Ruin Probabilities for a Sparre Andersen Model with Investments

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Abstract

We study a Sparre Andersen model in which the business activity of the company is described by a compound renewal process with drift assuming that the capital reserves are invested in a risky asset. The price of the latter is assumed to evolve according to a geometric Lévy process. We prove that the asymptotic behavior of the ruin probability depends to a large extent only on the properties of the price process.

Key words: Ruin probabilities, Sparre Andersen model, Actuarial models with investments, Renewal processes, Distributional equations

MSC 60G51, 60G70, 91G05

1. Introduction

In the classical collective risk theory the Sparre Andersen model (called also renewal risk model) is a generalisation of the Cramér–Lundberg model in which it is assumed that the claims count is a renewal process rather than a Poisson process, i.e. the times between consecutive claims are independent random variables which are not necessarily exponentially distributed, see, e.g., Grandell's book [4]. The word "classical" denotes the situation in which the reserves of the insurance company are kept in a bank account which does not earn any interest. In the modern literature more realistic models are considered, namely those in which the capital is invested, either fully or partially, in a risky financial asset. Given the long investment horizon which insurance companies have, an economically reasonable investment strategy is represented by a market index like the MSCI, the S&P500 or the DAX for example. This is actually standard practice in the financial industry, primarily due to the long term growth opportunities that these indices offer, see for example [12]. Although classical contributions model the dynamics of an index via geometric Brownian motions, a more realistic description of the relevant empirical features is provided by exponential Lévy processes, see for example [2], Ch. 8, and references therein. This motivates our choice in the following.

It was shown in [3] that the financial risk leads to a dramatic change in the behavior of the ruin probability $\Psi(u)$ as a function of the initial capital u: instead of its comfortable exponential decrease with the growth of the available capital, in case of the investment of the reserves in the stock market, the ruin probability $\Psi(u)$ decreases as a power function. Moreover, the ruin is imminent, $\Psi \equiv 1$, if the volatility of the stock market is too high. Because of the practical importance of this phenomenon, actuarial models with investments attracted a lot of attention. Most of the papers deal with asymptotic studies in settings stemming from the Cramér–Lundberg model and can be included in the beautiful mathematical framework of the generalised Ornstein–Uhlenbeck process suggested and studied by Paulsen, see [8] – [11]. Paulsen used the powerful method of distributional equations. A detailed analysis of the available results is beyond the scope of the present note and we refer the reader to the recent paper [6].

In contrast to the Cramér–Lundberg line of research there are very few results on ruin probabilities for Sparre Andersen type models with investments. The paper by Albrecher, Constantinescu, and Thomann [1] is one of the rare studies to the latter, see also references therein. Their approach is based on the techniques of continuous time Markov processes in a geometric Brownian motion setting. Fur-

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thermore, they work under the restrictive assumption that the Laplace transform of the law of interarrival times is a rational function.

In the present note we consider a Sparre Andersen type non-life insurance model with investment in a risky asset. The price of this asset evolves according to an exponential Lévy process, while the business activity of the company is described by a compound renewal process with positive drift and negative jumps. In such a model, the reserves of the company are not a Markov process any more and standard techniques cannot be applied. However, we obtain an asymptotic result which is essentially of the same form as the main result in [6]. Only very weak assumptions are required on the interarrival times, like the existence of an exponential moment, which is obviously fulfilled in the case of the exponential distribution or the Erlang distribution, and their mixtures. As in [6], the main assumption is that the cumulant generating function $H : \gamma \mapsto \ln \mathbf{E} e^{-\gamma V_1}$ of the log price process V has a strictly positive root β which does not lie on the boundary of the effective domain of H. Assuming that the claims are random variables, whose power of order β is integrable, the ruin probability decays with the rate $u^{-\beta}$ (see Theorem 2.3). As in [6], our proofs are based on recent progress in the theory of distributional equations.

2. The model

Let $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ be a stochastic basis on which there is given a non-deterministic Lévy process $R = (R_t)_{t \geq 0}$ with $\Delta R > -1$ and an independent compound renewal process $P = (P_t)_{t \geq 0}$ with drift $c > 0$ and negative jumps. We denote by T_n, $n \geq 1$, the successive jump instants of the latter process with the convention $T_0 = 0$, and put $F(t) := \mathbf{P}(T_1 \leq t)$. We associate with R the stochastic exponential $\mathcal{E}(R)$ which is, due to the assumption on the jumps, a strictly positive process.

We will study the process $X = X^u$, $u > 0$, which is defined as the solution of the non-homogeneous linear stochastic equation

$$X_t = u + \int_0^t X_{s-} dR_s + P_t. \tag{1}$$

The process X can be written in the "dot" notation of stochastic calculus as $X = u + X_- \cdot R + P$ or, in its "differential" form, as $dX_t = X_{t-} dR_t + dP_t$, $X_0 = u$. The compound renewal process P is usually represented in the form

$$P_t = ct + \sum_{i=1}^{N_t} \xi_i, \tag{2}$$

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where N is a counting renewal process with interarrival times $T_i - T_{i-1}$, $i \geq 1$, which form an i.i.d. sequence, independent of the i.i.d. sequence $\xi_i = \Delta P_{T_i}$, $i \geq 1$. The case considered here ($c > 0$ and $\xi_i < 0$) corresponds to the non-life insurance situation. For $R = 0$ we have the classical Sparre Andersen model (which we already excluded by the assumptions made).

In the actuarial context $X = X^u$ represents the dynamics of the reserves of an insurance company that invests the total of these reserves in a stock with the price process $S = \mathcal{E}(R)$ which has unit as its initial value. Having in mind that $dR = dS/S_-$, it is natural to call R *relative price process*, while $V := \ln \mathcal{E}(R)$ is usually referred to as the *log price process*. Thus the price process S which we introduced here as a stochastic exponential can alternatively be expressed as an ordinary exponential $S = \exp(V)$. For the interplay between stochastic and ordinary exponentials see Theorem 3.49 in [2]. Note that V is again a Lévy process. In general this process will have jumps of arbitrary size in both directions. Typical examples which are used in financial models are generalized hyperbolic processes and various subclasses such as hyperbolic, normal inverse Gaussian or variance gamma processes.

The processes R and P generate the filtration $\mathbf{F}^{R,P} = (\mathcal{F}_t^{R,P})_{t\geq 0}$. Define $\tau^u := \inf\{t : X_t^u \leq 0\}$, which denotes the time point of ruin. The main object of our interest is the ruin probability $\Psi(u) := \mathbf{P}(\tau^u < \infty)$ and its asymptotic behavior as the initial reserve u tends to infinity.

Let (a, σ^2, Π) be the Lévy triplet of R corresponding to the standard truncation function $h(x) := x I_{\{|x|\leq 1\}}$. Putting $\bar{h}(x) := x I_{\{|x|>1\}}$, we can write the canonical decomposition of R in the form

$$R_t = at + \sigma W_t + h * (\mu - \nu)_t + \bar{h} * \mu_t. \tag{3}$$

Here W is a standard Wiener process, $\mu(dt, dx)$ is the random measure of jumps of R, i.e. the Poisson random measure with compensator $\nu(dt, dx) = dt\Pi(dx)$.

For notations and basic facts we recommend the books [5], Ch. 2, and [2], Chs. 2 and 4. As in [6] the symbols $\Pi(f)$ or $\Pi(f(x))$ stand for the integral of f with respect to the Lévy measure Π. Recall that $\Pi(x^2 \wedge 1) < \infty$.

Due to the independence of R and P, these processes have no common jumps. The solution of the generalised stochastic exponential (1) can be represented (see [2], Proposition 3.48) as

$$X_t^u := \mathcal{E}_t(R)(u + \mathcal{E}_-^{-1}(R) \cdot P_t). \tag{4}$$

Since we assume that $\Delta R > -1$ and R is nondeterministic, the Lévy measure

Π is concentrated on the interval $]-1,\infty[$ and the Lévy characteristics σ^2 and Π do not vanish simultaneously.

Recall that the stochastic exponential $\mathcal{E}_t(R)$ has explicit form

$$\mathcal{E}_t(R) = e^{R_t - \frac{1}{2}\sigma^2 t + \sum_{s \leq t}(\ln(1 + \Delta R_s) - \Delta R_s)}.$$

The log price $V = \ln \mathcal{E}(R)$ can then be expressed as

$$V_t = at - \frac{1}{2}\sigma^2 t + \sigma W_t + h * (\mu - \nu)_t + (\ln(1 + x) - h) * \mu_t. \qquad (5)$$

Its Lévy triplet is (a_V, σ^2, Π_V), where

$$a_V = a - \frac{\sigma^2}{2} + \Pi(h(\ln(1 + x)) - h)$$

and $\Pi_V = \Pi\varphi^{-1}$, $\varphi : x \mapsto \ln(1 + x)$.

The cumulant generating function $H : q \to \ln \mathbf{E}\, e^{-qV_1}$ of the random variable V_1 admits an explicit expression

$$H(q) := -a_V q + \frac{\sigma^2}{2}q^2 + \Pi\big(e^{-q\ln(1+x)} - 1 + qh(\ln(1 + x))\big). \qquad (6)$$

Its effective domain $\operatorname{dom} H := \{q :\ H(q) < \infty\}$ is the set $\{J(q) < \infty\}$, where

$$J(q) := \Pi\big(I_{\{|\ln(1+x)|>1\}}\, e^{-q\ln(1+x)}\big) = \Pi\big(I_{\{|\ln(1+x)|>1\}}(1 + x)^{-q}\big). \qquad (7)$$

Its interior is the open interval $(\underline{q}, \bar{q})$ with

$$\underline{q} := \inf\{q \leq 0 :\ J(q) < \infty\}, \qquad \bar{q} := \sup\{q \geq 0 :\ J(q) < \infty\}.$$

Being a convex function, H is continuous and admits finite right and left derivatives on $(\underline{q}, \bar{q})$. If $\bar{q} > 0$, then the right derivative satisfies

$$D^+ H(0) = -a_V - \Pi(\bar{h}(\ln(1 + x))) < \infty,$$

though it may be equal to $-\infty$. We do not exclude this case.

In the formulation of the asymptotic results we shall always assume that $\bar{q} > 0$ and that the equation $H(q) = 0$ has a root $\beta \in]0, \bar{q}[$. Since H is not constant, such a root is unique. Clearly, it exists if and only if $D^+ H(0) < 0$ and $\limsup_{q \uparrow \bar{q}} H(q)/q > 0$. In the case where $\underline{q} < 0$ the condition $D^- H(0) > 0$ is

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necessary to ensure that $H(q) < 0$ for values $q < 0$ which are sufficiently small in absolute value.

If $J(q) < \infty$, then the process $m = (m_t(q))_{t \geq 0}$ with

$$m_t(q) := e^{-qV_t - tH(q)} \tag{8}$$

is a martingale and

$$\mathbf{E}\, e^{-qV_t} = e^{tH(q)}. \tag{9}$$

Lemma 2.1. *Suppose that $Ee^{\varepsilon T_1} < \infty$ for some $\varepsilon > 0$. Let $\beta \in]0, \bar{q}[$ be the root of the equation $H(q) = 0$. If $q \in [\beta, \bar{q}[$ is such that $H(q) \leq \varepsilon/2$, then*

$$\mathbf{E}\, \sup_{s \leq T_1} e^{-qV_s} < \infty. \tag{10}$$

Proof. Let us take $r \in]1, \bar{q}/q[$ sufficiently close to 1 to ensure the inequality $H(qr) - rH(q)) < (1/2)\varepsilon$. Then

$$\mathbf{E}\, m_t^r(q) = e^{t(H(qr) - rH(q))} \leq e^{(1/2)\varepsilon t}.$$

Using Doob's inequality we get that

$$\mathbf{E} \sup_{s \leq t} m_s(q) \leq \mathbf{E} \sup_{s \leq t} m_s^r(q) \leq C_r e^{t(H(qr) - rH(q))} \leq C_r e^{(1/2)\varepsilon t}.$$

Note that

$$e^{-qV_s} = m_s(q)e^{sH(q)} \leq e^{tH(q)} \sup_{s \leq t} m_s(q) \leq e^{(1/2)\varepsilon t} \sup_{s \leq t} m_s(q).$$

Therefore,

$$\mathbf{E} \sup_{s \leq T_1} e^{-qV_s} = \int_0^\infty e^{(1/2)\varepsilon t} \mathbf{E} \sup_{s \leq t} m_s(q) F(dt) \leq C_r \mathbf{E}\, e^{\varepsilon T_1} < \infty.$$

The lemma is proven. □

Corollary 2.2. *Let $Q := e^{-V} \cdot P_{T_1}$. If the conditions of the above lemma are fulfilled and $\mathbf{E}\, |\xi_1|^\beta < \infty$, then $\mathbf{E}\, |Q|^\beta < \infty$.*

Proof. Take $q \in]\beta, \bar{q}[$ such that $H(q) < \varepsilon/2$. Using an obvious estimate and applying the Hölder inequality with $r := q/\beta > 1$ and $p := r/(r-1)$, we get that

$$\mathbf{E}\left(\int_{[0,T_1]} e^{-V_t} dt\right)^\beta \leq \mathbf{E}\, T_1^\beta \sup_{s \leq T_1} e^{-\beta V_s} \leq (\mathbf{E}\, T_1^{p\beta})^{1/p} \left(\mathbf{E} \sup_{s \leq T_1} e^{-qV_s}\right)^{1/r} < \infty.$$

If $\beta \leq 1$, the elementary inequality $||a| + |b||^\beta \leq |a|^\beta + |b|^\beta$ implies that

$$\mathbf{E}\,|Q|^\beta \leq c^\beta \mathbf{E}\left(\int_{[0,T_1]} e^{-V_t}dt\right)^\beta + \mathbf{E}\,e^{-\beta V_{T_1}}\mathbf{E}\,|\xi_1|^\beta < \infty.$$

If $\beta > 1$, we use the inequality $||a| + |b||^\beta \leq 2^{\beta-1}(|a|^\beta + |b|^\beta)$ and arrive to the same conclusion. $\square$

Our main result is the following theorem.

Theorem 2.3. *Suppose that there is $\beta \in]0, \bar{q}[$ such that $H(\beta) = 0$, $\mathbf{E}|\xi_1|^\beta < \infty$. and $\mathbf{E}\,e^{\varepsilon T_1} < \infty$ for some $\varepsilon > 0$. If $\sigma \neq 0$ or the law of $|\xi_1|$ has unbounded support, then*

$$0 < \liminf_{u\to\infty} u^\beta \Psi(u) \leq \limsup_{u\to\infty} u^\beta \Psi(u) < \infty. \tag{11}$$

If $\sigma = 0$ and the law of $|\xi_1|$ has bounded support, (11) also holds except for the case where $0 < \Pi(|h|) < \infty$ and $\Pi(]-1,0[)\Pi(]0,\infty[) = 0$. In the latter case one needs the additional assumption that $\mathbf{P}(T_1 \leq t) > 0$ for any $t > 0$.

Remark. In [7] it is shown that in the model where S is a geometric Brownian motion, $\beta := 2a/\sigma - 1 \leq 0$, and $\mathbf{E}\,e^{\varepsilon T_1} < \infty$ for some $\varepsilon > 0$, the ruin probability $\Psi \equiv 1$.

3. Ruin problem: reduction to a discrete time case

Since we consider the non-life insurance model, ruin may happen only at jump-times of P. This allows us to monitor the process X^u only along the sequence T_n, i.e. to reduce the problem to a discrete time setting. For this purpose we introduce the discrete time processes V_{T_n} and $Y_n := -e^{-V_-} \cdot P_{T_n}$, $n = 1, 2, \ldots$. Then

$$X^u_{T_n} = e^{V_{T_n}}(u - Y_n).$$

Note that Y_n is a Markov chain and

$$Y_n = -\sum_{k=1}^n e^{-V_{T_{k-1}}} \int_{]T_{k-1},T_k]} e^{-(V_{s-} - V_{T_{k-1}})}dP_s \tag{12}$$

Define the stopping time $\theta^u := \inf\{n \in \mathbb{N}: Y_n \geq u\}$ with respect to the discrete time filtration $\mathbf{G} = (\mathcal{G}_n)_{n\geq 1}$, where $\mathcal{G}_n := \mathcal{F}^{R,P}_{T_n}$. In the considered model ruin happens only at the jump-times of P. Hence, $\{\tau^u < \infty\} = \{\theta^u < \infty\}$ and $\Psi(u) = \mathbf{P}(\theta^u < \infty)$.

Lemma 3.1. *If $Y_n \to Y_\infty$ almost surely as $n \to \infty$, where Y_∞ is a finite random variable unbounded from above, then for all $u > 0$*

$$\bar{G}(u) \leq \Psi(u) \leq \bar{G}(u)/\bar{G}(0), \tag{13}$$

where $\bar{G}(u) := \mathbf{P}(Y_\infty > u)$.

Proof. Let θ be an arbitrary stopping time with respect to the discrete-time filtration $\mathbf{G}$. As we assume that the finite limit Y_∞ exists, the random variable

$$Y_{\theta,\infty} := \begin{cases} -\lim_{N \to \infty} \int_{]T_\theta, T_{\theta+N}]} e^{-(V_{t-} - V_{T_\theta})} dP_t, & \theta < \infty, \\ 0, & \theta = \infty, \end{cases}$$

is well defined. On the set $\{\theta < \infty\}$

$$Y_{\theta,\infty} = e^{V_{T_\theta}}(Y_\infty - Y_\theta) = X_{T_\theta}^u + e^{V_{T_\theta}}(Y_\infty - u). \tag{14}$$

Let ζ be a $\mathcal{G}_\theta$-measurable random variable. Since the Lévy process V starts afresh at θ, the conditional distribution of $Y_{\theta,\infty}$ given $(\theta, \zeta) = (t, x) \in \mathbb{Z}_+ \times \mathbb{R}$ is the same as the distribution of Y_∞. It follows that

$$\mathbf{P}(Y_{\theta,\infty} > \zeta, \ \theta < \infty) = \mathbf{E}\,\bar{G}(\zeta)\,\mathbf{1}_{\{\theta < \infty\}}.$$

Thus, if $\mathbf{P}(\theta < \infty) > 0$, then

$$\mathbf{P}(Y_{\theta,\infty} > \zeta, \ \theta < \infty) = \mathbf{E}\,(\bar{G}(\zeta)\,|\,\theta < \infty)\,\mathbf{P}(\theta < \infty).$$

Noting that $\Psi(u) := \mathbf{P}(\theta^u < \infty) \geq \mathbf{P}(Y_\infty > u) > 0$, we deduce from this using (14) that

$$\begin{aligned} \bar{G}(u) &= \mathbf{P}(Y_\infty > u, \ \tau^u < \infty) = \mathbf{P}(Y_{\theta^u,\infty} > X_{\tau^u}^u, \ \tau^u < \infty) \\ &= \mathbf{E}\,(\bar{G}(X_{\tau^u}^u)\,|\,\tau^u < \infty)\,\mathbf{P}(\tau^u < \infty) \geq \bar{G}(0)\,\mathbf{P}(\tau^u < \infty) \end{aligned}$$

because $X_{\tau^u}^u \leq 0$ on $\{\tau^u < \infty\}$. So, we get the upper bound in (13). The lower bound is obvious. $\square$

In view of the above lemma to prove Theorem 2.3 we need to show that a finite limit Y_∞ exists and is unbounded from above.

4. The almost sure convergence of Y_n

Lemma 4.1. *Suppose that* $\mathbf{E}\, e^{\varepsilon T_1} < \infty$ *and* $\mathbf{E}\, |\xi_1|^{\beta \wedge \varepsilon \wedge 1} < \infty$ *for some* $\varepsilon > 0$ *and there is* $\beta > 0$ *such that* $H(\beta) = 0$. *Then the sequence* Y_n *tends almost surely to a finite random variable* Y_∞.

Proof. Take $p \in]0, \beta \wedge \varepsilon \wedge 1[$. According to (12)

$$Y_n - Y_{n-1} = -e^{-V_{T_{n-1}}} \int_{]T_{n-1}, T_n]} e^{-(V_{s-} - V_{T_{n-1}})} dP_s = M_1 ... M_{n-1} Q_n$$

where

$$M_j := e^{-(V_{T_j} - V_{T_{j-1}})}, \qquad Q_n = - \int_{]T_{n-1}, T_n]} e^{-(V_{s-} - V_{T_{n-1}})} dP_s.$$

Clearly, $M_1 ... M_{n-1} Q_n$ is the product of independent random variables, where M_j are identically distributed and so are the random variables Q_n. Note that

$$\mathbf{E}\, M_1^p = \mathbf{E}\, e^{-pV_{T_1}} = \int_0^\infty \mathbf{E}\, e^{-pV_t} F(dt) = \int_0^\infty e^{tH(p)} F(dt) < 1.$$

Also $\mathbf{E}\, |Q_1|^p < \infty$ (formally we can not use Corollary 2.2 because we require here a weaker integrability condition on ξ_1 but its simple proof needs only a minor change).

Since $\mathbf{E}\, M_1^p ... M_{j-1}^p |Q_j|^p = \rho^{j-1} \mathbf{E}\, |Q_1|^p$, we have that

$$\mathbf{E} \sum_{j \geq 1} |Y_j - Y_{j-1}|^p < \infty$$

and, therefore, $\sum_{j \geq 1} |Y_j - Y_{j-1}|^p < \infty$ a.s.. Consequently, $\sum_{j \geq 1} |Y_j - Y_{j-1}| < \infty$ a.s. and, hence, the sequence Y_n converges a.s. to the finite random variable $Y_\infty := \sum_{j \geq 1} (Y_j - Y_{j-1})$. $\square$

So, Y_∞ is the sum of an absolutely convergent series. Putting $A_0 := 1$ and $A_n := M_1 ... M_n$ we get that

$$Y_\infty = \sum_{n=0}^\infty A_n Q_{n+1} = Q_1 + M_1 \sum_{n=1}^\infty \frac{A_n}{A_1} Q_{n+1} =: Q_1 + M_1 Y_{1,\infty}.$$

The random variable $Y_{1,\infty}$ (we abbreviate here the sum of the series starting from $n = 1$) is independent of (Q_1, M_1) and has the same law as Y_∞. Using the language of the implicit renewal theory this means that Y_∞ is the solution of the distributional (or random) equation

$$Y_\infty \stackrel{d}{=} Q + M Y_\infty, \quad Y_\infty \text{ independent of } (M, Q), \tag{15}$$

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where $(M, Q) := (M_1, Q_1)$.

Note that

$$Y_\infty = Q_1 + A_1 Q_2 + \ldots + A_{n-1} Q_n + \sum_{k=n}^\infty A_k Q_{k+1} = Q_1 + A_1 Q_2 + \ldots + A_n Y_{n,\infty},$$

where

$$Y_\infty \overset{d}{=} Y_{n,\infty} := \sum_{k=n}^\infty \frac{A_k}{A_n} Q_{k+1}.$$

The following theorem adapted to our needs combines several results of the implicit renewal theory, see Th. A.6 in [6].

Theorem 4.2. *Suppose that for some $\beta > 0$,*

$$\mathbf{E}\, M^\beta = 1, \quad \mathbf{E}\, M^\beta \, (\ln M)^+ < \infty, \quad \mathbf{E}\, |Q|^\beta < \infty. \tag{16}$$

Then $\limsup u^\beta \bar{G}(u) < \infty$. *If the random variable Y_∞ is unbounded from above, then* $\liminf u^\beta \bar{G}(u) > 0$ *and in the case where the law of $\ln M$ is non-arithmetic, $\bar{G}(u) \sim C_+ u^{-\beta}$ where $C_+ > 0$.*

Proof of Theorem 2.3. We already know that the sequence of random variables Y_n converges a.s. to a finite random variable Y_∞ (see Lemma 4.1 whose conditions are weaker than those of Theorem 2.3). Moreover, we just proved that Y_∞ is the solution of distributional equation (15) and $E\,|Q|^\beta < \infty$ (Corollary 2.2). If Y_∞ is unbounded from above we can use Lemma 3.1 relating the asymptotic of the ruin probability with the tail behavior of the distribution function of Y_∞ and get the claimed result from the theorem cited above. In the next section we show, assuming that the almost sure limit Y_∞ exists, that Y_∞, indeed, is always unbounded, except the case where $\sigma = 0$, the random variable ξ_i is bounded, $0 < \Pi(|h|) < \infty$ and $\Pi(]-1, 0[)\Pi(]0, \infty[) = 0$ for which we need the additional assumption $\mathbf{P}(T_1 \le t) > 0$ for any $t > 0$. □

5. When is Y_∞ unbounded from above ?

We start from the following elementary consideration assuming that the limit Y_∞ exists and is finite.

Lemma 5.1. *If the random variables Q_1 and Y_n/A_n for some $n \ge 1$ are unbounded from above, then Y_∞ is unbounded from above.*

Proof. Take N such that $\mathbf{P}(Y_\infty > -N) > 0$. Since Y_n/A_n is unbounded from above and independent of the random variable $Y_{n,\infty}$ which has the same law as Y_∞, we get that

$$
\begin{aligned}
\mathbf{P}(Y_\infty > 0) &= \mathbf{P}(Y_n + A_n Y_{n,\infty} > 0) = \mathbf{P}(Y_n/A_n + Y_{n,\infty} > 0) \\
&\geq \mathbf{P}(Y_n/A_n \geq N, \ Y_{n,\infty} > -N) \\
&= \mathbf{P}(Y_n/A_n \geq N)\mathbf{P}(Y_{n,\infty} > -N) > 0.
\end{aligned}
$$

Thus, $\mathbf{P}(Y_{n,\infty} > 0) = \mathbf{P}(Y_\infty > 0) > 0$. Furthermore, if the random variable Q_1 is unbounded from above and $\mathbf{P}(Y_\infty > 0) > 0$, then Y_∞ is also unbounded from above. Indeed, take arbitrary $u > 0$, then

$$
\begin{aligned}
\mathbf{P}(Y_\infty > u) &\geq \mathbf{P}(Q_1 + A_1 Y_{1,\infty} > u, \ Y_{1,\infty} > 0) \geq \mathbf{P}(Q_1 > u, \ Y_{1,\infty} > 0) \\
&= \mathbf{P}(Q_1 > u)\mathbf{P}(Y_{1,\infty} > 0) = \mathbf{P}(Q_1 > u)\mathbf{P}(Y_\infty > 0) > 0.
\end{aligned}
$$

The lemma is proven. $\square$

We shall use the above lemma with $n = 1$ when $Y_1/A_1 = Q_1/A_1 = Q_1/M_1$ or with $n = 2$ when $Y_2/A_2 = Q_1/A_2 + Q_2/M_2$.

The arguments below use the following observation. Let ζ be a real-valued random variable and let η be a random variable with values in a Polish space. Let $\mathbf{P}_\zeta$ and $\mathbf{P}_\eta$ be their distributions and let $\mathbf{P}_{\zeta|x}$ be a regular conditional distribution of ζ given $\eta = x$. If for any real N the set $\{x \colon \mathbf{P}_{\zeta|x}([N, \infty)) > 0\}$ is not a $\mathbf{P}_\eta$-null set, then ζ is unbounded from above.

Lemma 5.2. *Let $K > 0$, $\sigma \neq 0$ and $t > s > 0$. Then the random variables*

$$
\zeta := K e^{\sigma W_t} - \int_0^t e^{\sigma W_r} dr, \qquad \tilde{\zeta} := K e^{\sigma(W_t - W_s)} - e^{\sigma W_t} \int_0^t e^{\sigma W_r} dr, \qquad (17)
$$

are unbounded from above.

Proof. Recall that the Wiener measure charges any open ball in the space $C_0([0, T])$ of continuous functions $x_. = (x_t)_{t \in [0,T]}$ with $x_0 = 0$ (in other words, it is of full support on this space). Let g be a continuous function on $C_0([0, T])$ such that $g(x^0_.) = N$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ such that $|g(x_.) - g(x^0_.)| < \varepsilon$ on the open ball $\{x_. \colon |x_. - x^0_.| < \delta\}$. In particular, $g(x_.) > N - \varepsilon$ on this ball. With this, the claims of the lemma are obvious. $\square$

Note that in the above lemma the sign of σ does not matter since $-W$ is again a Wiener process.

Let $\bar{V} := V - \sigma W$. Then

$$Q_1 = e^{-V_{T_1}}|\xi_1| - c\int_0^{T_1} e^{-V_r}dr \geq |\xi_1|e^{-\bar{V}_{T_1}}e^{-\sigma W_{T_1}} - c\sup_{r\leq T_1} e^{-\bar{V}_r}\int_0^{T_1} e^{-\sigma W_r}dr.$$

Taking into account the independence of ξ_1, T_1, $\bar{V}$ and W, we obtain using conditioning and Lemma 5.2, that Q_1 is unbounded from above.

The random variable Q_1/M_1 is unbounded from above if and only if $|\xi_1|$ has this property. To cover the general case we consider the random variable

$$Y_2/A_2 = e^{V_{T_2}-V_{T_1}}|\xi_1| + |\xi_2| - ce^{V_{T_2}}\int_0^{T_2} e^{-V_r}dr$$

$$\geq e^{V_{T_2}-V_{T_1}}|\xi_1| - ce^{V_{T_2}}\int_0^{T_2} e^{-V_r}dr. \tag{18}$$

The same arguments as above allow us to conclude that the random variable on the right-hand side is unbounded from above.

Thus, if $\sigma \neq 0$, the random variable Y_∞ is unbounded without further assumptions.

It remains to consider the case $\sigma = 0$. The further arguments are based on decompositions of Lévy processes without Gaussian component into sums of independent Lévy processes and the fact that a Poisson process has on a finite interval any number of jumps with strictly positive probability. The reasoning depends on the behavior of the Lévy measure.

1. Suppose that $\Pi(]-1,0[) > 0$ and, hence, $\Pi(]-1,-\varepsilon[) > 0$ for some $\varepsilon > 0$. We decompose V into a sum of two independent processes $V = V^{(1)}+V^{(2)}$, where $V^{(2)} = V - V^{(1)}$ and

$$V_s^{(1)} := I_{]-1,-\varepsilon[}\ln(1+x)*\mu_s + I_{]\varepsilon,\infty[}\ln(1+x)*\mu_s.$$

Note that $V^{(1)}$ is the sum of two independent compound Poisson processes with negative and positive jumps respectively. Using conditioning with respect to the random variables $V^{(2)}$, T_1, ξ_1, which are independent of $V^{(1)}$, we easily obtain that Q_1 is unbounded from above because for any $K > 0$ the random variable

$$\zeta := Ke^{-V_t^{(1)}} - \int_0^t e^{-V_r^{(1)}}dr, \quad K > 0, \tag{19}$$

is unbounded from above. The latter property is clear: we may consider trajectories where $V^{(1)}$ has only negative jumps and these jumps are concentrated near

t leading to large values for the first term and a small impact on the value of the integral.

If $|\xi_1|$ is unbounded, then Q_1/M_1 is unbounded from above and, hence, so is Y_∞.

If $|\xi_1|$ is bounded and $\Pi(]0,\infty[) > 0$ (hence for some sufficiently small $\varepsilon > 0$ both components of $V^{(1)}$ are nontrivial compound Poisson processes), we can check that Y_2/A_2 is unbounded from above. Indeed, from (18),

$$Y_2/A_2 \geq e^{V_{T_2} - V_{T_1}} \left(|\xi_1| - c e^{V_{T_1}} \int_0^{T_1} e^{-V_r} dr - c \int_{T_1}^{T_2} e^{-(V_r - V_{T_1})} dr \right).$$

Using conditioning, we reduce the problem to checking that for any $t > s > 0$ and $K > 0$ the random variable

$$e^{V_t^{(1)} - V_s^{(1)}} \left(K - e^{V_s^{(1)}} \int_0^s e^{-V_r^{(1)}} dr - \int_s^t e^{-(V_r^{(1)} - V_s^{(1)})} dr \right) \qquad (20)$$

is unbounded from above. The compound Poisson process $(V_r^{(1)})_{r \in [0,s]}$ has positive and negative jumps with intensities bounded away from zero and, therefore, for any $\varepsilon' > 0$ there is a non-null set Γ_1 on which

$$e^{V_s^{(1)}} \leq 1, \qquad \int_0^s e^{-V_r^{(1)}} dr \leq \varepsilon'.$$

The process $(V_r^{(1)} - V_s^{(1)})_{r \in [s,t]}$ is independent of $(V_r^{(1)})_{r \in [0,s]}$ and has the same jump properties. For an $\varepsilon' > 0$ and $N > 0$ there is a non-null set Γ_2 on which

$$e^{V_t^{(1)} - V_s^{(1)}} \geq N, \qquad \int_s^t e^{-(V_r^{(1)} - V_s^{(1)})} dr \leq \varepsilon'.$$

Taking $\varepsilon' \leq K/4$ we get that on the non-null set $\Gamma_1 \cap \Gamma_2$ the random variable defined in (20) is larger than $NK/2$ and, therefore, is unbounded from above.

2. Suppose that $\Pi(]0,\infty[) = 0$ and $\Pi(|h|) = \infty$. Then for any $N > 0$ there exists $\varepsilon > 0$ such that $\tilde{R}_\varepsilon := \Pi(I_{]-1,-\varepsilon[}|h|) \geq N$. Let us consider the decomposition $V = V^{(1)} + V^{(2)}$ into a sum of two independent processes, this time with

$$V_s^{(1)} := I_{]-1,-\varepsilon]} h * (\mu - \nu)_s + I_{]-1,-\varepsilon]}(\ln(1+x) - h) * \mu_s.$$

Again, even if $|\xi_1|$ is bounded, the random variable Y_2/A_2 is unbounded from above. Indeed, on the non-null set where the process $(V_r^{(1)} - V_s^{(1)})_{r \in [s,t]}$ has no jumps and

$$e^{V_s^{(1)}} \int_0^s e^{-V_r^{(1)}} dr \le K/2$$

the random variable given by (20) dominates the value

$$e^{\tilde{R}_\varepsilon(t-s)}\left(K/2 - \int_s^t e^{\tilde{R}_\varepsilon(s-r)} dr\right) \ge e^{N(t-s)}(K/2 - 1/N).$$

This implies the desired property.

The remaining subcase in which $\Pi(]-1,0[) > 0$ but $\Pi(]0,\infty[) = 0$ will be considered later.

3. Suppose that $\Pi(]-1,0[) = 0$ and $\Pi(h) = \infty$. Then for any $N > 0$ there exists $\varepsilon > 0$ such that $R_\varepsilon := \Pi(I_{]\varepsilon,1]}h) \ge N$. We consider again a decomposition $V = V^{(1)} + V^{(2)}$ in a sum of two independent processes. The process $V^{(1)}$ is now defined as

$$V_s^{(1)} \quad := \quad I_{]\varepsilon,1]}h * (\mu - \nu)_s + I_{]\varepsilon,1]}(\ln(1+x) - h) * \mu_s.$$

The set on which $V^{(1)}$ has no jumps on $[0,t]$ has strictly positive probability. On this set we derive the bound

$$Ke^{-V_t^{(1)}} - \int_0^t e^{-V_r^{(1)}} dr \ge Ke^{R_\varepsilon t} - \frac{1}{R_\varepsilon}e^{R_\varepsilon t} \ge e^{Nt}(K - 1/N) \qquad (21)$$

and we conclude as above that Q_1 is unbounded from above.

If $|\xi_1|$ is unbounded, then Q_1/M_1 is unbounded from above as well as Y_∞.

If $|\xi_1|$ is bounded we again can prove that Y_2/A_2 is unbounded from above. Indeed, on the set on which the process $V^{(1)}$ has no jumps on $[0,s]$ (hence, it is growing linearly on this interval) we conclude that

$$e^{V_t^{(1)}}\left(Ke^{-V_s^{(1)}} - \int_0^t e^{-V_r^{(1)}} dr\right) \ge e^{V_t^{(1)} - V_s^{(1)}}\left(K - \frac{1}{R_\varepsilon} - \int_s^t e^{V_s^{(1)} - V_r^{(1)}} dr\right). \quad (22)$$

Let us define $J = I_{]\varepsilon,1]}\ln(1+x) * \mu$, then $V_t^{(1)} - V_s^{(1)} = -R_\varepsilon(t-s) + J_t - J_s$ and $V_s^{(1)} - V_r^{(1)} \le R_\varepsilon(t-s) - (J_r - J_s)$. Since the increment $(J_r - J_s)$ may have arbitrary many jumps (all of a size larger than $\ln(1+\varepsilon)$) and this happens independently of $J_{r\in[0,s]}$, we get that the random variable given by (20) is unbounded from above.

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Lemma 5.3. *Suppose that* $\sigma = 0$, $\Pi(]-1,0[) = 0$ *and* $0 < \Pi(h) < \infty$. *If* $\mathbf{P}(T_1 \le t) > 0$ *for any* $t > 0$, *then* Y_∞ *is unbounded from above.*

Proof. In this case $V_t = -a_h t + L_t$ where the process $L_t := \ln(1+x) * \mu_t$ is increasing. The constant $a_h := \Pi(h) - a$ is strictly positive because the equality $\ln \mathbf{E} e^{-\beta V_1} = 0$ with $\beta > 0$ may hold only if $\mathbf{P}(V_1 < 0) > 0$. We have

$$Y_n := -e^{-V_-} \cdot P_{T_n} = \sum_{k=1}^{n} e^{a_h T_k - L_{T_k}} |\xi_k| - c \int_0^{T_n} e^{a_h s - L_s} ds. \qquad (23)$$

Let $\delta > 0$. Put $L_t^\delta := I_{\{x < \delta\}} \ln(1+x) * \mu_t$. Then for every $t \ge 0$

$$\mathbf{E} L_t^\delta = t \Pi(I_{\{x < \delta\}} \ln(1+x)) \downarrow 0, \quad \delta \downarrow 0.$$

Thus, for a sufficiently small δ the set $\{L_t^\delta \le 1\}$ is non-null. Also non-null is the set $\{L_t - L_t^\delta = 0\}$. Their intersection, due to the independence of L_t^δ and $L_t - L_t^\delta$, is non-null and the larger set $\{L_t \le 1\}$ is also non-null.

Take $K', \kappa > 0$ such that $\mathbf{P}(1/K' \le T_1 \le K') > 0$ and $\mathbf{P}(|\xi_1| > \kappa) > 0$. Then $\mathbf{P}(\inf_{k \le n} |\xi_k| > \kappa) > 0$ for each n due to the independence of the random variables ξ_k.

The independence of the interarrival times and the assumption on their distribution imply that for any n the set

$$\{1/K' \le T_1 \le K', \ T_k - T_{k-1} \le 2^{-(k-1)}, \ 1 < k \le n\}$$

is non-null as well as its intersection with the set $\{\inf_{k \le n} |\xi_k| > \kappa, L_{K'+2} \le 1\}$. On this intersection, which we denote by Γ_n, we minorate the sum in the right-hand side of (23) by replacing L_{T_k} by unit and majorate the integral by replacing L_s by zero. Taking into account that $T_n \le T_1 + 1 \le K' + 1$, we get in this way that

$$Y_n \ge \frac{\kappa}{e} \sum_{k=1}^{n} e^{a_h T_k} - \frac{c}{a_h} e^{a_h (T_1+1)} \ge \left(\frac{\kappa}{e} n - \frac{c}{a_h} e^{a_h}\right) e^{a_h T_1} \ge \left(\frac{\kappa}{e} n - \frac{c}{a_h} e^{a_h}\right) e^{a_h / K'},$$

where the last inequality holds for $n \ge c e^{a_h+1}/(\kappa a_h)$. Thus, for any fixed N we have on the set Γ_n the bound $Y_n \ge N$ when n is sufficiently large.

Take $y > 0$ such that $\mathbf{P}(Y_\infty > -y) > 0$. Due to the independence the set

$$\Gamma_n \cap \{T_{n+1} - T_n \le 1, \ Y_{n+1,\infty} > -y\}$$

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is non-null and on this set for sufficiently large n we have that

$$
\begin{aligned}
Y_\infty &= Y_{n+1} + A_{n+1} Y_{n+1,\infty} = Y_n + A_n Q_{n+1} + A_{n+1} Y_{n+1,\infty} \\
&= Y_n + e^{a_h T_{n+1} - L_{T_{n+1}}} |\xi_{n+1}| - c \int_{T_n}^{T_{n+1}} e^{a_h s - L_s} ds \\
&\quad + e^{a_h T_{n+1} - L_{T_{n+1}}} Y_{n+1,\infty} \\
&\geq Y_n - (c/a_h) e^{a_h(K'+2)} - e^{a_h(K'+2)} y \geq N - (c/a_h + y) e^{a_h(K'+2)}.
\end{aligned}
$$

Since N is arbitrary, the random variable Y_∞ is unbounded from above. □

Lemma 5.4. *Suppose that $\sigma = 0$, $\Pi(]0, \infty[) = 0$, and $0 < \Pi(|h|) < \infty$. If $\mathbf{P}(T_1 \leq t) > 0$ for any $t > 0$, then Y_∞ is unbounded from above.*

Proof. In this case $V_t = b_h t - L_t$ where the process $L_t := -\ln(1 + x) * \mu_t$ is increasing and the $b_h := a - \Pi(h) > 0$. We have

$$
Y_n := -e^{-V_-} \cdot P_{T_n} = -\sum_{k=1}^{n} e^{-b_h T_k + L_{T_k}} \xi_k - c \int_0^{T_n} e^{-b_h s + L_s} ds.
$$

As in the proof of the previous lemma, we get that for any $t \geq 0$ the set $\{L_t \leq 1\}$ is a non-null set, there are constants κ and K' such that $\mathbf{P}(|\xi_k| > \kappa) > 0$ for each k and $\mathbf{P}(1/K' \leq T_1 \leq K') > 0$. For any n the set

$$
\{1/K' \leq T_1 \leq K',\ T_k - T_{k-1} \leq 2^{-(k-1)},\ 1 < k \leq n\}
$$

is non-null and its intersection with the set $\{\inf_{k \leq n} |\xi_k| > \kappa, L_{K'+2} \leq 1\}$ is non-null. On this intersection, denoted Γ_n,

$$
Y_n \geq \kappa e^{-b_h T_1} \sum_{k=1}^{n} e^{-b_h(T_k - T_1)} - \frac{ce}{b_h} \geq \kappa e^{-b_h(K'+1)} n - \frac{ce}{b_h} \geq N
$$

for any N when n is sufficiently large.

Take $y > 0$ such that $\mathbf{P}(Y_\infty > -y) > 0$. Then the set

$$
\Gamma_n \cap \{T_{n+1} - T_n \leq 1,\ Y_{n+1,\infty} > -y\}
$$

is non-null and on this set for sufficiently large n we have that

$$
\begin{aligned}
Y_\infty &= Y_n + e^{-b_h T_{n+1} + L_{T_{n+1}}} |\xi_{n+1}| - c \int_{T_n}^{T_{n+1}} e^{-b_h s + L_s} ds \\
&\quad + e^{-b_h T_{n+1} + L_{T_{n+1}}} Y_{n+1,\infty} \\
&> Y_n - ce - ey \geq N - e(c + y).
\end{aligned}
$$

Since N is arbitrary, the random variable Y_∞ is unbounded from above. □

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