotes a standard Wiener process and p is a homogeneous Poisson random measure with compensator $q = \lambda \otimes H$ (cf. JS, II.1.20). We assume that H is a fixed measure on $(\mathbb{R}, \mathcal{B})$ with $\int (|x|^2 \wedge |x|)\,H(dx) < \infty$, $\int |\log(x + 1)| 1_{(-1,-\frac{1}{2})}(x)\,H(dx) < \infty$, and $H((-\infty,-1]) = 0$. The latter condition ensures that Z^1 does not jump to negative values (cf. Jacod (1979), (6.5)b). Defining

$$X_t := (\mu - r)t + \sigma W_t + \int x\,(p-q)(dt,dx)$$

for any $t \in \mathbb{R}_+$, Equation (4.4) can be rewritten as

$$dZ_t^1 = Z_{t-}^1\,dX_t,$$

which implies that Z^1 is given by the stochastic exponential of X, i.e. $Z^1 = Z_0^1\,\mathscr{E}(X)$ (cf. JS, I.4.61). Note that X is an integrable Lévy process with the characteristic triplet $(\mu -$

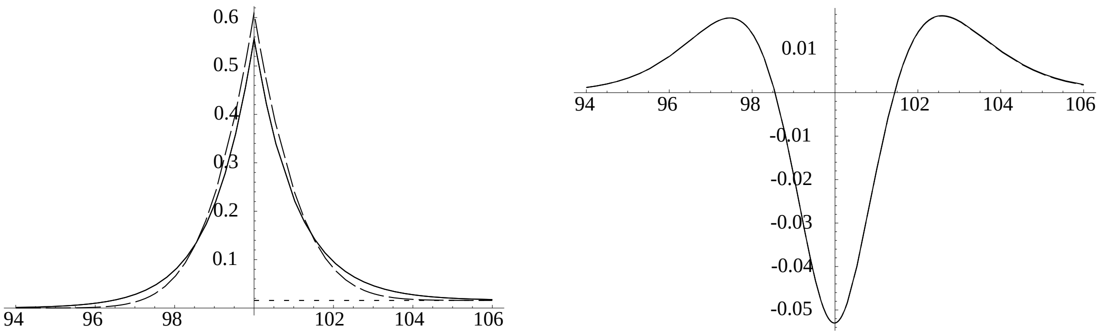

Figure 4.32: Time value and difference to Black-Scholes prices 1 day to maturity

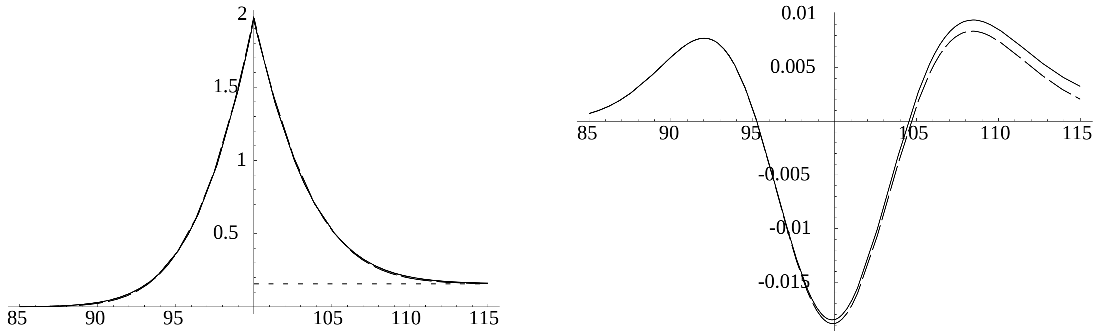

Figure 4.33: Time value and difference to Black-Scholes prices 10 days to maturity

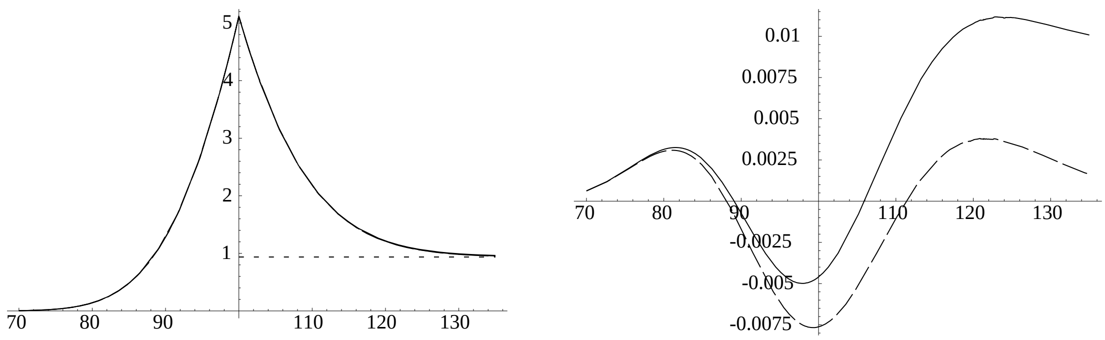

Figure 4.34: Time value and difference to Black-Scholes prices 60 days to maturity

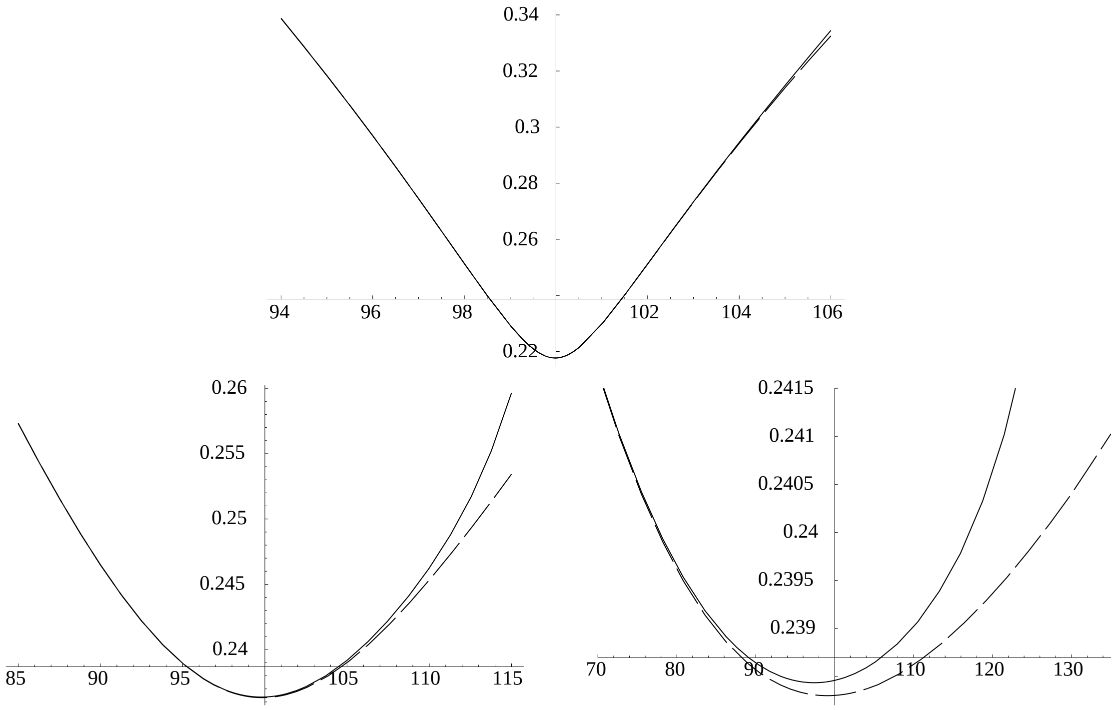

Figure 4.35: Implied Black-Scholes volatilities 1, 10, 60 trading days to maturity

$r, \sigma^2, H)^L$ in the sense of Definition 2.4 (cf. Theorem 2.3). By Lemma 2.22, $Z = (Z^0, Z^1)$ is an extended Grigelionis process with extended characteristics $(\varnothing, \varepsilon_{(1,Z_0^1)}, b, c, F, 0)^E$, where $b_t^0 = 0$, $b_t^1 = (\mu - r)Z_{t-}^1$, $c_t^{00} = c_t^{10} = c_t^{01} = 0$, $c_t^{11} = (\sigma Z_{t-}^1)^2$, $F_t(G) = \int 1_{G\setminus\{0\}}(0, Z_{t-}^1 x)\, H(dx)$ for any $t \in \mathbb{R}_+$, $G \in \mathcal{B}^2$.

Alternatively, we may consider a discounted stock price process of the form

$$Z_t^1 = Z_0^1 \exp\Big((\widehat{\mu} - r)t + \widehat{\sigma} W_t + \int_{[0,t]\times\mathbb{R}} x\,(\widehat{p} - \widehat{q})(dt, dx)\Big), \tag{4.5}$$

where $Z_0^1 \in \mathbb{R}_+^*$, $\widehat{\mu} \in \mathbb{R}$, $\widehat{\sigma} \in \mathbb{R}_+$, W denotes a standard Wiener process, and $\widehat{p}$ is a homogeneous Poisson random measure with compensator $\widehat{q} = \lambda \otimes \widehat{H}$. We assume that $\widehat{H}$ is a fixed measure on $(\mathbb{R}, \mathcal{B})$, satisfying $\int (|x|^2 \wedge |x|)\, \widehat{H}(dx) < \infty$ and $\int e^x 1_{[1,\infty)}(x)\, \widehat{H}(dx) < \infty$ for some $\varepsilon > 0$. Defining the Lévy process $\widehat{X}$ by

$$\widehat{X}_t := (\widehat{\mu} - r)t + \widehat{\sigma} W_t + \int x\,(\widehat{p} - \widehat{q})(dt, dx),$$

we can rewrite Equation (4.5) as

$$Z^1 = Z_0^1 \exp(\widehat{X}),$$

where $\widehat{X}$ is, as X above, an integrable Lévy process with the characteristic triplet $(\widehat{\mu} - r, \widehat{\sigma}^2, \widehat{H})^L$. By Itô's formula (cf. Theorem 2.25 and the following remark) we can now conclude that $Z = (Z^0, Z^1)$ is an extended Grigelionis process whose extended characteristics $(\varnothing, \varepsilon_{(1,Z_0^1)}, b, c, F, 0)^E$ are given by $b_t^0 = 0$, $b_t^1 = (\widehat{\mu} - r + \frac{\widehat{\sigma}^2}{2} + \int (e^x - 1 - x)\widehat{H}(dx))Z_{t-}^1$,

$c_t^{00} = c_t^{10} = c_t^{01} = 0$, $c_t^{11} = (\widehat{\sigma} Z_{t-}^1)^2$, $F_t(G) = \int 1_{G\setminus\{0\}}(0, Z_{t-}^1(e^x - 1))\, \widehat{H}(dx)$ for any $t \in \mathbb{R}_+$, $G \in \mathcal{B}^2$. The following lemma shows that the processes of type (4.4) and (4.5) are essentially the same.

Lemma 4.13 *1. Let $\widehat{\mu}, \widehat{\sigma}, \widehat{p}, \widehat{H}$ etc. be as above. Define*

$$\mu := \widehat{\mu} + \frac{\widehat{\sigma}^2}{2} + \int (e^x - 1 - x)\widehat{H}(dx),$$

$$\sigma := \widehat{\sigma},$$

$$H(G) := \int 1_G(e^x - 1)\, \widehat{H}(dx),$$

$$p([0,t] \times G) := \int_{[0,t]\times\mathbb{R}} 1_G(e^x - 1)\, \widehat{p}(ds, dx)$$

for any $t \in \mathbb{R}_+$, $G \in \mathcal{B}$. Then Z^1 from Equation (4.5) satisfies Equation (4.4).

2. *Conversely, let μ, σ, p, H etc. be as above. Define*

$$\widehat{\mu} := \mu - \frac{\sigma^2}{2} - \int (x - \log(x+1))\, H(dx),$$

$$\widehat{\sigma} := \sigma,$$

$$\widehat{H}(G) := \int 1_G(\log(x+1))\, H(dx),$$

$$\widehat{p}([0,t] \times G) := \int_{[0,t]\times\mathbb{R}} 1_G(\log(x+1))\, p(ds, dx)$$

for any $t \in \mathbb{R}_+$, $G \in \mathcal{B}$. Then Z^1 from Equation (4.4) satisfies Equation (4.5).

Remarks.

1. For $H = 0$, we are of course back in the Black-Scholes setting of Section 4.3.

2. If $\sigma^2 \neq 0$ and $H(G) = \int 1_G(e^x - 1)\lambda\, N(\alpha_J - \frac{1}{2}\sigma_J^2, \sigma_J^2)(dx)$ for any $G \in \mathcal{B}$, where $\lambda \in \mathbb{R}_+$, $\alpha_J \in \mathbb{R}$, $\sigma_J^2 \in \mathbb{R}_+$ are given parameters, then Equation (4.4) describes the process considered in Grünewald & Trautmann (1996), Equation (2).

3. The *hyperbolic stock price model* by Eberlein & Keller (1995) is of the form in Equation (4.5). Its introduction is based on the fact that hyperbolic distributions provide a very good fit for daily stock return data. Fix constants $\widehat{\mu} \in \mathbb{R}$, $\alpha, \delta \in \mathbb{R}_+^*$, $\beta \in (-\alpha, \alpha)$. Suppose that $\alpha - |\beta| \geq 4$. Define the mapping $g_{\alpha,\beta,\delta} : \mathbb{R} \to \mathbb{R}$ by

$$g_{\alpha,\beta,\delta}(x) := \frac{e^{\beta x}}{|x|}\Big(\int_0^\infty \frac{\exp(-\sqrt{2y + \alpha^2}|x|)}{\pi^2 y(J_1^2(\delta\sqrt{2y}) + Y_1^2(\delta\sqrt{2y}))}\, dy + e^{-\alpha|x|}\Big)1_{\mathbb{R}\setminus\{0\}}(x),$$

where J_1, Y_1 are Bessel functions of the first and second type, respectively. Then the stock price process in Eberlein & Keller (1995), Equation (24) is as in Equation (4.5) if we define $\widehat{\sigma} := 0$, $\widehat{H}(dx) := g_{\alpha,\beta,\delta}(x)\, dx$.

4. If $\int |x|^{2(1+\varepsilon)x} 1_{[1,\infty)}\, H(dx) < \infty$ or equivalently $\int e^{2(1+\varepsilon)x} 1_{[1,\infty)}\, \widehat{H}(dx) < \infty$ for some $\varepsilon > 0$, then the market (Z^0, Z^1) meets regularity condition (RC 1). This holds in particular for the models in the preceding remarks.

4.6.1 Derivative Pricing

For Z^1 as in Equation (4.4), the following lemma yields the dynamic of $Z = (Z^0, Z^1)$ under the pricing measure.

Lemma 4.14 *Let $T \in \mathbb{N}$. Assume that the integrability condition in Remark 4 above holds and that there exists a $\psi \in \mathbb{R}$ solving*

$$0 = \mu - r - \kappa\psi\sigma^2 + \int x(u'_\kappa(\psi x) - 1)\, H(dx). \tag{4.6}$$

Then Conditions 1–4 in Theorem 3.36 hold. Moreover, the extended characteristics $(\mathbb{N}^*, \varepsilon_{(0,Z_0^1)}, \widetilde{b}, c, \widetilde{F}, 0)^E$ of $Z = (Z^0, Z^1)$ relative to the equivalent martingale measure P^* leading to neutral price processes are given by $\widetilde{b} = 0$ and

$$\widetilde{F}_t(G) = \int 1_G(0, Z^1_{t-}x)u'_\kappa(\psi x)\, H(dx)$$

for any $t \in [0, T]$, $G \in \mathcal{B}^2$.

Remarks.

1. Under P^*, the discounted stock price process Z^1 has basically the same dynamic as with respect to P, but relative to $\widetilde{\mu}, \widetilde{\sigma}, \widetilde{H}$ instead of μ, σ, H, where $\widetilde{\mu} := 0$, $\widetilde{\sigma} := \sigma$, $\frac{d\widetilde{H}}{dH}(x) := u'_\kappa(\psi x)$ for any $x \in \mathbb{R}$ and the real number ψ is given by Equation (4.6).
2. If Z^1 is expressed as in Equation (4.5), we have to replace the parameters $\widehat{\mu}, \widehat{\sigma}, \widehat{H}$ with $\overline{\mu}, \overline{\sigma}, \overline{H}$ to obtain the P^*-dynamic of Z^1, where $\overline{\mu} := -\frac{\widehat{\sigma}^2}{2} - \int (e^x - 1 - x)\, \overline{H}(dx)$, $\overline{\sigma} := \widehat{\sigma}$, $\frac{d\overline{H}}{d\widehat{H}}(x) := u'_\kappa(\psi(e^x - 1))$ for any $x \in \mathbb{R}$. Again, ψ is given by Equation (4.6).
3. The existence condition in Lemma 4.14 is satisfied for the stock price process of Grünewald & Trautmann as well as that of Eberlein & Keller. Note that the pricing measure P^* for the hyperbolic model is not the same as the EMM P^ϑ by Eberlein & Keller, which is based on an Esscher transform. The latter corresponds to a transformation $\frac{d\overline{H}}{d\widehat{H}}(x) = e^{\vartheta x}$ of the Lévy measure (cf. Keller (1997), Lemma 21). Our transformation of the Lévy measure, on the other hand, is given by $\frac{d\overline{H}}{d\widehat{H}}(x) := u'_\kappa(\psi(e^x - 1))$ (cf. Remark 2). But note that for small values of x one may approximate $e^{\vartheta x} \approx 1 + \vartheta x \approx 1 - (\psi\kappa)x \approx u'_\kappa(\psi(e^x - 1))$ if $\vartheta \approx \psi\kappa$. This explains why the option prices derived by Eberlein & Keller coincide very well with our neutral values.

The following lemma is helpful for numerical computation of option prices and hedging strategies.

Lemma 4.15 *Let $T \in \mathbb{N}$ and $X^2 = g(Z^1_T)$, where $g : \mathbb{R} \to \mathbb{R}$ is a measurable mapping. Assume that there are $M_1, M_2 \in \mathbb{R}$ with $|g(x)| \leq M_1 + M_2|x|$ for any $x \in \mathbb{R}$. Define a mapping $\pi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ by $\pi(t, y) := E(g(y \exp(U_{T-t})))$, where U is a Lévy process with the characteristic triplet $(\overline{\mu}, \overline{\sigma}^2, \overline{H})^L$ as in Remark 2 above. Assume that Condition 5 in*

Theorem 3.36 holds (e.g. if the filtration is the canonical filtration of S^1 or its P-completion). Then the unique neutral price process Z^2 for the derivative with terminal value X^2 at T is of the form $Z^2_t = \pi(t, Z^1_t)$ for any $t \in [0,T]$. If π is of class C^2 on $[0,T) \times \mathbb{R}$, then the extended characteristics $(\varnothing, \varepsilon_{(1,Z^1_0,\pi(0,Z^1_0))}, \overline{b}, \overline{c}, \overline{F}, 0)^E$ of $\overline{Z} := (Z^0, Z^1, Z^2)$ are given by

$$\overline{b}_t = \begin{pmatrix} 0 \\ \\ (\mu - r)Z^1_{t-} \\ \\ \kappa\psi\sigma^2 Z^1_{t-} D_2\pi(t, Z^1_{t-}) \\ + \int (\pi(t, Z^1_{t-}(1+x)) - \pi(t, Z^1_{t-}))(u'_\kappa(\psi x) - 1)\, H(dx) \end{pmatrix}$$

$$\overline{c}_t = \begin{pmatrix} 0 & 0 & 0 \\ \\ 0 & (\sigma Z^1_{t-})^2 & (\sigma Z^1_{t-})^2 D_2\pi(t, Z^1_{t-}) \\ \\ 0 & (\sigma Z^1_{t-})^2 D_2\pi(t, Z^1_{t-}) & (\sigma Z^1_{t-} D_2\pi(t, Z^1_{t-}))^2 \end{pmatrix}$$

$$\overline{F}([0,t] \times G) = \int 1_{G \setminus \{0\}} \Big(0, xZ^1_{t-}, \pi(t, Z^1_{t-}(1+x)) - \pi(t, Z^1_{t-})\Big)\, H(dx)$$

for any $t \in [0,T]$, $G \in \mathcal{B}^3$, where $\psi \in \mathbb{R}$ is chosen as in Lemma 4.14.

Remark. One should still check that the regularity condition (RC 1) holds for the enlarged market from a theoretical point of view.

We perform numerical calculations for the hyperbolic stock price model by means of Lemma 4.15. The choice of the parameters $\widehat{\mu} = 0$, $\alpha = 100$, $\delta = 0.005$, $\beta = 0$ is guided by estimations for German stock data by Eberlein & Keller (cf. Keller (1997), p.89). For option pricing we compare the results to a Black-Scholes model (cf. Section 4.3), where the parameter $\widehat{\sigma} = 0.2387/\sqrt{250}$ is chosen such that the variance $\widehat{\sigma}^2 t$ of the return $\widehat{X}_t$ coincides in both models. Since $\widehat{X}$ is a Lévy process, it follows that the distribution of the daily *log-return* $Y_t := \log(Z^1_t) - \log(Z^1_{t-1}) = \widehat{X}_t - \widehat{X}_{t-1}$ does not depend on t. We denote by $f, f^*, \widetilde{f}$ the densities of the laws of Y_t relative to the given probability P, the pricing measure P^* leading to neutral derivative prices, and the equivalent martingale measure $\widetilde{P}$ corresponding to the Esscher transform (cf. Eberlein & Keller (1995)), respectively. The hardly visible dashed curve in the upper diagram of Figure 4.31 represents $\log(f)$. Since it is a hyperbola as opposed e.g. to the parabola in Figure 4.19, the distribution and hence the model is called *hyperbolic*. Very close to this line one can observe a solid curve marking $\log(f^*)$ as well as $\log(\widetilde{f})$, where the latter is also of hyperbolic shape. To emphasize the differences we plot $\log(f^*) - \log(f)$ (solid line) and $\log(\widetilde{f}) - \log(f)$ (dashed line) in the lower left diagram, as well as $\log(\widetilde{f}) - \log(f^*)$ in the lower right graph. Since the difference between $\log(f)$ and $\log(\widetilde{f})$ is small, one may expect derivative prices based on P^* and $\widetilde{P}$ to be very close, which is indeed the case as can be seen in Figures 4.32 – 4.34.

As in Section 4.4, we consider a European call option with strike price $K = 100$ expiring in 1, 10, 60 trading days, respectively. The dashed curve in the left-hand diagram of Figure 4.32 is the same as that in Figures 4.26 and 4.20. It represents the time value of the option in the Black-Scholes model as a function of the current stock price. The same holds for the dashed lines in the left-hand graphs of Figures 4.33 and 4.34, which are hardly visible since they are covered by solid curves. As before, the horizontal dotted line marks the lower arbitrage bound. The solid curves in the left diagrams correspond to the option price in the hyperbolic model based on both P^* and $\widetilde{P}$. Obviously, there is no big difference between neutral option prices and those from Eberlein & Keller. Even if one believes in neutral prices, one may therefore use the Esscher transform as an approximation since it is easier to compute. For the latter, it is not necessary to evaluate the Lévy-Khintchine formula numerically in order to obtain the return distribution under the pricing measure. In the diagrams on the right-hand side we indicate the difference of the hyperbolic option prices and the Black-Scholes value. The solid lines correspond to neutral prices, whereas the dashed curves now mark the values obtained by Esscher transform. Observe that the difference between hyperbolic and Black-Scholes prices is of about the same absolute size for all time horizons. Relative to the time value of the option, however, it only plays a role for short-lived options or possibly for options far in or out of the money.

As in Section 4.4, we also illustrate the differences by plotting implied Black-Scholes volatilities in Figure 4.35. The height of the horizontal axis indicates the annual volatility $\sqrt{250\widehat{\sigma}^2}$. The solid (resp. dashed) curves mark the implied volatilities from the Black-Scholes formula if we insert the neutral (resp. Eberlein & Keller) option prices from the hyperbolic model. As before, these curves should not be overinterpreted as a model test, since they are not data-based. They only indicate the kind of inconsistency a Black-Scholes economist would observe in a hyperbolic market. Note that the smile is significant for short-lived options and flattens out for long times to expiration.

Proofs

PROOF OF LEMMA 4.13. *1.* Using the integrability conditions on $\widehat{H}$ and the mean value theorem, one easily verifies that $\int(|e^x - 1|^2 \wedge |e^x - 1|)\,\widehat{H}(dx) < \infty$. By JS, II.1.33c, this implies that the mapping $w : \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $(\omega, t, x) \mapsto e^x - 1$ is in $G_{\mathrm{loc}}(\widehat{p})$. Since $q := \lambda \otimes H$ is the compensator of p, the same argument yields that the mapping $\Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $(\omega, t, x) \mapsto x$ is in $G_{\mathrm{loc}}(p)$. Since $\Delta w * (\widehat{p} - \widehat{q})_t = \int w(x)\,\widehat{p}(\{t\} \times dx) = \int x\,p(\{t\} \times dx) = \Delta x * (p - q)_t$ for any $t \in \mathbb{R}_+$, it follows from JS, I.4.19 that $w * (\widehat{p} - \widehat{q}) = x * (p - q)$ and hence $(Z_{t-}w) * (\widehat{p} - \widehat{q}) = (Z_{t-}x) * (p - q)$ (cf. JS, II.1.30). Application of Itô's formula as in Jacod (1979), (3.89) now yields that Z^1 satisfies Equation (4.5).

2. Using the integrability conditions on H and the mean value theorem, one easily verifies that $\int(|x|^2 \wedge |x|)\,\widehat{H}(dx) < \infty$, $\int e^x 1_{[1,\infty)}(x)\,\widehat{H}(dx) < \infty$, and that $\widehat{\mu}$ is well-defined. Since $\widehat{q} := \lambda \otimes \widehat{H}$ is the compensator of $\widehat{p}$, it follows that the right-hand side of Equation (4.5), which we denote by $\widetilde{Z}^1_t$, has all the properties in that paragraph. Application of Statement 1 in Lemma 4.13 yields that $\widetilde{Z}$ is also a solution to Equation (4.4). Since the SDE (4.4) has, up to indistinguishability, only one solution, it follows that $Z = \widetilde{Z}$ and the claim is proved. □

PROOF OF THE REMARKS. *2.* We want to show that the integrability conditions for H hold. It is obvious that $H((-\infty, 1]) = 0$. Moreover, $\int |\log(x+1)| 1_{(-1,-\frac{1}{2})}(X)\, H(dx) \leq \lambda \int |x|\, N(\alpha_J - \frac{1}{2}\sigma_J^2, \sigma_J^2)(dx) < \infty$. Finally, we have that $\int |x|\, H(dx) < \infty$ and $\int |x|^3 H(dx) < \infty$ because $x \mapsto e^{3|x|}$ is integrable for any normal distribution.

3. We want to show that the integrability conditions for $\widehat{H}$ hold. By Keller (1997), Equation (3.1) the distribution of $\widehat{X}_1$ has the Lebesgue density

$$f_{\alpha,\beta,\delta,\widehat{\mu}}(x) = \frac{\sqrt{\alpha - \beta^2}}{2\alpha\delta K_1(\delta\sqrt{\alpha - \beta^2})} \exp\Big(-\alpha\sqrt{\delta^2 + (x-\widehat{\mu})^2} + \beta(x - \widehat{\mu})\Big),$$

where K_1 is the modified Bessel function of the third type with index 1. For $x \to \pm\infty$, we have $f_{\alpha,\beta,\delta,\widehat{\mu}} = O(\exp(-\alpha|x| + \beta x)) = O(\exp(-4|x|))$. Therefore, $\int e^{3|x|}\, P^{\widehat{X}_1}(dx) < \infty$ and hence $\int |x|\, P^{\widehat{X}_1}(dx) < \infty$. By Lemma 2.2, Theorem 2.3 and Proposition 2.9, this implies that $\int (|x|^2 \wedge |x|)\, \widehat{H}(dx) < \infty$ and $\int e^{3x} 1_{[1,\infty)}(x)\, \widehat{H}(dx) < \infty$.

4. One easily verifies that the two integrability conditions are indeed equivalent if H and $\widehat{H}$ are related to each other as in Lemma 4.13. Note that $\int (|x|^2 \wedge |x|)\, F_t(dx) \leq ((Z^1_{t-})^2 \vee |Z^1_{t-}|) \int (|x|^2 \wedge |x|)\, H(dx)$ and similarly for b and c. In order to show that integrability condition (RC 1) holds, it suffices to prove $\sup_{s\in[0,t]} E((Z^1_{s-})^{2(1+\varepsilon)}) \leq \sup_{s\in[0,t]} E((Z^1_s)^{2(1+\varepsilon)}) < \infty$ for any $t \in \mathbb{R}_+$, where the first inequality follows from Fatou's lemma. Since $Z^1_t = Z^1_0 \exp(\widehat{X}_t)$, it remains to show that $\sup_{s\in[0,t]} E(\exp(2(1+\varepsilon)\widehat{X}_s)) < \infty$. This follows immediately from the integrability conditions for $\widehat{H}$ and Proposition 2.9.

For the particular models in the preceding remarks, the integrability conditions are shown above. □

PROOF OF LEMMA 4.14. Firstly note that Condition 5 (and of course Condition 6 for any particular claim) still have to be checked in order to apply Theorem 3.36. By Corollary 2.43 and Theorem 2.65, any local martingale has the representation property relative to the Lévy process X if the filtration is the canonical filtration of X (or the P-completion). Since Z^1 and X generate the same filtration, it follows that Condition 5 holds if the given filtration is the canonical filtration of S^1 or its P-completion.

For the predictable, locally bounded process φ defined by $\varphi_t = (0, \psi/Z^1_{t-})$, we have by definition of ψ that $Z^1_{t-}(\mu - r) - \kappa(\sigma Z^1_{t-})^2\varphi^1_t + \int Z^1_{t-}x(u'_\kappa(Z^1_{t-}\varphi^1_t x) - 1)\, H(dx) = 0$, which implies that φ is a u_κ-optimal strategy for $\mathfrak{A}$ (cf. Corollary 3.23). Hence, the Conditions 1 and 2 in Theorem 3.36 hold. For $x \to 0$, we have that

$$\begin{aligned} & u'_\kappa(\psi x)(\log(u'_\kappa(\psi x) - 1) + 1 \\ &= (1 + \psi x + O(x^2))((u'_\kappa(\psi x) - 1) + O((u'_\kappa(\psi x) - 1)^2) - 1) + 1 \\ &= (1 + \psi x + O(x^2))(\psi x + O(x^2) - 1) + 1 = O(x^2). \end{aligned}$$

Moreover, $u'_\kappa(\psi x)$ and hence $u'_\kappa(\psi x)(\log(u'_\kappa(\psi x))-1)+1$ is bounded from above. Finally, one easily shows that $y(\log(y)-1)\geq -1$ for any $y\in\mathbb{R}^*_+$ and hence $u'_\kappa(\psi x)(\log(u'_\kappa(\psi x))-1)+1\geq 0$. Together, this implies that $C^1:=\int(u'_\kappa(\psi x)(\log(u'_\kappa(\psi x))-1)+1)\,H(dx)<\infty$. Observe that C_T in Section 3.4, Remark 4 is of the form $C_T=T(\frac{\kappa^2}{2}\psi^2\sigma^2+C^1)<\infty$, which implies that Condition 3 in Theorem 3.36 holds.

Application of Corollary 3.38 and straightforward calculations yield that the extended P^*-characteristics of (Z^0,Z^1) are as claimed in Lemma 4.14. Since Z^1 is strictly positive, we can apply Lemma 2.22 to $\int_0^\cdot (Z^1_{t-})^{-1}\,dZ^1_t=\int(Z^1_{t-})^{-1}Z^1_{t-}\,dX_t=X$ and obtain that the P^*-extended characteristics of X are of the form $(\varnothing,\varepsilon_0,\widetilde{\mu},\widetilde{\sigma}^2,\widetilde{H},0)^E$ on $[0,T]$, where $\widetilde{\mu}=0$, $\widetilde{\sigma}=\sigma$, $\frac{d\widetilde{H}}{dH}(x)=u'_\kappa(\psi x)$. Hence, X is a P^*-Lévy process on $[0,T]$ with characteristic triplet $(\widetilde{\mu},\widetilde{\sigma}^2,\widetilde{H})^L$ (cf. Corollary 2.43 and the subsequent Remark 5). Observe that $\frac{d\widetilde{H}}{dH}$ is bounded and hence the integrability condition in Remark 4 holds for $\widetilde{H}$ instead of H as well. As in the proof of that remark, one may now conclude that $\sup_{s\in[0,T]}E^*((Z^1_{s-})^{2(1+\varepsilon)})<\infty$, which yields that the integrability condition in Section 3.4, Remark 5 holds. Hence, Condition 4 in Theorem 3.36 holds as well. □

PROOF OF THE REMARKS. *1.* This has already been shown in the proof of Lemma 4.14.

2. This follows from the first remark and application of Lemma 4.13.

3. Define a mapping $h:\mathbb{R}\to\mathbb{R}$ by $h(\psi):=-(\mu-r)\psi+\frac{1}{2}\kappa\sigma^2\psi^2-\int(u_\kappa(\psi x)-\psi x)\,H(dx)$. As in the proof of Theorem 3.22, one shows that h is a differentiable function with derivative $h'(\psi)=-(\mu-r)+\kappa\sigma^2\psi-\int x(u'_\kappa(\psi x)-1)\,H(dx)$ and that any of the three terms in the definition of h is a convex function of ψ. Moreover, using the dominated convergence theorem and the continuity of u'_κ, we conclude that h' is continuous. If $\sigma^2\neq 0$ as e.g. in the model considered by Grünewald & Trautmann (1996), then the increasing mapping $\psi\mapsto -(\mu-r)+\sigma^2\psi$ has arbitrarily small and large values. Since the third term is increasing as well, the same must be true for h'. By continuity of h', this implies that there exists a zero ψ of h' and we are done.

Now consider the model by Eberlein & Keller, where $h'(\psi)=-\mu-\int x(u'_\kappa(\psi x)-1)\,H(dx)$. For $\psi\to\infty$, monotone convergence yields that $\int_{\mathbb{R}_+}x(u'_\kappa(\psi x)-1)\,H(dx)\to -\int_{\mathbb{R}_+}x\,H(dx)=-\int(e^x-1)g_{\alpha,\beta,\delta}(x)\,dx=-\infty$, where the last equality follows from the fact that $g_{\alpha,\beta,\delta}(x)$ behaves as $\frac{1}{x^2}$ around 0 (cf. Eberlein & Keller (1995), p.295). Similarly, one proves that $\int_{\mathbb{R}_-}x(u'_\kappa(\psi x)-1)\,H(dx)\to-\infty$ and hence $h'(x)\to\infty$ for $\psi\to\infty$. In the same way, we obtain $h'(\psi)\to-\infty$ for $\psi\to-\infty$, which implies that there exists a zero ψ of h' in this case as well. □

PROOF OF LEMMA 4.15. By Theorem 3.36 we have that $Z^2_t=E^*(g(Z^1_t\exp(\widehat{X}_T-\widehat{X}_t))|\mathcal{F}_t)=\int g(Z^1_t\exp(\bar{\omega}_{T-t}))\,P^{*(\widehat{X}_{t+s}-\widehat{X}_t)_{s\geq 0}|\mathcal{F}_t}(d\bar{\omega})$ P-almost surely for any $t\in[0,T]$. Since $\widehat{X}$ is a P^*-Lévy process, we have that $P^{*(\widehat{X}_{t+s}-\widehat{X}_t)_{s\geq 0}|\mathcal{F}_t}=P^{*(\widehat{X}_s)_{s\geq 0}}$. As the P^*-characteristic triplet of $\widehat{X}$ is $(\overline{\mu},\overline{\sigma}^2,\overline{H})^L$, it follows that $Z^2_t=\pi(t,Z^1_t)$ for any $t\in[0,T]$. Application of Itô's formula (cf. Theorem 2.25) to the process $(t,Z^1_t)_{t\in\mathbb{R}}$ and the mapping $f:[0,T)\times\mathbb{R}_+\to\mathbb{R}^3$, $(t,z)\mapsto(1,z,\pi(t,z))$ yields that the extended characteristics of

$\overline{Z} = (Z^0, Z^1, Z^2)$ are of the form in Lemma 4.15, but with a different drift $\overline{b}^2$ for the process Z^2. By Corollary 3.38 applied to (Z^0, Z^1, Z^2) it follows that $\overline{b}^2_t - \kappa\psi\sigma^2 Z^1_{t-} D_2(t, Z^1_{t-}) - \int(\pi(t, Z^1_{t-}(1+x)) - \pi(t, Z^1_{t-}))(u'_\kappa(\psi x) - 1)\, H(dx)$ (for any $t \in [0,T]$) is the drift of Z^2 under the equivalent martingale measure P^* and hence 0. Therefore, $\overline{b}^2$ is indeed as claimed in Lemma 4.15. To be very strict, Itô's formula applies only to functions f that are defined and of class C^2 on $\mathbb{R} \times \mathbb{R}$. The way out is to argue by localization similarly as in the proof of Corollary 4.7. □

4.7 Bivariate Diffusion Models

A closer look at stock return data reveals that periods of violent price changes alternate with relatively calm intervals. This behaviour led to the introduction of ARCH and GARCH models on the one hand and bivariate diffusion settings on the other. For the latter, the volatility is modelled by a stochastic process following its own dynamic. We consider a market consisting of only one underlying besides the numeraire. Its discounted price process is assumed to satisfy the stochastic differential equations

$$\begin{aligned} dZ^1_t &= \mu(\sigma_t) Z^1_t\, dt + \sigma_t Z^1_t\, dW_t \\ d\sigma_t &= \alpha(\sigma_t)\, dt + \beta(\sigma_t)\sigma_t\, d\widehat{W}_t, \end{aligned} \tag{4.7}$$

where continuous functions $\mu, \alpha, \beta : \mathbb{R} \to \mathbb{R}$ as well as $Z^1_0, \sigma_0 \in \mathbb{R}^*_+$ are given and $W, \widehat{W}$ denote standard Wiener processes with correlation ρ (i.e. $\langle W, \widehat{W}\rangle_t = \rho t$ for any $t \in \mathbb{R}_+$). The second SDE descibes the dynamic of the stochastic volatility of Security 1. By Lemma 2.22 it follows that (Z^0, Z^1, σ) is an extended Grigelionis process whose extended characteristics $(\varnothing, \varepsilon_{(0,Z^1_0,\sigma_0)}, b, c, 0, 0)^E$ are given by

$$b_t = \begin{pmatrix} 0 \\ \mu(\sigma_t) Z^1_t \\ \alpha(\sigma_t) \end{pmatrix}$$

$$c_t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (\sigma_t Z^1_t)^2 & \rho\beta(\sigma_t) Z^1_t \sigma^2_t \\ 0 & \rho\beta(\sigma_t) Z^1_t \sigma^2_t & (\beta(\sigma_t)\sigma_t)^2 \end{pmatrix}$$

for any $t \in \mathbb{R}_+$. Note that the process Z^1 is P-almost surely $\mathbb{R}^*_+$-valued, because it is a stochastic exponential of a continuous process (cf. JS, I.4.64). Assume that σ is a $\mathbb{R}^*_+$-valued process as well.

4.7.1 Derivative Pricing

In order to compute neutral derivative prices, we need the following

Lemma 4.16 *With respect to the pricing measure P^* in Theorem 3.36, the extended characteristics $(\varnothing, \varepsilon_{(0,Z_0^1,\sigma_0)}, \widetilde{b}, c, 0, 0)^E$ of (Z^0, Z^1, σ) are given by $\widetilde{b}_t^0 = \widetilde{b}_t^1 = 0$, $\widetilde{b}_t^2 = \alpha(\sigma_t) - \rho\mu(\sigma_t)\beta(\sigma_t)$.*

Remark. The preceding lemma shows that, relative to P^*, the process (Z^0, Z^1, σ) has basically the same dynamic as with respect to P, but with $\widetilde{\mu} := 0$, $\widetilde{\alpha}(x) := \alpha(x) - \rho\mu(x)\beta(x)$, $\widetilde{\beta} := \beta$ instead of μ, α, β.

The following lemma helps to calculate option prices explicitly.

Lemma 4.17 *Let $T \in \mathbb{R}_+$ and $X^2 = g(Z_T^1)$, where $g : \mathbb{R} \to \mathbb{R}$ is a measurable mapping such that there are $M_1, M_2 \in \mathbb{R}$ with $|g(x)| \leq M_1 + M_2|x|$ for any $x \in \mathbb{R}$. Assume that, for any $z, x \in \mathbb{R}_+^*$, the martingale problem $(\varnothing, \varepsilon_{(z,x)}, \check{b}, \check{c}, 0, 0)^M$ in $\mathbb{R}^2$ has a unique solution-measure, where*

$$\check{b}_t(\bar{\omega}) := \begin{pmatrix} 0 \\ \alpha(\bar{\omega}_{t-}^2) - \rho\mu(\bar{\omega}_{t-}^2)\beta(\bar{\omega}_{t-}^2) \end{pmatrix}$$

$$\check{c}_t(\bar{\omega}) := \begin{pmatrix} (\bar{\omega}_{t-}^1 \bar{\omega}_{t-}^2)^2 & \rho\beta(\bar{\omega}_{t-}^2)\bar{\omega}_{t-}^1(\bar{\omega}_{t-}^2)^2 \\ \rho\beta(\bar{\omega}_{t-}^2)\bar{\omega}_{t-}^1(\bar{\omega}_{t-}^2)^2 & (\beta(\bar{\omega}_{t-}^2)\bar{\omega}_{t-}^2)^2 \end{pmatrix}$$

for any $(\bar{\omega}, t) = ((\bar{\omega}^1, \bar{\omega}^2), t) \in \mathbb{D}^2 \times \mathbb{R}_+$. Define a function $C_{bd} : \mathbb{R}_+^* \times \mathbb{R}_+^* \times [0, T] \to \mathbb{R}$ by

$$C_{bd}(z, x, t) := E\left(C_{BS}\left(z \exp\left(\rho \int_0^{T-t} \check{\sigma}_s \, d\check{W}_s - \frac{1}{2}\rho^2 \int_0^{T-t} \check{\sigma}_s^2 \, ds\right), \sqrt{1-\rho^2} \int_0^{T-t} \check{\sigma}_s^2 \, ds\right)\right),$$

where $C_{BS} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is defined as in Lemma 4.6 and $(\check{W}, \check{\sigma})$ is a solution to the stochastic differential equation

$$\check{\sigma}_0 = x, \quad d\check{\sigma}_s = (\alpha(\check{\sigma}_s) - \rho\mu(\check{\sigma}_s)\beta(\check{\sigma}_s)) \, ds + \beta(\check{\sigma}_s)\check{\sigma}_s \, d\check{W}_s \tag{4.8}$$

(cf. Remark 1 below). Then the neutral price process for the derivative with terminal value X^2 at time T is given by $Z_t^2 = C_{bd}(Z_t^1, \sigma_t, t)$ for any $t \in [0, T]$.

Remarks.

1. The last assumption in the preceding lemma means that $(\check{W}, \check{\sigma})$ is an adapted process on some filtered probability space (not necessarily the given one), $\check{W}$ is a standard Wiener process on that space, and $\check{\sigma}$ solves the SDE (4.8).

2. Recall that, by Theorem 2.30, martingale problems are closely related to stochastic differential equations. In particular note that if $(\check{Z}^1, \check{\sigma})$ is a solution process to the martingale problem $(\varnothing, \varepsilon_{(z,x)}, \check{b}, \check{c}, 0, 0)^M$, then $\check{\sigma}$ is a solution to the martingale problem $(\varnothing, \varepsilon_x, \check{b}^2, \check{c}^{22}, 0, 0)^M$. From Theorem 2.30, it follows that the law of $\check{\sigma}$ is a solution-measure to the SDE (4.8). This in turn implies that processes $(\check{W}, \check{\sigma})$ as in Lemma 4.17 necessarily exist.

3. If $\rho = 0$, then the definition of C_{bd} simplifies to $C_{bd}(z,x,t) = E(C_{BS}(z, \int_0^{T-t} \check{\sigma}_s^2\, ds))$ and the SDE for $\check{\sigma}$ is the same as Equation (4.7), which is solved by σ relative to P.

4. In the case $\rho = 0$, a second order Taylor approximation for $C_{BS}(z, \cdot)$ yields

$$C_{bd}(z,x,t) \approx C_{BS}(z,\Sigma) + \frac{1}{2} D_{22} C_{BS}(z,\Sigma) \mathrm{Var}\left(\int_0^{T-t} \check{\sigma}_s^2\, ds \right), \tag{4.9}$$

where $\check{\sigma}$ is as in Lemma 4.17 and $\Sigma := E(\int_0^{T-t} \check{\sigma}_s^2\, ds)$. Intuitively speaking, the neutral price of an option can be approximated by the Black-Scholes price where the constant variance is replaced with the mean over the remaining life time of the option. The second-order correction also takes the variability of the volatility into account.

4.7.2 Hedging

Consider now a markt with three securities 0, 1, 2 where Z^0, Z^1 are as in the previous subsection and Z^2 denotes the neutral price process of an option as in Lemma 4.17. Assume that you have sold one option and you want to hedge your risk.

Lemma 4.18 *Under the assumptions of Lemma 4.17, the u-optimal strategy for the hedging problem $\varphi^2 = -1$ is given by*

$$\varphi_t^1 = D_1 C_{bd}(Z_t^1, \sigma_t, t) + \rho \frac{\beta(\sigma_t)}{Z_t^1} D_2 C_{bd}(Z_t^1, \sigma_t, t) + \frac{1}{\kappa Z_t^1} \frac{\mu(\sigma_t)}{\sigma_t^2}$$

for any $t \in [0,T)$, where $\kappa := -u''(0)$ is the risk aversion of u. (As usual, φ^0 can be arbitrarily chosen.)

Remark. For $\rho = 0$ we have

$$\varphi_t^1 = D_1 C_{bd}(Z_t^1, \sigma_t, t) + \frac{1}{\kappa Z_t^1} \frac{\mu(\sigma_t)}{\sigma_t^2}.$$

Using the approximation in Remark 2 in the previous subsection, we obtain

$$\varphi_t^1 \approx D_1 C_{BS}(Z_t^1, \Sigma) + \frac{1}{2} D_{221} C_{BS}(z,\Sigma) \mathrm{Var}\left(\int_0^{T-t} \check{\sigma}_s^2\, ds \right) + \frac{1}{\kappa Z_t^1} \frac{\mu(\sigma_t)}{\sigma_t^2}, \tag{4.10}$$

where Σ and $\check{\sigma}$ are defined as in that remark.

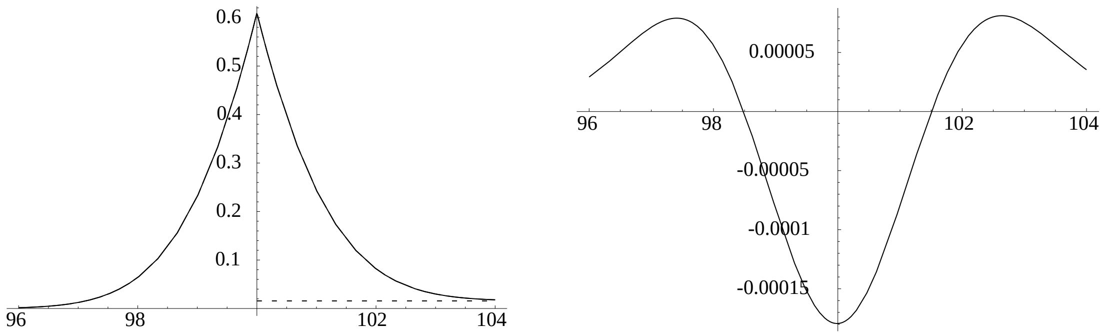

Figure 4.36: Time value and difference to Black-Scholes prices 1 day to maturity

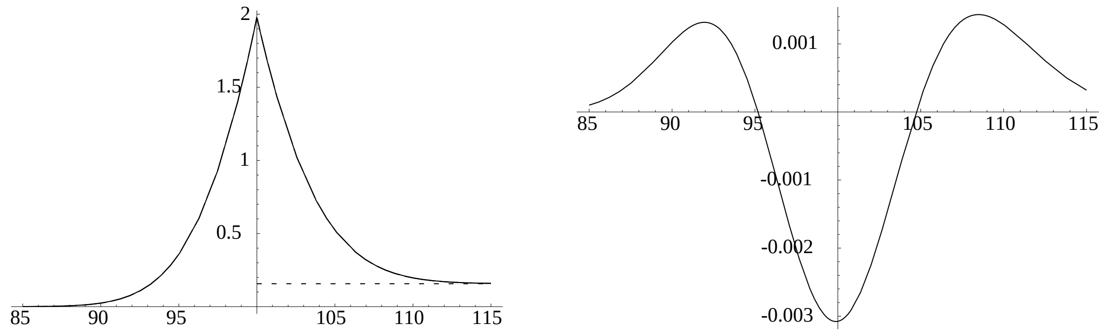

Figure 4.37: Time value and difference to Black-Scholes prices 10 days to maturity

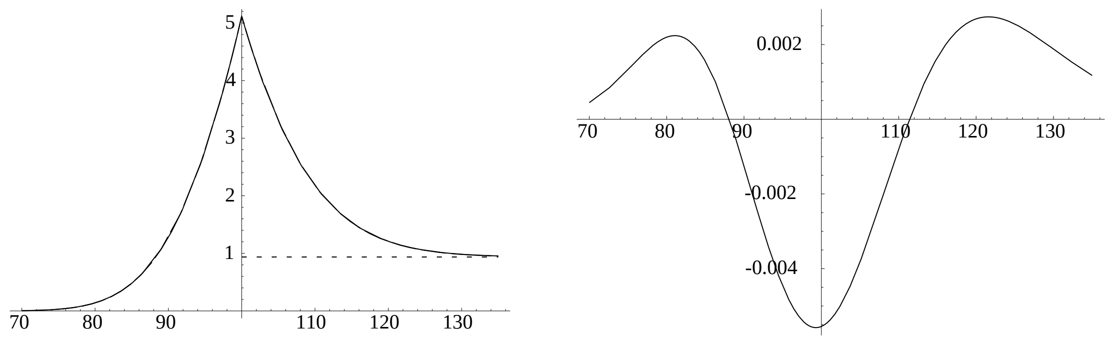

Figure 4.38: Time value and difference to Black-Scholes prices 60 days to maturity

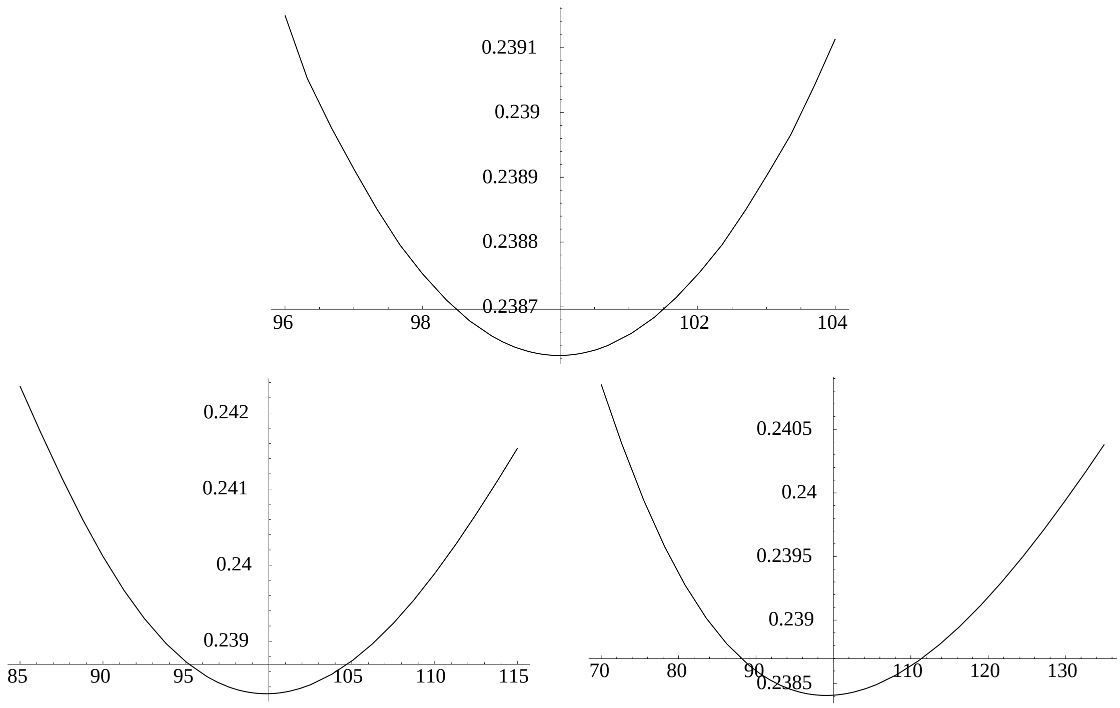

Figure 4.39: Implied Black-Scholes volatilities 1, 10, 60 trading days to maturity

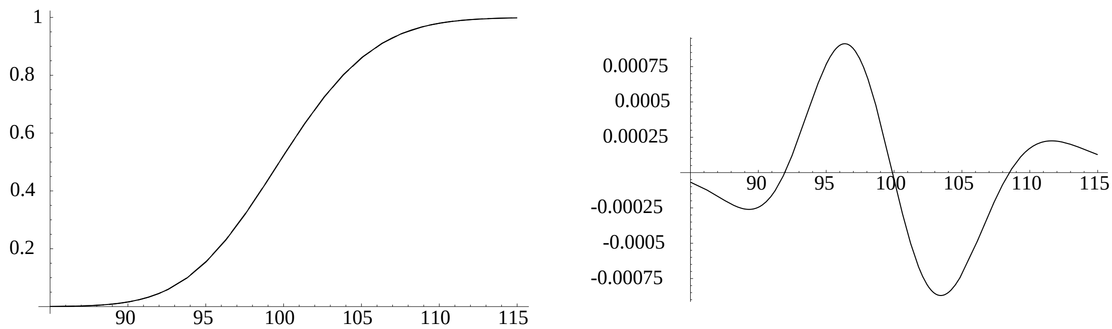

Figure 4.40: Hedging strategy and difference to Black-Scholes 10 days to maturity

4.7.3 Price Regions and Improved Derivative Models

Let us consider a market with underlyings 0, 1 as in the previous subsections and a derivative 2 as in Lemma 4.17. Price regions and improved derivative models for the price process Z^2 are based on $(\kappa, \overline{\rho}^2)$-consistent or -approximate price processes. As noted in Chapter 3, we do not know whether $(\kappa, \overline{\rho}^2)$-consistent processes always exist in the continuous-time framework, let alone how to compute them. Therefore, we focus on approximate prices.

Lemma 4.19 *Let the assumptions of Lemma 4.17 hold and fix $\kappa > 0$, $\overline{\rho}^2 \in \mathbb{R}$. With respect to the pricing measure $\widetilde{P}$ in Section 3.5 leading to $(\kappa, \overline{\rho}^2)$-approximate price processes, the extended characteristics $(\varnothing, \varepsilon_{(0,Z_0^1,\sigma_0)}, \widehat{b}, c, 0, 0)^E$ of (Z^0, Z^1, σ) are given by $\widehat{b}_t^0 = \widehat{b}_t^1 = 0$, $\widehat{b}_t^2 = \widetilde{b}_t^2 - \kappa\overline{\rho}^2(1-\rho^2)(\beta(\sigma_t))^2 D_2 C_{bd}(Z_t^1, \sigma_t, t)$ for any $t \in [0,T]$, where $\widetilde{b}$ is defined in Lemma 4.16.*

The following lemma helps in calculating option prices explicitly.

Lemma 4.20 *Suppose that the conditions in Lemma 4.17 hold and fix $\kappa > 0$, $\overline{\rho}^2 \in \mathbb{R}$. Moreover assume that, for any $(z,x) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, the martingale problem $(\varnothing, \varepsilon_{(z,x)}, \breve{b}, \breve{c}, 0, 0)^M$ in $\mathbb{R}^2$ has a unique solution-measure, where*

$$\breve{b}_t(\bar{\omega}) := \begin{pmatrix} 0 \\ \\ \alpha(\bar{\omega}_{t-}^2) - \rho\mu(\bar{\omega}_{t-}^2)\beta(\bar{\omega}_{t-}^2) \\ -1_{[0,T)}(t)\kappa\overline{\rho}^2(1-\rho^2)((\beta(\bar{\omega}_{t-}^2))^2 D_2 C_{bd}(\bar{\omega}_{t-}^1, \bar{\omega}_{t-}^2, t) \end{pmatrix}$$

$$\breve{c}_t(\bar{\omega}) := \begin{pmatrix} (\bar{\omega}_{t-}^1 \bar{\omega}_{t-}^2)^2 & \rho\beta(\bar{\omega}_{t-}^2)\bar{\omega}_{t-}^1(\bar{\omega}_{t-}^2)^2 \\ \\ \rho\beta(\bar{\omega}_{t-}^2)\bar{\omega}_{t-}^1(\bar{\omega}_{t-}^2)^2 & (\beta(\bar{\omega}_{t-}^2)\bar{\omega}_{t-}^2)^2 \end{pmatrix}$$

for any $(\bar{\omega}, t) = ((\bar{\omega}^1, \bar{\omega}^2), t) \in \mathbb{D}^2 \times \mathbb{R}_+$. Define a function $\widetilde{C}_{bd} : \mathbb{R}_+^* \times \mathbb{R}_+^* \times [0,T] \to \mathbb{R}$ by $\widetilde{C}_{bd}(z,x,t) := E(g(\breve{Z}_{T-t}^1))$, where $(\breve{Z}^1, \breve{\sigma})$ is a solution-process to the above martingale problem. Then the $(\kappa, \overline{\rho}^2)$-approximate price process for the derivative with terminal value X^2 at time T is given by $\widetilde{Z}_t^2 = \widetilde{C}_{bd}(Z_t^1, \sigma_t, t)$ for any $t \in [0,T]$.

Remarks.

1. Since, relative to $\widetilde{P}$, the dynamic of σ is affected by the price process Z^1, $\widetilde{C}_{bd}$ cannot be generally expressed in terms of C_{BS}, as is the case in Lemma 4.17.

2. Hedging strategies can now be computed as in Lemma 4.18, but with $\widetilde{C}_{bd}$ instead of C_{bd}. However, the following remark does not not make sense in this case.

4.7.4 Qualitative Comparison to Black-Scholes

Explicit computation of option prices and hedging strategies necessitates the numerical solution of stochastic differential equations. This is beyond our scope here. Instead, we will use the approximations (4.9) and (4.10) to illustrate how the results deviate qualitatively from the Black-Scholes case. As in the previous sections, we consider a European call option with strike price $K = 100$ expiring in 1, 10, 60 trading days, respectively. If we work with the second-order Taylor approximation for the neutral option price, then Equations (4.9) and (4.10) show that we need not specify the diffusion equation completely. It suffices to fix values for $E(\int_0^T \sigma_s^2\, ds)$ and $\mathrm{Var}(\int_0^T \sigma_s^2\, ds)$ for the three different time horizons. In the case that σ^2 follows a shifted Ornstein-Uhlenbeck process, one can evaluate these quantities explicitly. In order to obtain Figures 4.36 – 4.40, we inserted the values from a process of the form

$$d(\sigma^2)_t = -\alpha((\sigma^2)_t - (\overline{\sigma}^2))\, dt + \beta\, d\overline{W}_t \tag{4.11}$$

with $\overline{\sigma} = \sigma_0 := 0.2387/\sqrt{250}$, $\alpha := 0.1$, $\beta := 2 \cdot 10^{-5}$, and $\overline{W}$ denoting a standard Wiener process. $\overline{\sigma}$ and σ_0 are chosen such that the mean of σ_t^2 coincides with the fixed value in the Black-Scholes models we consider in Sections 4.4 and 4.6 for comparison. The value $\alpha = 0.1$ intuitively means that a volatility shock has a half-life of about $\alpha^{-1}\log(2) \approx 6.93$ trading days. We choose a very small value of β for two reasons. Firstly, the reader may already have observed that a shifted Ornstein-Uhlenbeck process is inadequate to model the positive quantity σ_t^2. Therefore, we consider Equation (4.11) only as a reasonable approximation in a neighbourhood of $\overline{\sigma}^2$. By choosing a small β we ensure that $(\sigma^2)_t$ hardly leaves this neighbourhood. Secondly, the validity of the Formulas (4.9) and (4.10) is restricted to small values of $\mathrm{Var}(\int_0^T \sigma_s^2\, ds)$, which is another reason to let β be small. As a consequence, we are almost back in the constant volatility setting of Section 4.3 and therefore the option prices and hedging strategies hardly differ from the Black-Scholes model.

Indeed, the diagrams on the left in Figures 4.36 – 4.38 show the time value of the call in the bivariate diffusion setting as well as the Black-Scholes price relative to volatility $\overline{\sigma}$ as a function of the current stock price. As in the Figures 4.20 – 4.22 and 4.32 – 4.34, the horizontal dotted lines mark lower arbitrage bounds. In the diagrams on the right we plot the tiny difference between the neutral option prices in the bivariate diffusion setting and their Black-Scholes counterparts. In contrast to the previous section, where we use realistic parameters for the hyperbolic distribution, these price differences on the right are not meant to contain quantitative information, since the diffusion model for the volatility is not obtained by statistical means. On the contrary, we choose β excessively small. Maybe surprisingly, the M-shape (or W-shape if you rotate the graphs) of the price differences as well as the implied Black-Scholes volatility smiles in Figure 4.39 look very similar to the corresponding curves in Figures 4.32 – 4.34 and 4.35 for the hyperbolic setting, although the models are of quite different kind. However, in the hyperbolic case the deviation from Black-Scholes is most pronounced for short-lived options, whereas in the bivariate diffusion setting the differences seem to reach their maximum later, as the size of the smile in Figure 4.39 and also the diagrams in Figures 4.36 – 4.38 indicate. Note that the hight of the abscissa in Figure 4.39 corresponds as in the previous sections to the annual volatility $\sqrt{250\overline{\sigma}^2}$ of the Black-Scholes model we use for comparison.

In the left-hand diagram of Figure 4.40 we plot the number of stocks in a hedging portfolio as a function of the current stock price. The task is to hedge -1 European call options 10 trading days before expiration. We make use of the approximation (4.10) in the case of infinite risk aversion κ. The curve in the right-hand diagram marks the difference between the hedging portfolio in the bivariate diffusion and the Black-Scholes model.

Let us stress once more that the Figures 4.36 – 4.40 can only give a qualitative picture. It would be disirable to compare prices and strategies for model parameters that were obtained by real – and preferably the same – market data.

Proofs

PROOF OF LEMMA 4.16. Note that the Conditions 1–6 in Theorem 3.36 depend on the particular model and have to be checked.

By Corollary 3.23, a strategy $\varphi = (\varphi_0, \varphi_1)$ is u_κ-optimal for $\mathfrak{A}$ if and only if $b^1_t - \kappa c^{11}_t \varphi^1_t = 0$, i.e. $\varphi^1_t = \frac{1}{\kappa}\frac{\mu(\sigma_t)}{\sigma^2_t Z^1_t}$. The shape of the extended P^*-characteristics of (Z^0, Z^1, σ) now follows from Corollary 3.38. □

Proposition 4.21 *Let W be a $\mathbb{R}$-valued standard Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$, and let $\mathcal{C}$ be a sub-σ-field of $\mathcal{F}$ that is independent of W. Moreover, denote by Y a continuous adapted process that is $\mathcal{C}$-measurable. Then we have for any $T \in \mathbb{R}_+$*

$$P^{\int_0^T Y_s\,dW_s|\mathcal{C}} = N\Big(0, \int_0^T Y_s^2\,ds\Big) \quad P\text{-almost surely.}$$

PROOF. *First step:* Suppose that the predictable process Y is of the form $Y = \sum_{i=1}^n 1_{]s_i,s_{i+1}]}\alpha_i$, where $0 \leq s_1 \leq \ldots \leq s_{n+1}$ are real numbers and $\alpha_1, \ldots, \alpha_n$ are $\mathcal{C}$-measurable random variables. Then $\int_0^T Y_s\,dW_s = \sum_{i=1}^n (W_{s_{i+1}\wedge T} - W_{s_i\wedge T})\alpha_i$. Fix $\omega \in \Omega$. If we set $g : \mathbb{D}^1 \to \mathbb{R}$, $\bar\omega \mapsto \sum_{i=1}^n (\bar\omega_{s_{i+1}\wedge T} - \bar\omega_{s_i\wedge T})\alpha_i(\omega)$, then

$$\begin{aligned} P^{\int_0^T Y\,dW|\mathcal{C}}(\omega) &= (P^W)^g \\ &= \ast_{i=1}^n N(0, ((s_{i+1}\wedge T) - (s_i \wedge T))(\alpha_i(\omega))^2) \\ &= N\Big(0, \int_0^T Y_s^2(\omega)\,ds\Big), \end{aligned}$$

where the asterisk denotes convolution.

Second step: For Y as in the assertion, there exists a sequence $(Y^k)_{k\in\mathbb{N}}$ of processes as in the first step with $Y^k_t \to Y_t$ uniformly on $[0,T]$ P-almost surely and $\int_0^T Y^k_s\,dW_s \to \int_0^T Y_s\,dW_s$ P-almost surely for $k \to \infty$ (cf. JS, I.4.44). Using the dominated convergence theorem, we obtain $\int f(x)\,P^{\int_0^T Y^k_s\,dW_s|\mathcal{C}}(dx) = E(f(\int_0^T Y^k_s\,dW_s)|\mathcal{C}) \stackrel{k\to\infty}{\longrightarrow} E(f(\int_0^T Y_s\,dW_s)|\mathcal{C}) = \int f(x)\,P^{\int_0^T Y_s\,dW_s|\mathcal{C}}(dx)$ P-almost surely for any bounded, continuous function $f : \mathbb{R} \to \mathbb{R}$. This means that $P^{\int_0^T Y^k_s\,dW_s|\mathcal{C}}(\omega)$ converges for P-almost all $\omega \in \Omega$ weakly to

$P^{\int_0^T Y_s\,dW_s|\mathcal{C}}(\omega)$. On the other hand, we have $\int_0^T (Y_s^k)^2\,ds \overset{k\to\infty}{\longrightarrow} \int_0^T (Y_s)^2\,ds$ P-almost surely, which implies that $N(0, \int_0^T (Y_s^k(\omega))^2\,ds) \overset{k\to\infty}{\longrightarrow} N(0, \int_0^T (Y_s(\omega))^2\,ds)$ weakly for P-almost all $\omega \in \Omega$ (by the continuity theorem, cf. Billingsley (1978), Theorem 26.3). The first step and the uniqueness of the weak limit now yield the claim. □

PROOF OF LEMMA 4.17. Note that the conditions 1–5 in Theorem 3.36 depend on the particular model and have to be checked.

By Theorem 3.36 we have that $Z_t^2 = E^*(g(Z_T^1)|\mathcal{F}_t) = \int g(\bar{\omega}_{T-t}^1)\, P^{*(Z_{t+s}^1,\sigma_{t+s})_{s\in\mathbb{R}_+}|\mathcal{F}_t}(d(\bar{\omega}^1,\bar{\omega}^2))$ P-almost surely for any $t \in [0,T]$. Note that Z^1 and σ are P^*-almost surely positive because P and P^* are equivalent. Fix $(\omega,t) \in \Omega \times [0,T]$. By the argument in Remark 2 there exists a probability space $(\check{\Omega}, \check{\mathcal{F}}, (\check{\mathcal{F}}_s)_{s\in\mathbb{R}_+}, \check{P})$ and adapted processes $\check{W}, \check{\sigma}$ on that space such that $\check{W}$ is a standard Wiener process and $\check{\sigma}$ solves the SDE (4.8) with $\check{\sigma}_0 = \sigma_t(\omega)$ instead of x. It is easy to show that one can choose the space such that it also supports another standard Wiener process $\overline{W}$ being independent of $(\check{W}, \check{\sigma})$. Now define the process $\check{Z}^1$ on that space by

$$\check{Z}_s^1 = Z_t^1(\omega)\, \mathscr{E}\left(\rho \int_0^{\cdot} \check{\sigma}_u\, d\check{W}_u + \sqrt{1-\rho^2}\int_0^{\cdot} \check{\sigma}_u\, d\overline{W}_u\right)_s$$

One easily shows that $(\check{Z}_s^1, \check{\sigma}_s)_{s\in\mathbb{R}_+}$ is a solution process to the martingale problem $(\varnothing, \varepsilon_{(Z_t^1(\omega),\sigma_t(\omega))}, \check{b}, \check{c}, 0, 0)^M$, which, by assumption, has a unique solution measure. Moreover, application of Lemma 2.33 yields that $P^{*(Z_{t+s}^1,\sigma_{t+s})_{s\in\mathbb{R}_+}|\mathcal{F}_t}(\omega)$ is a solution-measure to the same martingale problem (for the condition concerning $(\Omega,\mathcal{F})$ cf. Condition 5 in Theorem 3.36 and Remark 2 following Lemma 2.33). Hence, we obtain

$$\begin{aligned}
Z_t^2(\omega) &= \int g(\bar{\omega}_{T-t}^1)\, \check{P}^{*(\check{Z}_s^1,\check{\sigma}_s)_{s\in\mathbb{R}_+}}(d(\bar{\omega}^1,\bar{\omega}^2)) \\
&= \check{E}\Bigg(g\Bigg(Z_t^1(\omega)\exp\left(\rho\int_0^{T-t}\check{\sigma}_u\,d\check{W}_u - \frac{1}{2}\rho^2\int_0^{T-t}\check{\sigma}_u^2\,du\right) \\
&\qquad \exp\left(\sqrt{1-\rho^2}\int_0^{T-t}\check{\sigma}_u\,d\overline{W}_u - \frac{1}{2}(1-\rho^2)\int_0^{T-t}\check{\sigma}_u^2\,du\right)\Bigg)\Bigg)
\end{aligned}$$

By Proposition 4.21 we have that $\check{P}^{\int_0^{T-t}\check{\sigma}_u\,d\overline{W}_u|\sigma(\check{\sigma},\check{W})} = N(0, \int_0^{T-t}\check{\sigma}_u^2\,du)$ P-almost surely. In view of the definition of C_{BS}, this implies that

$$\begin{aligned}
&Z_t^2(\omega) \\
&= \int C_{BS}\left(Z_t^1(\omega)\exp\left(\rho\int_0^{T-t}\check{\sigma}_u\,d\check{W}_u - \frac{1}{2}\rho^2\int_0^{T-t}\check{\sigma}_u^2\,du\right), \sqrt{1-\rho^2}\int_0^{T-t}\check{\sigma}_u^2\,du\right) d\check{P} \\
&= C_{bd}(Z_t^1(\omega), \sigma_t(\omega), t),
\end{aligned}$$

which yields the claim. □

PROOF OF REMARK 4. Note that this remark only makes sense if the mapping $v \mapsto C_{BS}(z,v)$ is twice differentiable and the second order Taylor approximation

$$C_{BS}(z,v) \approx C_{BS}(z,\Sigma) + (v-\Sigma)D_2C_{BS}(z,\Sigma) + \frac{1}{2}(v-\Sigma)^2 D_{22}C_{BS}(z,\Sigma)$$

is reasonably good for the actual values of $v = \int_0^{T-t} \check{\sigma}_s^2\, ds$. The claim now follows immediately from $C_{bd}(z,x,t) = E(C_{BS}(z, \int_0^{T-t} \check{\sigma}_s^2\, ds))$. □

PROOF OF LEMMA 4.18. Firstly, note that we assume here without proof that the conditions in Theorem 3.36 hold for the model under consideration, and that the mapping C_{bd} is of class C^2.

By application of Itô's formula (cf. Theorem 2.25) to the process $(t, Z_t^1, \sigma_t)_{t\in\mathbb{R}_+}$ and the mapping $f : [0,T) \times \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}^4$, $(t,z,x) \mapsto (1, z, C_{bd}(z,x,t), x)$, we obtain for the extended characteristics $(\varnothing, \varepsilon_{(1,Z_0^1,Z_0^2,\sigma_0)}, \overline{b}, \overline{c}, 0, 0)^E$ of $\overline{Z} = (Z^0, Z^1, Z^2, \sigma)$

$$\overline{c}_t^{12} = D_1C_{bd}(Z_t^1, \sigma_t, t)(\sigma_t Z_t^1)^2 + D_2C_{bd}(Z_t^1, \sigma_t, t)\rho\sigma_t^2\beta(\sigma_t)Z_t^1 \tag{4.12}$$

$$\overline{c}_t^{23} = D_2C_{bd}(Z_t^1, \sigma_t, t)(\sigma_t\beta(\sigma_t))^2 + D_1C_{bd}(Z_t^1, \sigma_t, t)\rho\sigma_t^2\beta(\sigma_t)Z_t^1 \tag{4.13}$$

for any $t \in [0,T)$. By Lemma 4.1 the u_κ-optimal strategy for the hedging problem $\varphi_t^2 = -1$ is given by $\varphi_t^1 = \frac{\overline{c}_t^{12}}{\overline{c}_t^{11}} + \frac{1}{\kappa}\frac{\overline{b}_t^1}{\overline{c}_t^{11}}$, which is of the form in Lemma 4.18. To be very strict, Itô's formula can only be applied if f is defined and C^2 on $\mathbb{R}^3$. The way out is to argue by localization as in the proof of Corollary 4.7. □

PROOF OF THE REMARK. Note that this remark only makes sense if the mapping $(z,v) \mapsto C_{BS}(z,v)$ is of class $C^{1,2}$. The approximation for φ^1 then follows by differentiation. □

PROOF OF LEMMA 4.19. Note that the assumptions leading to Definition 3.43 and to Lemma 4.17 depend on the particular model and have to be checked.

Similarly as in the proof of Lemma 4.18, one verifies that the strategy φ in step 3 on page 123 is given by $\varphi_t^1 = -\overline{\rho}^2(D_1C_{bd}(Z_t^1,\sigma_t,t) + \rho\frac{\beta(\sigma_t)}{Z_t^1}D_2C_{bd}(Z_t^1,\sigma_t,t)) + \frac{1}{\kappa Z_t^1}\frac{\mu(\sigma_t)}{\sigma_t^2}$ for any $t \in [0,T]$. By step 5 on page 123, the density process $\widetilde{L}^T$ of $\widetilde{P}$ is given by $\widetilde{L}^T = \mathscr{E}(-\int_0^\cdot(\kappa\varphi_s^1)\, d(Z^{1,c})_s^T - \int_0^\cdot(\kappa\overline{\rho}^2)\, d(Z^{2,c})_s^T)$. We may therefore apply Lemma 2.27 and Girsanov's Theorem 2.26 to obtain the extended $\widetilde{P}$-characteristics of (Z^0, Z^1, Z^2, σ) and hence (Z^0, Z^1, σ). Note that $\widehat{b}^0 = \widehat{b}^1 = 0$ already follows from the fact that (Z^0, Z^1) are $\widetilde{P}$-local martingales (cf. step 7 on page 123). Moreover, we conclude from (4.12), (4.13) and (2.8) that

$$\begin{aligned}\widehat{b}_t^2 \;&=\; b_t^2 - \kappa\varphi_t^1\rho\sigma_t^2\beta(\sigma_t)Z_t^1 \\ &\quad - \kappa\overline{\rho}^2\Big(D_2C_{bd}(Z_t^1,\sigma_t,t)(\sigma_t\beta(\sigma_t))^2 + D_1C_{bd}(Z_t^1,\sigma_t,t)\rho\sigma_t^2\beta(\sigma_t)Z_t^1\Big)\end{aligned}$$

for any $t \in [0,T]$. (Observe that, here, the superscript 2 corresponds to σ, whereas in (4.12), (4.13), this was the case for the superscript 3.) Sorting terms yields the claim. $\square$

PROOF OF LEMMA 4.20. Note that the assumptions mentioned in the proof of Lemma 4.19 must be shown for any particular model.

By Definition 3.43 we have that $\widetilde{Z}^2_t = \widetilde{E}(g(Z^1_T)|\mathcal{F}_t) = \int g(\bar{\omega}^1_{T-t})\, \widetilde{P}^{(Z^1_{t+s},\sigma_{t+s})_{s\in\mathbb{R}_+}|\mathcal{F}_t}(d(\bar{\omega}^1,\bar{\omega}^2))$ P-almost surely for any $t \in [0,T]$. Note that Z^1 and σ are $\widetilde{P}$-almost surely positive because P and $\widetilde{P}$ are equivalent. Fix $(\omega,t) \in \Omega \times [0,T]$. The application of Lemma 2.33 yields that $\widetilde{P}^{(Z^1_{t+s},\sigma_{t+s})_{s\in\mathbb{R}_+}|\mathcal{F}_t}(\omega)$ is a solution-measure to the martingale problem $(\varnothing, \varepsilon_{(Z^1_t(\omega),\sigma_t(\omega))}, \breve{b}, \breve{c}, 0, 0)^M$. Since this martingale problem is assumed to have a unique solution-measure, the claim follows. $\square$

4.8 Keller's Model

This section is an exception in that we do not present any pricing measure, derivative prices, or hedging strategies. We only want to show that advanced models incorporating a number of features of real financial time series can often be easily expressed in terms of extended characteristics and hence within the framework of Chapter 3. However, explicit numerical calculations of prices, strategies etc. are rarely easily produced in models with complicated dependence structures and are beyond our scope here.

As an example we consider a market model by Keller (1997), Subsection 4.4.1, which consists, as in the previous sections, of a bank account S^0 and a stock S^1. Since real markets are often closed at night, no trade takes place in this period. In Keller's continuous-time model this is taken into account by shrinking the nights to intervals of length zero at integer times. Hence, an overnight price change corresponds to a jump at the respective integer time. The money market account is given by $S^0_t = \exp(r_d t + r_n[t])$, where $r_d, r_n \in \mathbb{R}$ are the *intraday* and the *overnight interest rate*, respectively. Similarly, the stock price process satisfies $S^1_t = S^1_0 \exp(R^d_t + R^n_t)$, where R^d is the *intraday return process*. The *overnight return process* R^n is assumed to be of the form $R^n_t = \sum_{k=1}^{[t]} \Delta R^n_k$, where $(\Delta R^n_k)_{k\in\mathbb{N}^*}$ is a sequence of i.i.d. random variables whose distribution Q satisfies $\int e^{|x|}\,Q(dx) < \infty$. During business hours the stock price jumps randomly at random times. More specifically, let $R^d := \sum_{l\in\mathbb{N}} 1_{\{\tau_l \le t\}}\xi_l$, where $(\tau_l)_{l\in\mathbb{N}}$ is an increasing sequence of stopping times with $\tau_l \uparrow \infty$ P-almost surely and $(\xi_l)_{l\in\mathbb{N}}$ a sequence of $\mathbb{R}\setminus\{0\}$-valued random variables. Conditionally on the past, the distribution of jump times and sizes is given by

$$P^{(\tau_{l+1},\xi_{l+1})|(\tau_m,\xi_m)_{m\in\{0,\dots,l\}}} = (\mathrm{Exp}_{\varphi_{l+1}} * \varepsilon_{\tau_l}) \otimes N(0, h_{l+1})$$

for any $l \in \mathbb{N}$, where $\mathrm{Exp}_{\varphi_{l+1}}$ denotes an exponential distribution with parameter φ_{l+1} and the processes $(h_l)_{l\in\mathbb{N}}$, $(\varphi_l)_{l\in\mathbb{N}}$ are recursively defined by

$$\begin{aligned} h_0 &:= 0, \quad h_{l+1} := v_0 + \alpha\xi_l^2 + \beta h_l, \\ \varphi_0 &:= 0, \quad \varphi_{l+1} := \lambda_0 + \gamma(\tau_l - \tau_{l-1}) + \delta\varphi_l \end{aligned}$$

for some fixed constants $v_0, \lambda_0 \in \mathbb{R}_+^*$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}_+$, $\xi_0 := 0$, $\tau_{-1} := \tau_0 := 0$. Moreover, R_n and $(\tau_l, \xi_l)_{l\in\mathbb{N}}$ are assumed to be independent. In this market model the return process exhibits normally distributed jumps seperated by exponential waiting times. The activity or volatility of the market is reflected firstly by the variance h_l of the jump height and secondly by the parameter φ_l of the distribution for the waiting time $\tau_l - \tau_{l-1}$ between successive jumps. The recursive definition of these parameters intuitively means that periods of high resp. low activity are likely to persist. This is the type of observed market behaviour that also led to the development of ARCH- and GARCH-models. The following lemma shows how to express this market model in the language of Chapter 2.

Lemma 4.22 *As usual, we define the discounted price processes by* $Z^0 := S^0/S^0 = 1$, $Z^1 := S^1/S^0$. *Assume that the filtration of the underlying stochastic basis is the canonical filtration of* S^1 *(or equivalently* (S^0, S^1), (Z^0, Z^1), Z^1*). Then* $Z := (Z^0, Z^1)$ *is an extended Grigelionis process whose extended characteristics* $(\mathbb{N}^*, \varepsilon_{(1,S_0^1)}, b, 0, F, K)^E$ *are given by*

$$K_t(G) = \int 1_G(0, Z^1_{t-}(\exp(x - r_n) - 1))\, Q(dx),$$

$$F_t(G) = \sum_{l\in\mathbb{N}} 1_{]\tau_l,\tau_{l+1}]}(t)\varphi_{l+1} \int 1_G(0, Z^1_{t-}(e^x - 1))\, N(0, h_{l+1})(dx),$$

$$b^0_t = 0,$$

$$b^1_t = \int x^1\, F_t(d(x^1, x^2)) - Z^1_{t-} r^d$$

for any $t \in \mathbb{R}_+$, $G \in \mathcal{B}^2$.

Proofs

PROOF OF LEMMA 4.22. *First step:* We will show that R^d is an extended Grigelionis process with extended characteristics $(\mathbb{N}^*, \varepsilon_0, b^d, 0, F^d, 0)^E$, where

$$\begin{aligned} F^d_t(G) &:= \sum_{l\in\mathbb{N}} 1_{]\tau_l,\tau_{l+1}]}(t)\, \varphi_{l+1}\, N(0, h_{l+1})(G), \\ b^d_t &:= \int x\, F^d_t(dx) \end{aligned}$$

for any $t \in \mathbb{R}_+$, $G \in \mathcal{B}$.

By definition we have that $R^d = x * \mu$, where the random measure μ is given by $\mu(ds, dx) := \sum_{l\in\mathbb{N}} 1_{\{\tau_l<\infty\}}\, \varepsilon_{(\tau_l,\xi_l)}(ds, dx)$. Since $\xi_l \neq 0$, it follows that the smallest filtration $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$ for which μ is optional (in the sense of JS, III.1.25) coincides with the canonical filtration of R^d. Relative to this filtration, the compensator ν of μ is given by

$$\begin{aligned} &\nu([0, t] \times G) \\ &= \sum_{l\in\mathbb{N}} \iint_0^t \frac{1_{[0,\tau_{l+1}]}(s)\, 1_G(x)}{(\mathrm{Exp}_{\varphi_{l+1}} * \varepsilon_{\tau_l})([s, \infty))} (\mathrm{Exp}_{\varphi_{l+1}} * \varepsilon_{\tau_l})(ds)\, N(0, h_{l+1})(dx) \end{aligned}$$

for any $t \in \mathbb{R}_+$, $G \in \mathcal{B}$ (cf. JS, III.1.33). Since $(\mathrm{Exp}_{\varphi_{l+1}} * \varepsilon_{\tau_l})(A) = \int_A 1_{[\tau_l,\infty)}(s)\varphi_{l+1} \exp(-\varphi_{l+1}(s-\tau_l))\,ds$, it follows that $\nu([0,t]\times G) = \int_0^t F_s^d(G)\,ds$ for any $t \in \mathbb{R}_+$, $G \in \mathcal{B}$. Therefore we can write R^d as

$$R_t^d = \int_0^t \int x\, F_s^d(dx)\,ds + x * (\mu - \nu)_t,$$

where the first term is predictable and of finite variation. Since μ is the measure of jumps of R^d, we can conclude that, relative to $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$, R^d is an extended Grigelionis process with the above characteristics. Since R^d and R^n are independent, it follows from Lemma 2.23 that this is also true relative to the canonical filtration of Z^1 (which is generated by R^d and R^n).

Second step: By Lemma 2.20, R^n is an extended Grigelionis process with extended characteristics $(\mathbb{N}^*, \varepsilon_0, 0, 0, 0, Q)^E$.

Third step: Note that $Z_t^0 = 1$ and $Z_t^1 = S_0^1 \exp(R^d + R^n - r^d t - r^n[t])$ for any $t \in \mathbb{R}_+$. We will now apply Itô's formula (cf. Theorem 2.25) to the extended Grigelionis process $Y = (t, [t], R_t^d, R_t^n)_{t\in\mathbb{R}_+}$ and the mapping $f : \mathbb{R}^4 \to \mathbb{R}^2$, $(y_1, y_2, y_3, y_4) \mapsto (1, S_0^1 \exp(y_3 + y_4 - r^d y_1 - r^n y_2))$. Straightforward calculations yield that the extended characteristics of $(Z^0, Z^1) = f(Y)$ are indeed of the claimed form.

It remains to check the integrability conditions in Remark 1 following Theorem 2.25 to make sure that $f(Y)$ is a special semimartingale. Firstly note that $\int |x|\, K_t(dx)$ is finite for any $t \in \mathbb{N}^*$ because $\int e^{|x|}\, Q(dx) < \infty$. Moreover, the local boundedness of Z^1 and $\int e^{|x|}\, N(0,\sigma^2)(dx) < \infty$ for any $\sigma \in \mathbb{R}_+$ imply that $\int_0^{\tau_l} \int |x|\, F_s(dx)\,ds < \infty$ P-almost surely for any $l \in \mathbb{N}$. Since by assumption $\tau_l \uparrow \infty$ P-almost surely, we have in particular $\int_0^t \int (|x|^2 \wedge |x|)\, F_s(dx)\,ds < \infty$ and the proof is complete. $\square$

4.9 Interest Rate Models

Practically all the examples considered so far consist of or are inspired by stock price models. This is not to suggest that our approach only works or is mainly aimed at this kind of market. On the contrary, the general framework in Chapter 3 gives no preference to any particular kind of security. To demonstrate this we consider now short-term interest rate models and their implications on zero-coupon bond prices. More specifically, we focus on the Vasicek and the Cox-Ingersoll-Ross model (cf. Björk (1997), Section 3). The setting is as follows. The only underlying in our market is a short-term fixed-income investment $S_t^0 = \exp(\int_0^t r_s\,ds)$ (i.e. satisfying $dS_t^0 = S_t^0 r_t\,dt$), which will also serve as the numeraire. In contrast to the previous sections, the instantaneous interest rate is now a continuous stochastic process $(r_t)_{t\in\mathbb{R}_+}$, which is assumed to be a solution to the diffusion-type SDE

$$dr_t = \mu(r_t)\,dt + \sigma(r_t)\,dW_t, \tag{4.14}$$

where $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}_+$ are given continuous functions, W denotes a standard Wiener process and $r_0 \in \mathbb{R}$ is fixed. If we let $\mu(r_t) := \beta(\vartheta - r_t)$ and $\sigma(r_t) := \sigma$

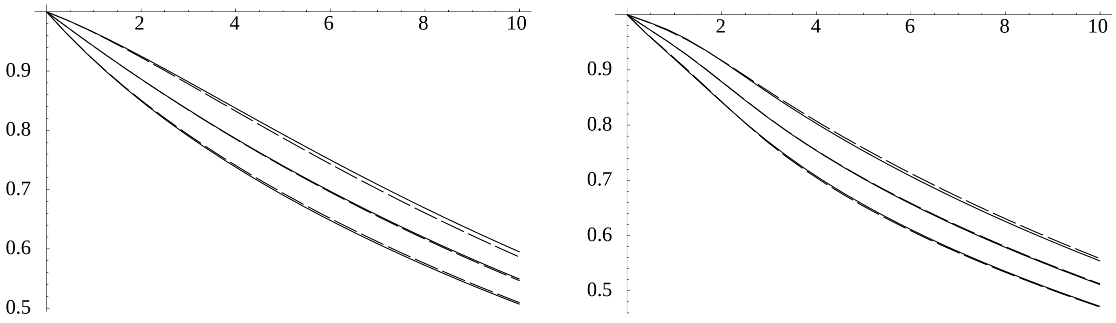

Figure 4.41: Bond prices without and with external supply

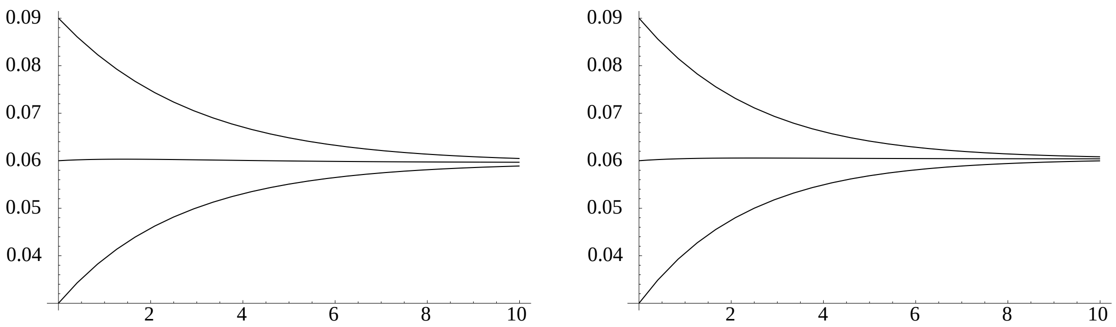

Figure 4.42: Forward rates without external supply

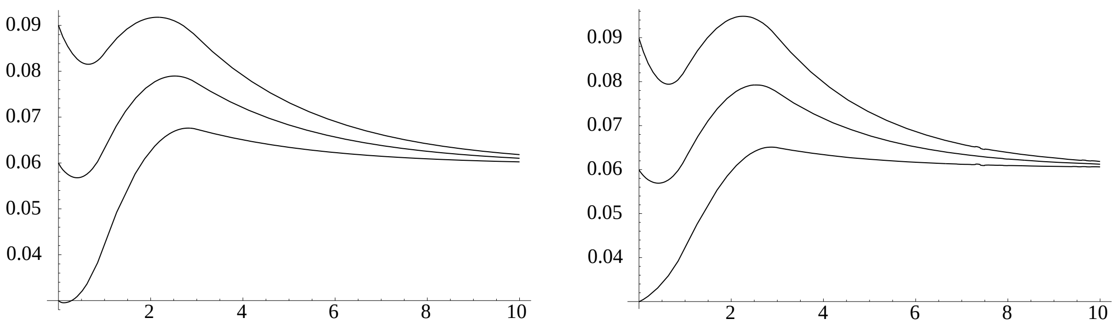

Figure 4.43: Forward rates with external supply

for some $\beta, \vartheta, \sigma \in \mathbb{R}_+$, we obtain the *Vasiček model*. In this case, the solution to the SDE (4.14) is a shifted Ornstein-Uhlenbeck process. If, on the other hand, the coefficients are chosen as $\mu(r_t) := \alpha - \beta r_t$ and $\sigma(r_t) := \sigma\sqrt{r_t \vee 0}$ for some $\alpha, \beta \in \mathbb{R}_+$, $\sigma \in \mathbb{R}_+^*$, then Equation (4.14) yields the so-called *Cox-Ingersoll-Ross model*. The discounted numeraire is, as usually, given by $Z^0 := S^0/S^0 = 1$. The stochastic process $(r_t)_{t\in\mathbb{R}_+}$ is, by Lemma 2.22, an extended Grigelionis process whose extended characteristics $(\varnothing, \varepsilon_{r_0}, b, c, 0, 0)^E$ are given by $b_t = \mu(r_t)$, $c_t = (\sigma(r_t))^2$ for any $t \in \mathbb{R}_+$.

Remark. In the Vasicek- and the Cox-Ingersoll-Ross case the SDE (4.14) and equivalently the corresponding martingale problem in the sense of Theorem 2.30 has a unique solution-measure. In the Cox-Ingersoll-Ross model the solution always stays positive, whereas this is not the case in the Vasicek model.

4.9.1 Pricing of Zero Coupon Bonds

Since the numeraire Z^0 is the only security in the market, one easily sees that Conditions 1–4 in Theorem 3.36 are met and that the pricing measure equals the given probability measure P. Assume from now on that the filtration of the underlying stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ is the canonical filtration of S^0 (or equivalently r) or its P-completion. By Theorem 2.65 this implies that Condition 5 in Theorem 3.36 holds as well. To us, *zero-coupon bonds* are securities yielding a payout 1 at some future time t. As their discounted terminal value is given by the $\mathcal{F}_t$-measurable random variable $(S_t^0)^{-1}$, we can treat them as derivatives in the sense of Section 3.4. Since the dynamic of the model is unchanged under the pricing measure, we obtain the well-known zero-coupon bond price formulas as shown in the following

Lemma 4.23 *Fix $t_1, \ldots, t_n \in \mathbb{R}$ and denote by $X^i := (S_{t_i}^0)^{-1}$ the discounted terminal payout of a zero-coupon bond maturing at time t_i for $i = 1, \ldots, n$. Then we have*

1. *In the Vasiček model the processes $Z^1, \ldots, Z^n$ (resp. $S^1, \ldots, S^n$ in undiscounted terms), defined by*

$$S_t^i = S_t^0 Z_t^i := \exp(-(t_i - t)R(t_i - t, r_t)) \tag{4.15}$$

   for any $i \in \{1, \ldots, n\}$, $t \in [0, t_i]$, are neutral derivative price processes for the bonds maturing at times $t_1, \ldots, t_n$, where

$$R(\tau, r) := \left(\vartheta - \frac{\sigma^2}{2\beta^2}\right) - \frac{1}{\beta\tau}\left(\left(\vartheta - \frac{\sigma^2}{2\beta^2} - r\right)(1 - e^{-\beta\tau}) - \frac{\sigma^2}{4\beta^2}(1 - e^{-\beta\tau})^2\right)$$

   for any $\tau \in \mathbb{R}_+$, $r \in \mathbb{R}$. Moreover, $Z^1, \ldots, Z^n$ are the only neutral price processes such that the market $\overline{Z} := (Z^0, \ldots, Z^n)$ meets regularity condition (RC 1).

2. *In the Cox-Ingersoll-Ross model there are unique neutral bond price processes Z^1, $\ldots, Z^n$ (resp. $S^1, \ldots, S^n$ in undiscounted terms) for the bonds maturing at times*

$t_1, \ldots, t_n$. *These are given by*

$$\begin{aligned} S_t^i = S_t^0 Z_t^i \quad := \quad & \exp\left(\frac{2\alpha}{\sigma^2}\log\left(\frac{2\gamma\exp(\frac{1}{2}(t_i - t)(\gamma+\beta))}{\gamma-\beta+(\gamma+\beta)\exp(\gamma(t_i-t))}\right)\right. \\ & \left. - r_t \frac{2(\exp(\gamma(t_i-t))-1)}{\gamma-\beta+(\gamma+\beta)\exp(\gamma(t_i-t))}\right), \end{aligned}$$

where $\gamma := \sqrt{\beta^2+2\sigma^2}$. *Moreover, the extended market* $\overline{Z} := (Z^0, \ldots, Z^n)$ *meets regularity condition (RC 1).*

4.9.2 Improved Bond Pricing

Approximate derivative prices in the sense of Sections 3.5 and 3.6 are based on alternative pricing measures. Since these are equivalent to the original probability measure P, Girsanov's theorem yields that, relative to these distributions, the extended characteristics of r are of the form $(\varnothing, \varepsilon_{r_0}, \widetilde{b}, c, 0, 0)^E$ for some drift process $\widetilde{b}$. The fact that this new drift $\widetilde{b}_t$ is no more necessarily a deterministic function of r complicates explicit numerical computations. Of course the same is true for consistent derivative prices in the sense of Sections 3.5 and 3.6, but here we face the additional and more serious obstacle that we do not yet know how to obtain these prices at all in a continuous-time setting. Therefore it is surprising that, under assumptions that are close to $(\kappa, \rho^1, \ldots, \rho^n)$-consistency, we end up with a simple dynamic of the short-term interest rate under the corresponding pricing measure, namely the *Hull-White model* (cf. Björk (1997), Section 3). The general setting is as follows. Fix $t_1, \ldots, t_n \in \mathbb{R}_+$ and $X^1, \ldots, X^n$ as in Lemma 4.23. Moreover, let $\kappa > 0$, $\rho^1, \ldots, \rho^n \in \mathbb{R}$. Instead of considering consistent or approximate price processes corresponding to constant external supply $\rho^1, \ldots, \rho^n$, we focus on stochastic external supply $\rho^1/\widetilde{Z}_t^1, \ldots, \rho^n/\widetilde{Z}_t^n$ in the sense of Remark 3 following Definition 3.47, where $\widetilde{Z}^1, \ldots, \widetilde{Z}^n$ are the still unknown bond prices or the neutral processes from Lemma 4.23. This looks like a very different thing, but in fact is not. The discounted bonds are generally securities of very low volatility and drift. Therefore, $\widetilde{Z}_t^i$ should usually be a quite good approximation of the initial bond price $\widetilde{Z}_0^i$. So $(\kappa, \rho^1/\widetilde{Z}_t^1, \ldots, \rho^n/\widetilde{Z}_t^n)$-consistent (resp. approximate) prices correspond to an external supply that is not exactly deterministic and constant, but nevertheless remains close to the fixed vector $(\kappa, \rho^1/\widetilde{Z}_0^1, \ldots, \rho^n/\widetilde{Z}_0^n) \in \mathbb{R}^n$. The reason for this approximation becomes apparent in the following

Lemma 4.24 *Let* $T := \sup\{t_1, \ldots, t_n\}$.

1. *Suppose we work with the Vasi cek model for the short rate. Define the mapping* $\widetilde{\vartheta} : \mathbb{R}_+ \to \mathbb{R}$ *by*

$$\widetilde{\vartheta}(t) := \vartheta + \kappa\frac{\sigma^2}{\beta^2}\sum_{i=1}^n \rho^i 1_{[0,t_i)}(t)(1-e^{-\beta(t_i-t)})$$

and processes $\widetilde{S}^i = \widetilde{Z}^i S^0$ *for* $i = 1\ldots, n$ *by*

$$\widetilde{S}_t^i := \exp\left(\int_t^{t_i}(1-e^{-\beta(t_i-s)})\left(\frac{\sigma^2}{2\beta^2}(1-e^{-\beta(t_i-s)})-\widetilde{\vartheta}(s)\right)ds - r_t\frac{1}{\beta}(1-e^{-\beta(t_i-s)})\right)$$

for any $t \in [0, t_i]$ *and* $\widetilde{Z}^i_t := \widetilde{Z}^i_{t_i}$ *for any* $t \in [t_i, \infty)$*. Then we have*

(a) $(\widetilde{Z}^1, \dots, \widetilde{Z}^n)$ *are* $(\kappa, \rho^1/\widetilde{Z}^1_t, \dots, \rho^n/\widetilde{Z}^n_t)$*-consistent bond price processes for the zero-coupon bonds maturing at* $t_1, \dots, t_n$*. Moreover, the market* $(Z^0, \widetilde{Z}^1, \dots, \widetilde{Z}^n)$ *meets regularity condition (RC 1).*

(b) *By* $\frac{dP^*}{dP} := L_\infty$ *with* $L := \mathscr{E}(\frac{\beta}{\sigma^2}\int_0^\cdot(\widetilde{\vartheta}(t) - \vartheta)\, dr^c_t)$ *we define a probability measure* $P^* \sim P$*. With this choice* $\widetilde{Z}^1, \dots, \widetilde{Z}^n$ *are* P^**-martingales.*

(c) *Relative to* P^**,* $(r_t)_{t\in\mathbb{R}_+}$ *is an extended Grigelionis process whose extended characteristics* $(\varnothing, \varepsilon_{r_0}, \widetilde{b}, c, 0, 0)^E$ *are given by* $\widetilde{b}_t = \beta(\widetilde{\vartheta}(t) - r_t)$ *for any* $t \in \mathbb{R}_+$*.*

(d) $(\widetilde{Z}^1, \dots, \widetilde{Z}^n)$ *are also* $(\kappa, \rho^1/Z^1, \dots, \rho^n/Z^n)$*-approximate bond price processes for the zero-coupon bonds maturing at* $t_1, \dots, t_n$*, where* $Z^1, \dots, Z^n$ *here denote the processes from Statement 1 in Lemma 4.23.*

2. *Suppose we work with the Cox-Ingersoll-Ross model for the short rate* r*. Define the mapping* $\widetilde{\beta} : \mathbb{R}_+ \to \mathbb{R}$ *by*

$$\widetilde{\beta}(t) := \beta - \kappa\sigma^2 \sum_{i=1}^n \rho^i 1_{[0,t_i)}(t) \frac{2(\exp(\gamma(t_i - t)) - 1}{\gamma - \beta + (\gamma + \beta)\exp(\gamma(t_i - t))}$$

and processes $\widetilde{S}^i = \widetilde{Z}^i S^0$ *for* $i = 1 \dots, n$ *by*

$$\widetilde{S}^i_t := \exp(A^i(t) - B^i(t) r_t)$$

for any $t \in [0, t_i]$ *and* $\widetilde{Z}^i_t := \widetilde{Z}^i_{t_i}$ *for any* $t \in [t_i, \infty)$*, where the function* $B^i : [0, T] \to \mathbb{R}$ *is the unique solution to the initial value problem*

$$B^i(T) = 0, \;\; dB^i(t) = \left(\widetilde{\beta}(t) B^i(t) + \frac{1}{2}\sigma^2 (B^i(t))^2 - 1\right) dt \tag{4.16}$$

and the mapping $A^i : [0, T] \to \mathbb{R}$ *is given by*

$$A^i(t) := -\int_t^T \alpha B^i(s)\, ds$$

for any $i \in \{1, \dots, n\}$*,* $t \in [0, T]$*. Then we have*

(a) $(\widetilde{Z}^1, \dots, \widetilde{Z}^n)$ *are* $(\kappa, \rho^1/Z^1_t, \dots, \rho^n/Z^n_t)$*-approximate bond price processes for the zero-coupon bonds maturing at* $t_1, \dots, t_n$*, where* $Z^1, \dots, Z^n$ *denote the processes from Statement 2 in Lemma 4.23. Moreover, the market* $(Z^0, \widetilde{Z}^1, \dots, \widetilde{Z}^n)$ *meets regularity condition (RC 1).*

(b) *By* $\frac{dP^*}{dP} := L_\infty$ *with* $L := \mathscr{E}(\frac{1}{\sigma^2}\int_0^\cdot(\beta - \widetilde{\beta}(t))\, dr^c_t)$ *we define a probability measure* $P^* \sim P$*. With this choice* $\widetilde{Z}^1, \dots, \widetilde{Z}^n$ *are* P^**-martingales.*

(c) *Relative to* P^**,* $(r_t)_{t\in\mathbb{R}_+}$ *is an extended Grigelionis process whose extended characteristics* $(\varnothing, \varepsilon_{r_0}, \widetilde{b}, c, 0, 0)^E$ *are given by* $\widetilde{b}_t = \alpha - \widetilde{\beta}(t) r_t$ *for any* $t \in \mathbb{R}_+$*.*

The short-term interest rate dynamic under the pricing measure P^* is basically the same as relative to P, but with time-dependent drift parameters $\widetilde{\vartheta}(t)$, $\widetilde{\beta}(t)$ instead of the fixed values ϑ, β. Therefore, we obtain a special case of the Hull-White extension of the Vasicek- resp. the Cox-Ingersoll-Ross model (cf. Björk (1997), Chapter 3). Since the term structure in Hull-White-type settings is affine, they are computationally well tractable (cf. Björk (1997), Subsection 3.4).

For the explicit construction of an improved bond price model one may now proceed as indicated in Subsection 1.2.6 or Section 3.6 with the definition of $(p^1, \ldots, p^n)$-consistent prices. One observes current bond prices $S_0^1, \ldots, S_0^n$ on the real market and chooses supply parameters $\rho^1, \ldots, \rho^n$ in such a way that the theoretical $(\kappa, \rho^1/Z_t^1, \ldots, \rho^n/Z_t^n)$-approximate prices from Lemma 4.24 and the observed bond prices coincide. This procedure is known as *inverting the yield curve* (cf. Björk (1997), Subsection 3.5).

The common approach to the inversion of the yield curve faces a theoretical problem. The set of all Hull-White-type dynamics under the pricing measure that are consistent with the given Vasicek- or Cox-Ingersoll-Ross model is obtained by substituting deterministic time-dependent drift parameters $\beta(t)(\vartheta(t) - r_t)$ (resp. $\alpha(t) - \beta(t)r_t)$) for the fixed values $\beta(\vartheta - r_t)$ (resp. $\alpha - \beta r_t)$) and letting the diffusion coefficient remain unchanged. Since we are given only a finite number of initial bond prices, there is a great degree of flexibility for the functions β etc., and their actual choice is often made ad-hoc. Our approach, on the other hand, is based on concrete assumptions and the number of free parameters $\rho^1, \ldots, \rho^n$ is equal to that of observable bond prices.

Let us put it another way. Suppose you are looking for a term structure model that is consistent with the observed zero-coupon bond prices. If you believe that the short-term interest rate is well described by the Vasicek model (resp. the Cox-Ingersoll-Ross model), that speculators on the market invest in u_κ-optimal portfolios and that the external supply/demand of any bond is approximately constant through time, then relative to the pricing measure the short-term interest rate dynamic is of the particular Hull-White form given in Lemma 4.24. Note that if you actually invert the yield curve in this manner, the resulting pricing measure, bond prices and drift parameters depend neither on the risk aversion κ nor at all on the choice of the utility function u. Indeed, the only property of u entering the Radon-Nikodým density L_∞ is the risk aversion κ. But arguing as in Definition 3.46, one may assume $\kappa = 1$ w.l.o.g.

In Figures 4.41 – 4.43 we examine the Vasicek- and the Cox-Ingersoll-Ross model numerically. The solid curves in Figure 4.41 correspond to the Vasicek model with parameters $\vartheta = 0.0616$, $\beta = 0.3636$, $\sigma = 0.00229$, whereas the dashed lines are based on a Cox-Ingersoll-Ross model with $\alpha = 0.0254$, $\beta = 0.4105$, $\sigma = 0.0898$. In both cases time is measured in years. The parameter sets have been estimated by Annette Ehret for the same 30-year set of interest rate data. The left-hand diagram in Figure 4.41 shows neutral bond prices as a function of time to maturity in years. The upper, middle, resp. lower curve corresponds to an initial annual interest rate r_0 of 3%, 6%, 9%, respectively. Now we consider a bond market with two bonds maturing at $t_1 = 1$ and $t_2 = 3$ whose external supplies equal

$\rho^1/Z^1 = -1$ and $\rho^2/Z^2 = 0.3$ (relative to $\kappa = 1$). Intuitively speaking, this means that for a period of one year there is a strong net investment in bonds, whereas for the subsequent two years many traders seem to finance themselves by shorting bonds. The resulting $(1, \rho^1/Z^1, \rho^2/Z^2)$-approximate bond prices and, in addition, the neutral bond price of any further bond introduced in this enlarged market $(Z^0, \widetilde{Z}^1, \widetilde{Z}^2)$ are shown in the right-hand diagram of Figure 4.41. Again, the upper, middle, lower curve correspond to an initial interest rate r_0 of 3%, 6%, 9%, respectively.

The differences become much more apparent if we turn to *forward rates*. Strictly speaking, forward rates are only defined in a market with a continuum of bonds for any terminal date, and thus not in our setting with its finite amount of securities. However, in the markets considered above there exists a function $f : \mathbb{R}_+ \to \mathbb{R}$ such that the initial price of any bond that is already in the market or is newly introduced to its neutral price is given by $\exp(-\int_0^t f(s)\,ds)$, where $t \in \mathbb{R}_+$ denotes its maturity. This is in line with the usual definition of forward rates at time 0 (cf. Björk (1997), Section 2). Figure 4.42 shows the forward rates corresponding to the left-hand diagram in Figure 4.41, i.e. to a market where any bond is traded at its neutral price. Here as well as in Figure 4.43, the left-hand diagram belongs to the Vasicek model, whereas the right-hand graph shows Cox-Ingersoll-Ross forward rates. Obviously, the upper curve now relates to the high initial interest rate and vice versa. A comparison of the diagrams shows that the forward rates are very similar, but converge slightly more quickly to an average value in the Cox-Ingersoll-Ross model. In Figure 4.43 the forward rates for the $(1, \rho^1/Z^1, \rho^2/Z^2)$-approximate market $(Z^0, \widetilde{Z}^1, \widetilde{Z}^2)$ are given. They correspond to the right-hand diagram in Figure 4.41 or, in other words, to a bond market with non-zero supply exactly for the two bonds maturing at $t_1 = 1$ and $t_2 = 3$. As one may expect, the forward rate is comparatively small for the period with net supply and high for the subsequent time of excess demand of bonds.

Promoted by the Heath-Jarrow-Morton approach to fixed-income markets, it is very popular to model these with a continuum of bonds for any conceivable maturity. From a theoretical point of view this contradicts our approach, which is based on just the finite number of assets that are really traded in the market. However, it may still be an interesting question to what extent the notions and results from Chapter 3 can be extended to a setting with an infinite number of securities.

Proofs

PROOF OF THE REMARK. For the Vasicek case this follows immediately from Corollary 2.41. The statements for the Cox-Ingersoll-Ross model follow from Ikeda & Watanabe (1989), Example IV.8.2. □

PROOF OF LEMMA 4.23. *1. First step:* By Lamberton & Lapeyre (1996), Subsection 6.2.1 the random variables $X^1, \dots, X^n$ are integrable with expectation $E(X^i) = E(\exp(-\int_0^{t_i} r_s\, ds)) = \exp(-t_i R(t_i, r_0))$ for $i = 1, \dots, n$. Moreover, we have $E(X^i|\mathcal{F}_t) = (S_t^0)^{-1} E(\exp(\int_t^{t_i} r_s\, ds)|\mathcal{F}_t) = (S_t^0)^{-1} \exp(-(t_i - t) R(t_i - t, r_t))$ for any $i \in \{1, \dots, n\}$, $t \in [0, t_i]$. In particular, the processes $Z^1, \dots, Z^n$ defined by Equation (3.11) are continuous semimartingales of the form in Equation (4.15). In particular, the existence conditions in Remark 6 following Theorem 3.36 are met, which implies that these processes are neutral derivative price processes.

Second step: We will now show that the extended market $\overline{Z} = (Z^0, \dots, Z^n)$ meets integrability condition (RC 1). Fix $i, j \in \{1, \dots, n\}$. We define a mapping $f : [0, t_i \wedge t_j) \times \mathbb{R}_+^* \times \mathbb{R} \to \mathbb{R}^2$ of class C^2 by $f(t, x, r) := (\frac{1}{x} \exp(-(t_i - t)R(t_i - t, r)), \frac{1}{x} \exp(-(t_j - t)R(t_j - t, r)))$. For $t < t_i \wedge t_j$, we thus have $(Z_t^i, Z_t^j) = f(t, S_t^0, r_t)$. If $(\varnothing, \varepsilon_{(1, Z_0^1, \dots, Z_0^n)}, \overline{b}, \overline{c}, 0, 0)^E$ denotes the extended characteristics of $\overline{Z}$, then we have $\overline{b} = 0$ since $\overline{Z}$ is a P-local martingale, and $\overline{c}_t^{ij} = 0$ for $t \geq t_i \wedge t_j$ because Z^i is constant on $[t_i, \infty)$. In order to compute $\overline{c}_t^{ij}$ for $t < t_i \wedge t_j$, we apply Itô's formula (cf. Theorem 2.25) to the extended Grigelionis process $(t, S_t^0, r_t)_{t \in \mathbb{R}_+}$ and the mapping f above. This yields

$$\overline{c}_t^{ij} = D_3 f^1(t, S_t^0, r_t) \sigma^2 D_3 f^2(t, S_t^0, r_t) = Z_t^i Z_t^j \frac{\sigma^2}{\beta^2} (1 - e^{-\beta(t_i - t)})(1 - e^{-\beta(t_j - t)})$$

for any $t < t_i \wedge t_j$. Since f is not really defined and C^2 on $\mathbb{R}^3$, and hence Theorem 2.25 is not literally applicable, we refer the reader to the proof of Corollary 4.7 for an exact argumentation by localization. Since $(\overline{c}_t^{ij})^2 \leq \frac{\sigma^4}{\beta^4} (Z_t^i)^2 (Z_t^j)^2$, Schwarz's inequality yields that $E(|\overline{c}_t^{ij}|^2) \leq \frac{\sigma^4}{\beta^4} \sup\{E((Z_s^k)^4) : k \in \{1, \dots, n\}, s \in [0, T]\}$ for any $i, j \in \{1, \dots, n\}$, $t \in \mathbb{R}_+$. Since $Z_t^i = E(X^i | \mathcal{F}_t)$, Jensen's inequality implies that $E((Z_t^i)^4) \leq E((X^i)^4) = E(\exp(-\int_0^t 4r_s \, ds))$. Since $(4r_t)_{t \in \mathbb{R}_+}$ solves the same SDE as $(r_t)_{t \in \mathbb{R}_+}$ but with $4\beta, 4\sigma$ instead of β, σ, it follows from Lamberton & Lapeyre (1996), Subsection 6.2.1 that $E(\exp(-\int_0^{t_i} 4r_s \, ds)) = \exp(-t_i \widetilde{R}(t_i, 4r_0))$, where $\widetilde{R}$ is defined as R in Lemma 4.23, but with $4\vartheta, 4\sigma$ instead of ϑ, σ. As a uniform upper bound, we thus have

$$E((Z_t^i)^4) \leq \exp\left(T\left(4\vartheta + \frac{16\sigma^2}{2\beta^2}\right) + \frac{1}{\beta}\left(4\vartheta + \frac{16\sigma^2}{2\beta^2} + 4|r_0| + \frac{16\sigma^2}{4\beta^2}\right)\right) =: M \in \mathbb{R}_+$$

for any $i \in \{1, \dots, n\}$ and any $t \in \mathbb{R}_+$. The integrability condition (RC 1) now follows easily.

Third step: We will show that there are no further neutral price processes such that the extended market meets regularity condition (RC 1). Otherwise, let $(\widetilde{Z}^1, \dots, \widetilde{Z}^n)$ be such processes. By Corollary 3.23 and the u_κ-optimality of the empty portfolio, it follows that $\widetilde{Z} - \widetilde{Z}_0$ is a local martingale. Since any local martingale has the representation property relative to the continuous process r (cf. Theorem 2.65), it follows that $\widetilde{Z} - \widetilde{Z}_0$ has no discontinuous local martingale part, i.e. $\widetilde{Z} = \widetilde{Z}_0 + \widetilde{Z}^C$. The regularity condition (RC 1) implies that $\langle \widetilde{Z}^{i,c}, \widetilde{Z}^{i,c} \rangle_T =: \int_0^T \widetilde{c}_t^{ii} \, dt$ is integrable and hence $\widetilde{Z}^{i,c}$ is a square-integrable martingale for $i = 1, \dots, n$ (cf. JS, I.4.50). Since $\widetilde{Z}_0^i$ is $\mathcal{F}_0$-measurable and hence deterministic, $\widetilde{Z}^i$ is a martingale as well. It follows that Z^i and $\widetilde{Z}^i$ are martingales with the same terminal random variable X^i, and hence $Z = \widetilde{Z}^i$ for $i = 1, \dots, n$.

2. First step: In the Cox-Ingersoll-Ross model, the interest rate process is positive and hence $0 \leq X^i \leq 1$ for $i = 1, \dots, n$. Therefore, it follows from Theorem 3.36 that there exist unique neutral derivative price processes $Z^1, \dots, Z^n$ which are given by

$$Z_t^i = E(X^i|\mathcal{F}_t) = (S_t^0)^{-1} E\Big(\exp\Big(-\int_t^{t_i} r_s\, ds\Big)\Big|\mathcal{F}_t\Big).$$

The explicit formula for the expected value can be found in Lamberton & Lapeyre (1996), Subsection 6.2.2.

Second step: We will show that the extended market $\overline{Z}$ meets integrability condition (RC 1). Fix $i, j \in \{1, \dots, n\}$. We define a mapping $f : [0, t_i \wedge t_j) \times \mathbb{R}_+^* \times \mathbb{R} \to \mathbb{R}^2$ of class C^2 by

$$\begin{aligned}
&f(t,x,r) := \\
&\Big(\frac{1}{x}\exp\Big(\frac{2\alpha}{\sigma^2}\log\Big(\frac{2\gamma\exp(\frac{1}{2}(t_i-t)(\gamma+\beta))}{\gamma-\beta+(\gamma+\beta)\exp(\gamma(t_i-t))}\Big) - r\frac{2(\exp(\gamma(t_i-t))-1)}{\gamma-\beta+(\gamma+\beta)\exp(\gamma(t_i-t))}\Big), \\
&\frac{1}{x}\exp\Big(\frac{2\alpha}{\sigma^2}\log\Big(\frac{2\gamma\exp(\frac{1}{2}(t_j-t)(\gamma+\beta))}{\gamma-\beta+(\gamma+\beta)\exp(\gamma(t_j-t))}\Big) - r\frac{2(\exp(\gamma(t_j-t))-1)}{\gamma-\beta+(\gamma+\beta)\exp(\gamma(t_j-t))}\Big)\Big)
\end{aligned}$$

By the same arguments as in the second step in the first part of the proof, we obtain for the extended characteristics $(\varnothing, \varepsilon_{(1,Z_0^1,\dots,Z_0^n)}, \overline{b}, \overline{c}, 0, 0)^E$ of $\overline{Z}$ that $\overline{b} = 0$, $\overline{c}_t^{ij} = 0$ for $t \geq t_i \wedge t_j$ and

$$\overline{c}_t^{ij} \;=\; Z_t^i Z_t^j \sigma^2 r_t \frac{2(\exp(\gamma(t_i-t))-1)}{\gamma-\beta+(\gamma+\beta)\exp(\gamma(t_i-t))}\frac{2(\exp(\gamma(t_j-t))-1)}{\gamma-\beta+(\gamma+\beta)\exp(\gamma(t_j-t))}$$

for any $t \in [0, t_i \wedge t_j)$. Since $|Z^i| \leq 1$ and

$$\begin{aligned}
\sup\Big\{\Big|\frac{2(\exp(\gamma(t_i-t))-1)}{\gamma-\beta+(\gamma+\beta)\exp(\gamma(t_i-t))}\Big| : i \in \{1,\dots,n\}, t \in [0,t_i]\Big\}& \\
\leq \quad \frac{2(\exp(\gamma T)-1)}{\gamma-\beta} =: M \in \mathbb{R}_+,&
\end{aligned}$$

we conclude that $|\overline{c}_t^{ij}|^2 \leq \sigma^4 M^4 (r_t)^2$ for any $i, j \in \{1, \dots, n\}$, $t \in \mathbb{R}_+$. If we set $L := \frac{\sigma^2}{4\beta}(1 - e^{-\beta t})$, then it follows from Lamberton & Lapeyre (1996), p.131 that r_t/L is χ^2-distributed with $4\alpha/\sigma^2$ degrees of freedom and noncentrality parameter $\zeta := \frac{4r_0\beta}{\sigma^2(e^{\beta t}-1)}$. Therefore

$$\begin{aligned}
E\Big(\Big(\frac{r_t}{L}\Big)^2\Big) &= \mathrm{Var}\Big(\frac{r_t}{L}\Big) + \Big(E\Big(\frac{r_t}{L}\Big)\Big)^2 = 2\frac{4\alpha}{\sigma^2} + 4\zeta + \Big(\frac{4\alpha}{\sigma^2}+\zeta\Big)^2 \\
&= \frac{16}{\sigma^2}\Big(\frac{\alpha}{2} + \frac{r_0\beta}{e^{\beta t}-1} + \Big(\alpha + \frac{r_0\beta}{e^{\beta t}-1}\Big)^2\Big)
\end{aligned}$$

(cf. Johnson & Kotz (1970b), p.134, Equation (13)) and hence

$$\begin{aligned}
E(r_t^2) &= \sigma^2\Big(\frac{\alpha(1+2\alpha)}{2\beta^2}(1-e^{-\beta t})^2 + \frac{r_0}{\beta}e^{-\beta t}(1-e^{-\beta t})(1+2\alpha) + r_0^2 e^{-2\beta t}\Big) \\
&\leq \sigma^2\Big(\frac{\alpha(1+2\alpha)}{2\beta^2} + \frac{r_0}{\beta}(1+2\alpha) + 2r_0^2\Big) =: \widetilde{M} \in \mathbb{R}_+
\end{aligned}$$

for any $t \in \mathbb{R}_+$. It follows that $E(|\overline{c}_t^{ij}|^2) \leq \sigma^4 M^4 \widetilde{M}$ for any $i, j \in \{1, \dots, n\}$, $t \in \mathbb{R}_+$, which yields (RC 1). $\square$

PROOF OF LEMMA 4.24. *1. First step:* For any $i \in \{1, \dots, n\}$ define the functions $A^i : [0,T] \to \mathbb{R}$, $B^i : [0,T] \to \mathbb{R}$ by

$$\begin{aligned} A^i(t) &= \int_{t\wedge t_i}^{t_i} (1 - e^{-\beta(t_i - s)}) \Big(\frac{1}{2}\frac{\sigma^2}{\beta^2}(1 - e^{-\beta(t_i - s)}) - \widetilde{\vartheta}(s) \Big) \, ds \\ B^i(t) &= \frac{1}{\beta}(1 - e^{-\beta(t_i - t)}) 1_{[0,t_i]}(t). \end{aligned}$$

Observe that $A^i(t) = -\int_t^T (\beta \widetilde{\vartheta}(s) B^i(s) - \frac{1}{2}\sigma^2 (B^i(s))^2) \, ds$ and $B^i(t) = \int_t^{t_i \wedge t} (\beta B^i(s) + 1) \, ds$ for any $t \in [0,T]$. Fix $i, j \in \{1, \dots, n\}$. We define a mapping $f : \mathbb{R}_+^* \times \mathbb{R}^5 \to \mathbb{R}^3$ of class C^2 by $f(x, r, u_1, v_1, u_2, v_2) = (\frac{1}{x}\exp(u_1 - rv_1), \frac{1}{x}\exp(u_2 - rv_2), r)$. For any $t \leq t_i \wedge t_j$ we have $(\widetilde{Z}_t^i, \widetilde{Z}_t^j, r) = f(S_t^0, r_t, A^i(t), B^i(t), A^j(t), B^j(t))$. For $i = j$ application of Itô's formula (cf. Theorem 2.25) yields that $\widetilde{Z}^i$ is a continuous Grigelionis process on $[0, t_i]$ and hence on $\mathbb{R}_+$ because $\widetilde{Z}^i = (\widetilde{Z}^i)^{t_i}$. Therefore, $\widetilde{Z} = (Z^0, \widetilde{Z}^1, \dots, \widetilde{Z}^n, r)$ is a continuous Grigelionis process as well. We will now compute its extended characteristics $(\varnothing, \varepsilon_{(1, \widetilde{Z}_0^1, \dots, \widetilde{Z}_0^n, r_0)}, \widehat{b}, \widehat{c}, 0, 0)^E$ by application of Itô's formula to $f_{i,j}$ and the Grigelionis process $(S_t^0, r_t, A^i(t), B^i(t), A^j(t), B^j(t))_{t\in[0,T]}$ for any $i, j \in \{1, \dots, n\}$. This yields

$$\begin{aligned} \widehat{b}_t^i &= \widetilde{Z}_t^i \Big(-\frac{1}{S_t^0} S_t^0 r_t - B^i(t)\beta(\vartheta - r_t) + \beta\widetilde{\vartheta}(t) B^i(t) - \frac{1}{2}\sigma^2 (B^i(t))^2 \\ &\qquad + r_t(\beta B^i(t) + 1) + \frac{1}{2}\sigma^2 (B^i(t))^2 \Big) \\ &= \widetilde{Z}_t^i \beta(\widetilde{\vartheta}(t) - \vartheta) B^i(t) \quad \text{for any } t < t_i \\ \widehat{c}_t^{ij} &= \widetilde{Z}_t^i \widetilde{Z}_t^j B^i(t) B^j(t) \sigma^2 \quad \text{for } t < t_i \wedge t_j \end{aligned} \tag{4.17}$$

Since the $\widetilde{Z}^i$ are constant after t_i, we have $\widehat{b}_t^i = 0$ for $t \geq t_i$ and $\widehat{c}_t^{ij} = 0$ for $t \geq t_i \wedge t_j$. For the diffusion coefficients related to the last component r in $(Z^0, \widetilde{Z}^1, \dots, \widetilde{Z}^n, r)$, which is indexed by $n+1$, we obtain

$$\widehat{c}_t^{i,n+1} = -\widetilde{Z}_t^i B^i(t) \sigma^2 \quad \text{for } t < t_i$$

and $\widehat{c}_t^{i,n+1} = 0$ for $t \geq t_i$. Let us once more remark that f is, strictly speaking, not of class C^2 on $\mathbb{R}^6$ and one may lead an exact proof by localization as in Corollary 4.7.

Second step: A straightforward calculation yields that $\widehat{b}_t^i - \kappa \sum_{j=1}^n \widehat{c}_t^{ij} \rho^j / \widetilde{Z}_t^j = 0$ for any $i \in \{1, \dots, n\}$, $t \in \mathbb{R}_+$. By the remark following Corollary 3.23, $(0, \rho^1/\widetilde{Z}^1, \dots, \rho^n/\widetilde{Z}^n)$ is a u_κ-optimal strategy for $\mathfrak{A}$ in the market $\widetilde{Z} = (Z^0, \widetilde{Z}^1, \dots, \widetilde{Z}^n)$. Therefore, $\widetilde{Z}^1, \dots, \widetilde{Z}^n$ are $(\kappa, \rho^1/\widetilde{Z}^1, \dots, \rho^n/\widetilde{Z}^n)$-consistent proce processes in the sense of Definition 3.40 and Remark 3 in Section 3.6.

Third step: Note that ϑ and B are bounded deterministic functions. In view of the shape of $\widehat{b}, \widehat{c}$ in the first step, integrability condition (RC 1) follows if we can show $\sup_{t\in[0,T]}$

$E((\widetilde{Z}^i_t)^4) < \infty$ for any $i \in \{1, \ldots, n\}$. Observe that

$$\widetilde{S}^i_t = S^i_t \cdot \exp\Big(-\int_t^{t_i} (1 - e^{-\beta(t_i - s)})(\widetilde{\vartheta}(s) - \vartheta)\, ds \Big)$$

for any $t \in [0, t_i)$, where S^i is defined as in Lemma 4.23. Therefore, $|\widetilde{Z}^i| \le |Z^i| K$ for some $K \in \mathbb{R}_+$, namely $K := \exp(\int_0^{t_i} (1 - e^{-\beta(t_i - s)}) |\widetilde{\vartheta}(s) - \vartheta|\, ds)$, where Z^i is the discounted neutral price process $Z^i = S^i / S^0$. The claim $\sup_{t\in[0,T]} E((\widetilde{Z}^i_t)^4) < \infty$ now follows from the estimate in the second step in the proof of Lemma 4.23.

Fourth step: As noted in the remark before Subsection 4.9.1, the martingale problem $(\varnothing, \varepsilon_{r_0}, b, c, 0, 0)^M$ in $\mathbb{R}$ with $b(\bar{\omega})_t = \beta(\vartheta - \bar{\omega}_{t-})$, $c(\bar{\omega}_t) = \sigma$ for any $(\bar{\omega}, t) \in \mathbb{D}^1 \times \mathbb{R}_+$ has a unique solution-measure, namely the distribution P^r of r. Suppose for the moment that $(\Omega, \mathcal{F})$ equals the Skorohod space $(\mathbb{D}^1, \mathcal{D}^1)$ and r is the canonical process on $\mathbb{D}^1$. Since $\widetilde{\vartheta} - \vartheta$ is a bounded function, it follows from Theorem 2.31 that $\mathscr{E}(\int_0^\cdot \frac{\beta}{\sigma^2}(\widetilde{\vartheta}(s) - \vartheta)\, dr^c_s)$ is the density process of a probability measure. This implies $E(\mathscr{E}(\int_0^\cdot \frac{\beta}{\sigma^2}(\widetilde{\vartheta}(s) - \vartheta)\, dr^c_s)_\infty) = E(\mathscr{E}(\int_0^\cdot \frac{\beta}{\sigma^2}(\widetilde{\vartheta}(s) - \vartheta)\, dr^c_s)_T) = 1$, where $T = \sup\{t_1, \ldots, t_n\}$. (Since this expectation depends only on the distribution of r, it follows that the equality also holds if the underlying space is not $(\mathbb{D}^1, \mathcal{D}^1)$, as long as r is a solution-process to the above martingale problem.) This shows that P^* is a well-defined probability measure equivalent to P.

Fifth step: An application of Girsanov's theorem (cf. Theorem 2.26 and Lemma 2.27) yields that the extended characteristics $(\varnothing, \varepsilon_{(\widetilde{Z}^1_0, \ldots, \widetilde{Z}^n_0, r_0)}, \check{b}, \check{c}, 0, 0)^E$ of $(\widetilde{Z}^1, \ldots, \widetilde{Z}^n, r)$ relative to P^* satisfy the following equations: $\check{b}^i = 0$ for $i = 1, \ldots, n$, $\check{b}^{n+1}_t = \beta(\widetilde{\vartheta}(t) - r_t)$ for any $t \in \mathbb{R}_+$. In particular, $\widetilde{Z}^1, \ldots, \widetilde{Z}^n$ are P^*-local martingales and Statement 1c in Lemma 4.24 holds.

Sixth step: Note that r^c is a multiple of Brownian motion. Since $\widetilde{\vartheta}$ is a deterministic (and piecewise continuous) function, it follows that L is a lognormally distributed random variable (cf. Proposition 4.21) and hence $E(L^2) < \infty$. In view of the third step, this implies $\sup_{t\in[0,T]} E^*((\widetilde{Z}^i_t)^2) = \sup_{t\in[0,T]} E((\widetilde{Z}^i_t)^2 L) \le \sup_{t\in[0,T]} (E((\widetilde{Z}^i_t)^4) E(L^2))^{1/2} < \infty$. By Equation (4.17) and the boundedness of B^i, B^j, we obtain $E^*(\langle \widetilde{Z}^i, \widetilde{Z}^i \rangle_T) < \infty$ for $i = 1, \ldots, n$, and hence $\widetilde{Z}^i$ is a square-integrable P^*-martingale (cf. JS, I.4.50c).

Seventh step: Application of Itô's formula similarly as in the second step of the proof of Lemma 4.23, but to the process (Z^i, t) instead of (Z^i, Z^j), yields that $d\langle Z^{i,c}, r^c \rangle_t = -Z^i_t \frac{\sigma^2}{\beta}(1 - e^{-\beta(t_i - t)})\, dt = -Z^i_t \sigma^2 B^i(t)\, dt$ for $i = 1, \ldots, n$ and $t \le t_i$. Since the local martingale $Z^{i,c}$ has the representation property relative to r (cf. Theorem 2.65), it follows that $dZ^{i,c}_t = H^i_t\, dr^c_t$ for some $H^i \in L^2_{\mathrm{loc}}(r^c)$. Obviously, we have $d\langle Z^{i,c}, r^c \rangle_t = H^i_t\, d\langle r^c, r^c \rangle_t = H^i_t \sigma^2\, dt$ and therefore $H^i_t = -Z^i_t B^i(t)$ for λ-almost all $t \in \mathbb{R}_+$. This implies that, for given $(\kappa, \rho/Z^1, \ldots, \rho/Z^n)$, the local martingale $\widetilde{N}$ in step 5 on page 123 is of the form $\widetilde{N}_t = -\kappa \int_0^t \sum_{i=1}^n \rho^i / Z^i_t\, dZ^{i,c}_t = \kappa \int_0^t \sum_{i=1}^n \rho^i B^i(s)\, dr^c_s = \frac{\beta}{\sigma^2} \int_0^t (\widetilde{\vartheta}(s) - \vartheta)\, dr^c_s$. Therefore, $\widetilde{P}$ in step 6 on page 123 equals P^*. Statement 1d now follows from Statement 1c.

2. Since the proof is similar to the Vasicek case, we only sketch the single steps.

First step: Firstly note that $\widetilde{\beta}$ is bounded. Therefore, there exists some $m, M \in \mathbb{R}_+$ with $\widetilde{\beta}(t)x + \frac{1}{2}\sigma^2 x^2 - 1 > 0$ for any $x \in [M, \infty)$, $t \in [0, T]$, and $\widetilde{\beta}(t)x + \frac{1}{2}\sigma^2 x^2 - 1 < 0$ for any $x \in [0, m)$, $t \in [0, T]$. Since the coefficients of the integral equation (4.16) are locally Lipschitz, it follows that the initial value problem for B^i has a unique solution staying in $[0, M]$ for $t \in [0, T]$. Since r is also non-negative in the Cox-Ingersoll-Ross model, it follows that $0 \leq \widetilde{S}^i \leq 1$ for any $i \in \{1, \dots, n\}$. As in the first part of the proof, we conclude that $\widetilde{Z} = (Z^0, \widetilde{Z}^1, \dots, \widetilde{Z}^n, r)$ is a continuous Grigelionis process with extended characteristics $(\varnothing, \varepsilon_{(1, \widetilde{Z}_0^1, \dots, \widetilde{Z}_0^n, r_0)}, \widehat{b}, \widehat{c}, 0, 0)^E$ given by

$$\begin{aligned}
\widehat{b}_t^i &= \widetilde{Z}_t^i \Big(-\frac{1}{S_t^0} S_t^0 r_t - B^i(t)(\alpha - \beta r_t) + \alpha B^i(t) \\
&\qquad + r_t \Big(-\widetilde{\beta}(t) B^i(t) - \frac{1}{2}\sigma^2 (B^i(t))^2 + 1 + \frac{1}{2}\sigma^2 (B^i(t))^2 \Big) \Big) \\
&= \widetilde{Z}_t^i (\beta - \widetilde{\beta}(t)) B^i(t) r_t \quad \text{for any } t < t_i \\
\widehat{c}_t^{ij} &= \widetilde{Z}_t^i \widetilde{Z}_t^j B^i(t) B^j(t) \sigma^2 r_t \quad \text{for } t < t_i \wedge t_j,
\end{aligned}$$

moreover $\widehat{b}_t^i = 0$ for $t \geq t_i$ and $\widehat{c}_t^{ij} = 0$ for $t \geq t_i \wedge t_j$. For the coefficients related to the last component r in $(Z^0, \widetilde{Z}^1, \dots, \widetilde{Z}^n, r)$, we obtain $\widehat{c}_t^{i,n+1} = -\widetilde{Z}_t^i B^i(t) \sigma^2 r_t$ for $t \leq t_i$ and $\widehat{c}_t^{i,n+1} = 0$ for $t \geq t_i$.

Second step: As in the fourth step of the first part of the proof, one shows that P^* is a well-defined probability measure equivalent to P.

Third step: As in the fifth step of the first part of the proof, one shows that $\widetilde{Z}^1, \dots, \widetilde{Z}^n$ are P^*-local martingales and $\check{b}_t^{n+1} = \alpha - \widetilde{\beta}(t) r_t$ is the P^*-drift coefficient of r for any $t \in \mathbb{R}_+$. Since $0 \leq S^i \leq 1$ and $S^0 \geq 1$, it follows that $\widetilde{Z}^i = \widetilde{S}^i / S^0$ also assumes only values in $[0, 1]$. This implies that $\widetilde{Z}^1, \dots, \widetilde{Z}^n$ are bounded P^*-martingales.

Fourth step: In view of the shape of $\widehat{b}, \widehat{c}$ in the first step, regularity condition (RC 1) follows if we can show $\sup_{t \in \mathbb{R}_+} E(r_t^2) < \infty$. This is shown in the second step of the proof of Lemma 4.23 for the Cox-Ingersoll-Ross model.

Fifth step: As in the seventh step of the first part of the proof, one shows that P^* equals the measure $\widetilde{P}$ in step 6 on page 123. Since $\widetilde{Z}^1, \dots, \widetilde{Z}^n$ are P^*-martingales, they are $(\kappa, \rho/Z^1, \dots, \rho/Z^n)$-approximate price processes. $\square$

Appendix A

Notions from Stochastic Calculus

Conditional Expectation

As in JS, we define conditional expectations for any real-values random variable, even if it is not integrable or non-negative, by

$$E(X|\mathcal{G}) := \begin{cases} E(X^+|\mathcal{G}) - E(X^-|\mathcal{G}) & \text{on the set where } E(|X||\mathcal{G}) < \infty \\ +\infty & \text{elsewhere,} \end{cases}$$

where $X^+ := X \vee 0$, $X^- := -(X \wedge 0)$.

Locally bounded predictable processes

Locally bounded predictable processes are often taken as a natural class of integrands for stochastic integrals (cf. JS, Section I.4). According to Dellacherie (1980), p.132, Lenglart has shown that for predictable processes pathwise boundedness on any compact interval suffices to ensure local boundedness. Since we could not find any reference, we prove this result below.

Lemma A.1 *Let H be a predictable process such that H_0 is bounded. Then H is locally bounded if and only if $\sup_{s\in[0,t]} |H_s| < \infty$ P-almost surely for any $t \in \mathbb{R}_+$.*

PROOF. By Jacod (1979), (1.1) we may assume w.l.o.g. that the stochastic basis is complete. The "only if"-part is obvious.

Assume that $\sup_{s\in[0,t]} |H_s| < \infty$ P-almost surely for any $t \in \mathbb{R}_+$. Define an increasing sequence $(T_n)_{n\in\mathbb{N}}$ of stopping times by $T_n := \inf\{t \in \mathbb{R}_+ : |H_t| \geq n\}$. Fix $n \in \mathbb{N}$ for the moment. Since H^{T_n} is predictable, the random set $A_n := \{(\omega, t) \in \Omega \times \mathbb{R}_+ : |H_t^{T_n}(\omega)| \in [n, \infty)\}$ is predictable as well. For the stopping time S_n defined by $S_n(\omega) := \inf\{t : (\omega, t) \in A_n\}$, we have $[S_n] \subset A_n$. By JS, I.2.13 this implies that S_n is predictable. Hence, there exists an announcing sequence $(S_{n,k})_{k\in\mathbb{N}}$ for S_n. For any $n \in \mathbb{N}$, we define $R_n := T_n \wedge \sup\{S_{1,n}, \ldots, S_{n,n}\}$. Then $(R_n)_{n\in\mathbb{N}}$ is an increasing sequence of stopping times with $R_n \uparrow \infty$ P-almost surely for $n \to \infty$ and $\sup_{t\in\mathbb{R}_+} |H_t^{R_n}| \leq |H_0| + n$. Hence, H is locally bounded. $\square$

Random Measures and Stochastic Integrals

Definition A.2 1. A *random measure* on $\mathbb{R}_+ \times \mathbb{R}^d$ is a family $\mu = (\mu(\omega; dt, dx) : \omega \in \Omega)$ of non-negative measures on $(\mathbb{R}_+ \times \mathbb{R}^d, \mathcal{B}_+ \otimes \mathcal{B}^d)$ satisfying $\mu(\omega; \{0\} \times \mathbb{R}^d) = 0$ for any $\omega \in \Omega$ (cf. JS, Definition II.1.3).

2. For any $(\mathcal{P} \otimes \mathcal{B}^d)$-measurable (i.e. *predictable*) mapping $W : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ the *integral process* $W * \mu$ is defined pathwise by

$$W * \mu_t(\omega) := \begin{cases} \int_{[0,t]\times\mathbb{R}^d} W(\omega, s, x)\, \mu(\omega; ds, dx) & \text{if } \int_{[0,t]\times\mathbb{R}^d} |W(\omega, s, x)|\, \mu(\omega; ds, dx) < \infty \\ +\infty & \text{else} \end{cases}$$

(cf. JS, II.1.5).

3. μ is called *predictable* if $W * \mu$ is predictable for any predictable mapping $W : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ (cf. JS, II.1.6).

Definition A.3 1. For any $\mathbb{R}^d$-valued càdlàg, adapted process X, the *random measure of jumps* μ^X is defined by

$$\mu^X(\omega; dt, dx) = \sum_s 1_{\mathbb{R}^d\setminus\{0\}}(\Delta X_s(\omega))\varepsilon_{(s,\Delta X_s(\omega))}(dt, dx)$$

(cf. JS, II.1.16).

2. A predictable random measure ν (which turns out to be uniquely defined up to a P-null set) is called *compensator* of μ^X if $E(W * \nu_\infty) = E(W * \mu^X_\infty)$ for any predictable mapping $W : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ (cf. JS, II.1.8).

Remarks.

1. Even if $|W| * \mu^X_t = \infty$ and $|W| * \nu_t = \infty$ so that the difference $W * \mu^X_t - W * \nu_t$ does not make sense, it is still possible to define a stochastic integral $W * (\mu^X - \nu)$ for a large class of $(\mathcal{P} \otimes \mathcal{B}^d)$-measurable mappings W, namely for $W \in G_{\mathrm{loc}}(\mu^X)$. For details, we refer to JS, Definition II.1.27.

2. We use a different notation than JS for integrals:

   (a) By $\int_0^t H_s\, dX_s$, we refer to the Stieltjes or stochastic integral of the real-valued process H with respect to the real-valued process X. The stochastic integral is denoted $(H{\cdot}X)_t$ is JS.

   (b) H can also be $\mathbb{R}^d$-valued. Then, the integral is also $\mathbb{R}^d$-valued with components $\int_0^t H^i_s\, dX_s$ for $i \in \{1, \dots, d\}$.

(c) If H is a $\mathbb{R}^d$-valued, predictable process and X a $\mathbb{R}^d$-valued semimartingale, we use the notation $\int_0^t H_s \cdot dX_s$ to denote $\sum_{i=1}^d \int_0^t H_s^i \, dX_s^i$. If X a $\mathbb{R}^d$-valued continuous local martingale, then $\int_0^t H_s \cdot dX_s$ can be defined for a larger class of integrands, namely $L^2_{\text{loc}}(X)$ (cf. JS, III.4.5, where the notation $(H \cdot X)_t$ is used).

(d) We often denote stochastic integrals with respect to random measures by $\int_{[0,t]\times E} W(s,x)\,\mu(ds,dx)$ and $\int_{[0,t]\times E} W(s,x)\,(\mu-\nu)(ds,dx)$. The notation in JS is $W * \mu_t$ and $W * (\mu-\nu)_t$, respectively. If W is $\mathbb{R}^d$-valued, then the integrals should be read componentwise.

Discrete-Time Models

Any discrete-time model can be naturally embedded in a continuous-time framework in the following manner. Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}}, P)$ be a discrete stochastic basis and $(X_n)_{n\in\mathbb{N}}$ an adapted process on that space. Define $\mathcal{F}_t := \mathcal{F}_{[t]}$ and $X_t := X_{[t]}$ for any $t \in \mathbb{R}_+$. Then $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ is a continuous stochastic basis and $(X_t)_{t\in\mathbb{R}_+}$ a càdlàg, adapted process on that space. Conversely, we make the following

Definition A.4 We call a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ *discrete* if $\mathcal{F}_t := \mathcal{F}_{[t]}$ for any $t \in \mathbb{R}_+$. Likewise, we say that a càdlàg process $(X_t)_{t\in\mathbb{R}_+}$ is *discrete* if $X_t := X_{[t]}$ P-almost surely for any $t \in \mathbb{R}_+$.

For details cf. JS, Subsection I.1.f.

Absolute Continuity of Measures

Definition A.5 1. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ be a filtered probability space and P' another probability measure on $(\Omega, \mathcal{F})$. We say that P' is *locally absolutely continuous* with respect to P, and we write $P' \overset{\text{loc}}{\ll} P$, if $P'|_{\mathcal{F}_t} \ll P|_{\mathcal{F}_t}$ for any $t \in \mathbb{R}_+$ (cf. JS, III.3.2). The up to indistinguishability unique P-martingale Z with $Z_t = dP'|_{\mathcal{F}_t}/dP|_{\mathcal{F}_t}$ is called the *density process* of P' relative to P (cf. JS, III.3.4).

2. We say that P, P' are *locally equivalent* ($P' \overset{\text{loc}}{\sim} P$) if $P' \overset{\text{loc}}{\ll} P$ and $P \overset{\text{loc}}{\ll} P'$.

Canonical Filtration

Definition A.6 If X is a càdlàg process, we call $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$, defined by $\mathcal{G}_t = \cap_{s>t}\sigma(X_u : u \in [0,s])$ for any $t \in \mathbb{R}_+$, the *canonical filtration* of X or the filtration *generated* by X.

Skorohod Space

Definition A.7 1. By $\mathbb{D}^d := \mathbb{D}(\mathbb{R}^d)$, we denote the space of all càdlàg functions $\mathbb{R}_+ \to \mathbb{R}^d$ (called *Skorohod space*, cf. JS, VI.1.1).

2. The mapping $X : \mathbb{D}(\mathbb{R}^d) \times \mathbb{R}_+ \to \mathbb{R}$, $(\bar{\omega}, t) \mapsto \bar{\omega}_t$ is called *canonical process* on $\mathbb{D}(\mathbb{R}^d)$.

3. The filtration generated by the canonical process X is denoted $(\mathcal{D}_t^d)_{t\in\mathbb{R}_+} := (\mathcal{D}(\mathbb{R}^d)_t)_{t\in\mathbb{R}_+}$. Moreover, we set $\mathcal{D}^d := \mathcal{D}(\mathbb{R}^d) := \mathcal{D}_{\infty-}(\mathbb{R}^d)$ (cf. JS, VI.1.1).

4. We denote the predictable σ-field on $\mathbb{D}^d \times \mathbb{R}_+$ by $\mathcal{P}^d$.

The space S^1

Definition A.8 Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ be a filtered probability space and $d \in \mathbb{N}^*$. For any $\mathbb{R}^d$-valued, càdlàg (here including a limit at infinity), adapted process X, we define

$$\|X\|_{S^1} := E(\|X\|_\infty^*).$$

Moreover, we set $S^1 := \{X \ \mathbb{R}^d\text{-valued, càdlàg, adapted process: } \|X\|_{S^1} \leq \infty\}$. By Dellacherie & Meyer (1982), VII.64, S^1 is a Banach space.

Martingale Representation Property

Definition A.9 A local martingale M has the *representation property* relative to a $\mathbb{R}^d$-valued semimartingale X if it is of the form

$$M = M_0 + \int_0^\cdot H_s \cdot dX_s^c + \int_{[0,\cdot]\times\mathbb{R}^d} W(s,x)\,(\mu^X - \nu)(ds, dx)$$

for some $H \in L^2_{\text{loc}}(X^c)$ and $W \in G_{\text{loc}}(\mu^X)$, where μ^X denotes the measure of jumps of X and ν its compensator.

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General Notation

$\mathbb{N}, \mathbb{N}^*$	$\{0, 1, 2, 3, \ldots\}$, $\{1, 2, 3, \ldots\}$
$\mathbb{R}, \mathbb{R}_+, \mathbb{R}_+^*, \overline{\mathbb{R}}_+, \mathbb{R}_-$	$(-\infty, \infty), [0, \infty), (0, \infty), [0, \infty], (-\infty, 0]$
$\mathbb{R}^d$	the Euclidean d-dimensional space
$\mathbb{R}^{d \times d}$	the set of real $d \times d$-matrices
$\mathbb{Q}, \mathbb{Q}^+$	the set of rational numbers, $\mathbb{Q} \cap \mathbb{R}_+$
$\dot\bigcup_{i \in I} A_i$	the disjoint union of the sets A_i
$A - b, bA$	$\{a - b : a \in A\}, \{ba : a \in A\}$
$\mathscr{L}(E, F)$	the space of linear, continuous mappings $E \to F$
$x \cdot y$	the scalar product of $x, y \in \mathbb{R}^d$
$A^\top$	the transposed of the matrix A
$x \wedge y, x \vee y$	$\inf(x, y), \sup(x, y)$
x^i	the ith component of $x \in \mathbb{R}^d$ or the ith power of $x \in \mathbb{R}$
$[x]$	the integer part of $x \in \mathbb{R}_+$
$\lvert x \rvert$	the Euclidean norm of $x \in \mathbb{R}^d$
$\lvert A \rvert$	the number of elements of the (countable) set A
$\lVert A \rVert$	the operator norm of the matrix A
$x_n \uparrow x, x_n \downarrow x$	$(x_n)_{n \in \mathbb{N}}$ increases (resp. decreases) and $\lim_{n\to\infty} x_n = x$
$\limsup$	limit superior of a sequence of numbers or sets
$\operatorname{ess\,sup}$	P-essential superior limit
$\min f$	the minimum of a function f
$o(\ldots), O(\ldots)$	Landau order symbols
$f^{-1}(A)$	the inverse image of A
$f\vert_A$	the restriction of the mapping f to the set A
f'	the derivative of a real function f
∂A	the boundary of a set A (but compare p.106)
Df	the derivative of a differentiable mapping f
$D_i f, D_{(i,j)} f$	partial derivatives of f
1_A	the indicator function of the set A
A^C	the complement of the set A
$\otimes$	product of σ-fields and measures
$*$	convolution of probability measures (but compare Definition A.2)
$\mathfrak{P}(A)$	the power set of A
$\mathcal{B}, \mathcal{B}_+, \mathcal{B}^d, \mathcal{B}(A)$	Borel-σ-fields on $\mathbb{R}, \mathbb{R}_+, \mathbb{R}^d, A$

$\sigma(\mathcal{E}), \sigma(X_s : s \in A)$	the σ-field generated by $\mathcal{E}$ or by $\{X_s : s \in A\}$
$\mathcal{F}_{t-}$,$\mathcal{F}_{\infty-}$	$\sigma(\cup_{s\in[0,t)}\mathcal{F}_s), \sigma(\cup_{s\in\mathbb{R}_+}\mathcal{F}_s)$
$\mathcal{F}^P$	the P-completion of the σ-field $\mathcal{F}$
$\mathcal{F}_t^P$	the σ-field generated by $\mathcal{F}_t$ and the P-null sets of $\mathcal{F}^P$
$\mathcal{P}$	the predictable σ-field
ε_x	the Dirac measure sitting in the point x
$\lambda, \lambda\vert_{[0,1]}, \lambda\vert_{\mathbb{R}_+}$	the Lebesgue measure on $\mathbb{R}$ and its restriction to $[0, 1]$ resp. $\mathbb{R}_+$
$N(\mu, \sigma^2)$	the normal distribution with mean μ and variance σ^2
$S_\alpha(\sigma, \beta, \mu)$	the stable distribution with parameters $\alpha, \sigma, \beta, \mu$
P^X	the distribution of the random variable X
$P^{X\vert\mathcal{G}}$	the regular conditional distribution of X given the σ-field $\mathcal{G}$
$P \ll P'$	P is absolutly continuous with respect to P'.
$P \sim P'$	The probability measures P, P' are equivalent.
$\frac{dP'}{dP}$	the Radon-Nikodým density of P' relative to P
$E(X), \mathrm{Var}(X)$	expected value and variance of X
$\mathrm{Va}(X)_t$	the total variation of the process X on $[0, t]$
$\Vert X\Vert_{L^p}$	$(E(\vert X\vert^p))^{\frac{1}{p}}$
$\Vert X\Vert_t^*$	$\sup\{\vert X_s\vert : s \leq t\}$
S^p	$\{x \in \mathbb{R}^{p+1} : \vert x\vert = 1\}$ (but compare Definition A.8)
$X_t(\omega)$	$X(\omega, t)$
X_{t-}	$\lim_{s\to t, s<t} X_s$
ΔX_t	$X_t - X_{t-}$
$[S, T],]S, T]$, etc.	stochastic intervals
$[T]$	the graph of a stopping time, i.e. $[T, T]$
X^T	the process X stopped at time T, i.e. $X_t^T = X_{T\wedge t}$
X^{T-}	the process X stopped strictly before T, i.e. $X_t^{T-} = 1_{[0,T)}(t)X_t + 1_{[0,T)^C}(t)X_{T-}$
$\langle M, N\rangle$	the predictable quadratic covariation of the local martingales M, N
X^c	the continuous local martingale part of the semimartingale X
M^d	$M - M^c$ for a local martingale M
$\mathscr{E}(X)$	the stochastic exponential of the semimartingale X
$\mathscr{V}, \mathscr{V}^d, \mathscr{V}^{d\times d}$	càdlàg, adapted processes in $\mathbb{R}, \mathbb{R}^d, \mathbb{R}^{d\times d}$, starting in 0, whose components are of finite variation
$\mathscr{A}_{\mathrm{loc}}^+$	càdlàg, adapted processes, starting in 0, that are locally integrable and increasing
$\mathscr{A}_{\mathrm{loc}}$	càdlàg, adapted processes, starting in 0, that are of locally integrable variation

Index of Symbols

Index of Terminology