Valuation in Illiquid Markets

Ernst Eberlein

Department of Mathematical Stochastics
University of Freiburg
Eckerstrasse 1
79104 Freiburg
Germany

e-mail: eberlein@stochastik.uni-freiburg.de

April 27, 2015

Valuation in illiquid markets

Ernst Eberlein

Abstract

What is the value of a financial instrument in an illiquid market? The classical valuation theory which is based on the law of one price assumes implicitly that market participants can trade freely in both directions at the same price. In the absence of perfect liquidity the law of one price should be replaced by a two price theory where the terms of trade depend on the direction of the trade. A static as well as a continuous time theory for two price economies is discussed. The two prices are termed bid and ask or lower and upper price but they should not be confused with the vast literature relating bid-ask spreads to transaction costs or other frictions involved in modeling financial markets. The bid price arises as the infimum of test valuations given by certain market scenarios whereas the ask price is the supremum of such valuations. The two prices correspond to nonlinear expectation operators. Specific dynamic models which are driven by purely discontinuous Lévy processes are considered.

This article emerged from papers written jointly with D. Madan, M. Pistorius, W. Schoutens and M. Yor (2014) [9, 10].

1 Introduction

Market liquidity may be loosely defined as the ease with which assets can be bought or sold. For a specific market, the degree of liquidity depends on a number of factors. One not so obvious but rather crucial factor is trust. When trust evaporates liquidity too will dry up. Changes in liquidity often occur as a result of a change in risk perception. The start of the 2007–2009 financial crisis illustrates this in a dramatic way. After a long period with plentiful trust and liquidity the market realized in July 2007 that there was a substantial default risk in portfolios underlying the tranches of Collateralized Debt Obligations (CDOs) in which a number of banks were heavily invested.

These portfolios had been classified as subprime. The initially AAA rated CDO tranches became totally illiquid within a short period. Consequently the values of these assets had to be written down when the portfolios of these investors were revalued.

One can try to categorize markets according to their liquidity. A rough scale would start with those instruments, that are traded at organized exchanges, being considered the most liquid. Examples are the exchanges for equity shares and for derivatives. The next category in such a liquidity scale would be bonds. It is evident that there are big differences within this asset class. The rating, (by which we do not necessarily mean the ratings issued by rating agencies), typically has some influence on the liquidity of the instrument, but ratings do not take specific account of liquidity. Rather, ratings quantify credit risk. Liquidity risk is a second order risk and often reacts to changes in the credit standing of the bond. Sovereign bonds feature at the top of this asset class but there are many corporate bonds that outperform sovereign bonds issued by governments of countries with a fragile economy. In decreasing order of liquidity another category in the liquidity scale are structured products that are only Over-the-Counter traded. Bespoke financial contracts which usually cannot be sold to a third party at all constitute a huge and even less liquid asset class. Insurance contracts would be typical examples of this kind, but note that there is a reinsurance market where some part of the risk can be traded. At the end of our liquidity scale one could place real estate. Real estate can be classified as a financial instrument since it is included within the portfolio of many institutional investors.

Among the many problems posed by a lack of liquidity is the question: What is the value of assets under these market conditions? We prefer here to use the word *value* instead of *price*. Pricing puts one primarily into the perspective of a trader who wants to buy or to sell a security. Valuation is a much broader notion. Investors often have a very long investment horizon. Nevertheless they need a value for each asset when they revalue their portfolio. This is necessary for active portfolio management on a daily, weekly or monthly basis or to produce a balance sheet even when they do not have any intention to sell. The same consideration applies to liabilities on the balance sheet of a financial institution.

The classical valuation theory which is based on the *law of one price* assumes implicitly that market participants can trade freely in both directions at the same price. This means in particular that the market is able to accept any amount of assets which are traded at the current market price whatever the direction of the trade is. In the absence of perfect liquidity the law of one price can no longer be justified. It should be replaced by a *two price theory* where the terms of trade depend on the direction of the trade. Let us illustrate this fundamental change of view by considering as an example sovereign bonds issued by a government that has already accumulated substantial debt and continues to produce a big budget deficit. At some point there will be a trigger whereby the market starts to worry about a possible default of the issuer and the partial or even total losses that could occur. As a consequence of the perception of an increased risk these bonds become less attractive for cautious investors and the price quotes will go down. Investors that hold these bonds on the asset side of their balance sheet will have to write the value of this position down. In fact the current mark to market accounting rules force them to do so even though they might not have the intention to sell the position at the reduced price which could afterwards recover again. Markets often overreact or even panic. A striking example where overreaction caused enormous financial damage was the situation of insurers – in their role as institutional investors – after the burst of the internet bubble in 2000. Stock quotes which had reached record levels in the internet rally fell sharply. In order to stimulate markets and economies interest rates were pushed down by monetary authorities and thus the discount factors applied to long term liabilities went up with the consequence that liabilities in the balance sheets of those insurers increased. At the same time they had to write down the investments in stock on the asset side of their balance sheet. In deed, it was reported that supervising agencies in some countries asked insurance companies in this situation to close the gap in their asset-liability relation. They did this by selling blue chips which were their most liquid assets. These sales contributed to drive stock markets down further. Billions of euros and dollars – wealth of investors in life insurance – were destroyed unnecessarily. The situation reached such a point that some insurance companies had to raise fresh capital. In this context it is worth recalling that the current mark to market accounting rule was inspired by the savings and loan crisis that had shaken the United States in the late 80s. According to the accounting principles at that time banks had been allowed to carry initial values of credits forward although, it was evident their debtors were already insolvent. Mark to market as an accounting principle represents the other extreme and may not be the optimal solution either.

Now let us turn to the balance sheet of the issuer of the government bond referred to above. The position is now on the liability side of the balance sheet. Following the mark to market accounting rule a lower market quote would reduce the debt and hence generates a profit for the issuer. These so-called “windfall” profits have been observed in 2008. Indeed, major US banks reported hundreds of millions US dollars of profits caused by a reduction of their debt positions when their ratings were downgraded in the crisis and price quotes for bonds fell. Pursuing this line of thought an issuer of bonds might even register record profits just prior to its own default. Does this make any sense? Certainly not, unless one is prepared to accept the idea that windfall profits help a distressed company to dress up its balance sheet. An issuer in distress cannot exploit the low price for its debt position. Funds to buy bonds back will not be available in such a situation. If the issuer could take advantage of the low price and reduce the debt, the market would certainly realize the new situation and the quote for the bond would jump up immediately. Thus the lower price at which the bonds are offered in the market has no practical relevance for the issuer. On the balance sheet the debt position should be reported essentially at the initial value which it had before the deterioration of the credit status. This is roughly the sum which the issuer in the event of survival will ultimately have to redeem at maturity. As a consequence we are led to a two price valuation for such a financial instrument. Figure 1 shows how the two values should evolve qualitatively as a function of the default probability of the issuer of the bond.

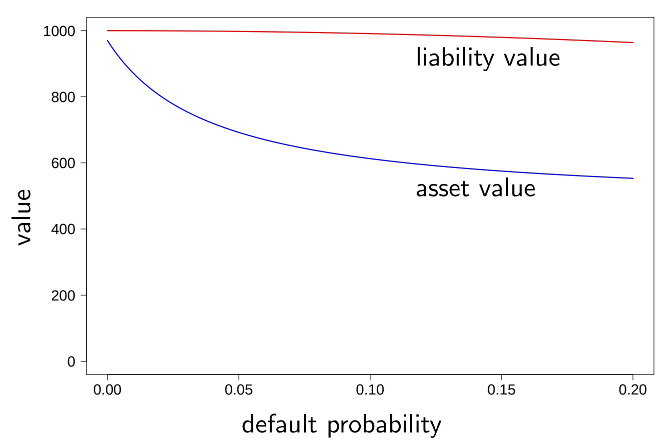

Figure 1 Asset and liability value as a function of default probabilities

The structure of the paper is as follows. In the next section we present the two price valuation approach in the context of a simple one period model. The corresponding dynamic theory in a continuous time model is developed in section 4. In section 3 we discuss briefly the Feynman-Kac representation on which the dynamic two price valuation approach is based from a technical point of view.

2 One period two price theory

The two price approach is based on the notion of acceptability of a cash flow. From the point of view of mathematics the outcome of a risky position is described by a random variable X defined on an appropriate probability space $(\Omega, \mathcal{F}, P)$. In a perfectly liquid market the current value of this position can be determined once one has a pricing kernel given by a risk neutral probability measure Q. The value is given by $E^Q[X]$, the expectation of X under Q. In this and the next section for simplicity we assume that the interest rate is zero. One classifies the position X as *acceptable* in case its average outcome is nonnegative, i.e. $E^Q[X] \geq 0$. In a liquid market the pricing operator given by the risk neutral measure Q can be derived from market data. To be more precise, after making appropriate model assumptions for the random variable X one can infer the actual parameters from historic price data or actual prices and then switch from this distribution to a risk neutral distribution. There is a vast literature on this and in particular on the problems which arise in the context of the measure change. We do not discuss any further details on this issue here. In case there is a derivative market with the quantity of interest as underlying, one would calibrate the assumed model by using price data from the derivative market. This direct approach allows to avoid any measure change since derivative prices are assumed to be risk neutral and the inferred Q is the natural candidate.

As mentioned above there are large financial markets which are not liquid enough to produce reliable price data. Given the uncertainty about the right valuation operator, instead of a unique Q one should take a whole set $\mathcal{M}$ of possible probability measures or scenarios $Q \in \mathcal{M}$ into account. The set $\mathcal{M}$ will usually have elements which are not risk neutral measures, but we assume that there is at least one risk neutral measure in the set. Once the risk neutral probability measure which is denoted by Q^* is determined, this distinguished element of $\mathcal{M}$ will enable us to derive in addition to two new values also a classical risk neutral value for the security to be valued. A risk neutral measure Q^* will later be used as a basis for the construction of appropriate sets $\mathcal{M}$. In this context we want to mention that in incomplete markets the set of risk neutral measures can be very large. It has been shown in Eberlein and Jacod (1997) [7] that in certain models for incomplete markets, in particular in exponential models driven by pure jump Lévy processes, the values derived from the set of risk neutral measures can span the full no-arbitrage interval (excluding the boundaries). The latter is the interval in which security values must necessarily lie as a result of arbitrage considerations.

Once an appropriate set $\mathcal{M}$ of probability measures has been chosen, a position X is considered to be *acceptable* if the average outcome of X is nonnegative under all $Q \in \mathcal{M}$, which can equivalently be expressed as

$$\inf_{Q\in\mathcal{M}} E^Q[X] \geq 0. \tag{1}$$

The first versions of this concept have been considered in Artzner, Delbaen, Eber and Heath (1999) [1] and Carr, Geman and Madan (2001) [3]. Because of the infimum this formula defines a nonlinear valuation operator which represents a relatively unpleasant object from the mathematical point of view. However under only slightly more restrictive assumptions this valuation can be made operational due to the following link between acceptability and concave distortions. Assume the set $\mathcal{M}$ is convex and the operator given by

$$\varrho(X) = -\inf_{Q\in\mathcal{M}} E^Q[X] = \sup_{Q\in\mathcal{M}} E^Q[-X] \tag{2}$$

is law invariant and comonotone, i.e. $\varrho(X)$ is a spectral risk measure. Comonotonicity means that the operator is additive for comonotone random variables. The later makes sense since a risk measure should add up the risks given by random positions which are perfectly correlated. Under these additional assumptions one can show (see, e.g., Cherny and Madan (2009) [4]) that there exists a concave distortion Ψ such that $\varrho(X)$ can be represented as an integral with respect to the distorted distribution function F of X (under Q^*)

$$\varrho(X) = -\int_{-\infty}^{+\infty} x d\Psi(F(x)). \tag{3}$$

The effect of Ψ is that it shifts probability mass to the left. Unfavourable outcomes of X thus get a higher probability weight and consequently favourable outcomes will occur with a lower probability. In other words one can say that the distorted valuation operator produces – as it should – a more conservative value than the undistorted operator. Acceptability means then just that the integral is nonnegative. In addition the corresponding set of probability measures $\mathcal{M}$, the so-called supporting set, can be easily described by the distortion function Ψ, namely

$$\mathcal{M} = \{Q \in \mathcal{P} \mid \widehat{\Psi}(Q^*(A)) \leq Q(A) \leq \Psi(Q^*(A)) \ \ (A \in \mathcal{F})\}$$

where $\widehat{\Psi}(x) := 1 - \Psi(1 - x)$.

There are many families of distortions which can be used in this context. An excellent choice is the family termed *minmaxvar* which has a real, nonnegative parameter γ (see Cherny and Madan (2009) [4]). It is given by

$$\Psi^\gamma(x) = 1 - \left(1 - x^{\frac{1}{1+\gamma}}\right)^{1+\gamma} \qquad (0 \leq x \leq 1, \ \ \gamma \geq 0),$$

where γ can be interpreted as a *stress parameter*. Higher γ means more deviation from the identity, thus stronger concavity and consequently more distortion of the underlying distribution function F. Figure 2 shows the distortion Ψ^γ for various values of γ.

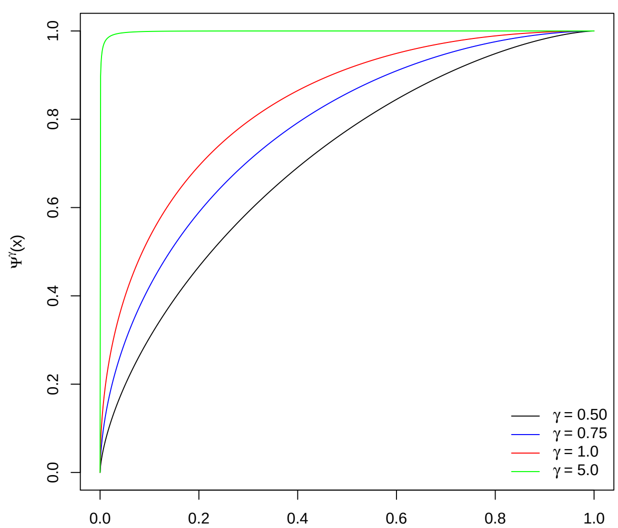

Figure 2

There is a rather intuitive statistical justification for this particular family minmaxvar. Assume for this γ to be an integer. Consider independent draws of a random variable Z, given by $Z_1, \ldots, Z_{\gamma+1}$, such that the maximum of these $\gamma + 1$ variables has the same distribution as X. Then $\varrho_\gamma(x) = -E[Y]$, where Y is a random variable which has the same distribution as the minimum of $Z_1, \ldots, Z_{\gamma+1}$. Thus the minmaxvar distortion leads to a distorted expectation operator where one is conservative in valuation of cash flows in a double sense, first by replacing the distribution of X by the distribution of the maximum of $\gamma + 1$ independent draws of a random variable Z and then by replacing the expectation of X by the expectation of a variable Y which has the same distribution as the minimum of the $\gamma + 1$ draws of Z.

Now let us consider a nonnegative random cash flow X on the left side of the balance sheet, i.e. an asset. The best value for this position from the perspective of the market is the largest value $b(X)$ such that $X - b(X)$ is acceptable. The market would not accept any higher price in case we try to sell this position. Applying the definition of acceptability as given in (1) we see that

$$b(X) = \inf_{Q \in \mathcal{M}} E^Q[X]. \tag{4}$$

We call $b(X)$ the *bid* or *lower value* of X. If X is a nonnegative random cash flow on the right side of the balance sheet, i.e. a liability, then the best value for this position is the smallest value $a(X)$ such that $a(X) - X$ is acceptable. As an immediate consequence we see that

$$a(X) = \sup_{Q \in \mathcal{M}} E^Q[X], \tag{5}$$

$a(X)$ is called the *ask* or *upper value* of X. Let us underline that although we use the notion of bid and ask here, we do not have the bid and ask prices for securities in highly liquid markets in mind. In first approximation the bid-ask spread in very liquid markets can be considered as a deterministic quantity which consists of fixed transaction costs and the cost for liquidity providers. In this sense the use of the notions of lower and upper value would be less confusing. The spread which is generated by the two values in (4) and (5) is a dynamic quantity and depends on the degree of liquidity. When the market turns less liquid one would consider even more possible scenarios, which means $\mathcal{M}$ would increase and as a consequence the spread between lower and upper value widens. On the contrary an improvement of liquidity might encourage market participants to no longer taking some of the worst scenarios into account. The smaller set $\mathcal{M}$ leads to a decreased spread. Expressed in terms of the parameter γ of the *minmaxvar* distortion considered above, this means to increase γ as liquidity weakens and to decrease it again as soon as liquidity improves.

In the valuation formulas above only nonnegative cashflows are considered. However there are many random cashflows such as swaps which can end with a positive or a negative payoff. For completeness we point out that the two value approach produces the right valuation for these cash flows as well. Represent the position X as the difference of the positive part of X minus its negative part. In case X is an asset, one sees easily that its value, the bid, is nothing but the bid of the positive part of X minus the ask of the negative part. On the other side if X represents a liability, then the best value, the ask, is the ask of the positive part minus the bid of the negative part.

Using distortions one gets the following explicit formulas for lower and upper values of a cashflow X

$$\begin{aligned} b(X) &= \int_{-\infty}^{+\infty} x d\Psi(F_X(x)) \\ a(X) &= -\int_{-\infty}^{+\infty} x d\Psi(1 - F_X(-x)) \end{aligned} \tag{6}$$

In the following it will be sufficient to derive bid values since (see (2)) the ask value is nothing but the negative of the bid value of the corresponding negative cash flow. For specific cash flows these valuation formulas become even more explicit. As an example let us consider the bid $bC(K,T)$ of a call option with strike K and maturity T. It can be derived in the form (see Madan and Cherny (2010) [13])

$$bC(K,T) = \int_K^{\infty} (1 - \Psi(F_{S_T}(x)))dx, \tag{7}$$

where S_T denotes the random payoff at maturity.

3 Some remarks on the Feynman–Kac representation

Before we are able to develop a dynamic two price theory, i.e. a theory where the two prices evolve in time, we have to comment on some recent results concerning the Feynman–Kac formula since this formula will be used as a basic ingredient. In the seminal paper by Black and Scholes (1973) [2], the present value of an option was derived as the solution of a partial differential equation, namely the heat equation. The alternative approach in terms of a purely stochastic representation of the option value as expectation with respect to a risk-neutral measure, i.e. as a result within martingale theory, emerged only later. Key references for the application of martingale theory in the context of the valuation of derivatives are Harrison and Kreps (1979) [11] and Harrison and Pliska (1981) [12]. There is a deep relation between the two approaches which come from fairly disjoint mathematical disciplines namely partial differential equations (PDEs) and the theory of stochastic processes. This bridge is given by the Feynman–Kac formula.

Let us consider a derivative given by its payoff function g. As an example we could consider a call option with $g(x) = (S_0e^x - K)^+$. Here we assume that the price process of the underlying quantity is given by an exponential model $S_t = S_0 \exp(L_t)$ with driving process L. Assuming for simplicity that the interest rate is zero, the fair value of this option at time t is given by $E[g(L_T - L_t + x)]$ where E denotes the expectation operator under the risk-neutral probability, T is the time of maturity and x the value of the driving process at time t. The class of driving processes we are interested in is the class of time-inhomogeneous Lévy processes which has very successfully been used in many areas in financial modeling in recent years. See e.g. Eberlein and Kluge (2005) [8] for its use in interest rate theory. This class of processes includes processes with jumps or even pure jump processes such as generalized hyperbolic, normal inverse Gaussian, Variance Gamma or CGMY processes. The Feynman–Kac representation says that under appropriate assumptions the expectation above can be obtained as the solution of the following parabolic equation

$$\begin{aligned} \partial_t u + \mathcal{A}_{T-t} u = 0 \quad (r = 0), \\ u(0) = g. \end{aligned} \tag{8}$$

Here $\mathcal{A}$ denotes the pseudo-differential operator given by the negative of the infinitesimal generator $\mathcal{L}$ of the driving process L. As soon as L is a process with jumps, equation (8) becomes a partial integro-differential equation (PIDE). For the solution $u(t, x)$ of equation (8) Feynman–Kac reads now formally as

$$u(T - t, x) = E[g(L_T - L_t + x)]. \tag{9}$$

There are many settings where the Feynman–Kac formula holds in the literature, mainly for diffusion processes. To be able to exploit it in finance where one considers models which are driven by more realistic processes as mentioned above, some basic requirements are necessary. In (8) one has to consider unbounded domains since the domain is the range of the driving process L. Furthermore, the initial condition should cover a large range of payoff functions. Assuming polynomial boundedness or Lipschitz-continuity for g – as is done in some versions in the literature – would exclude a priori a number of standard payoffs. Furthermore what is desirable from the point of view of numerical solutions of (8) is the existence of a variational or weak solution. Viscosity solutions are less adequate. An approach which fulfills these specific requirements and allows to consider rather flexible models is given in Eberlein and Glau (2014) [6]. For further details we refer the interested reader to this reference where as a side result the intimate relationship between PIDE (or PDE) based valuation methods and Fourier based methods is discussed. Eberlein (2013) [5] is a recent survey article on the latter.

4 Continuous time two price theory

We consider now a continuous time model where the underlying uncertainty is given by a pure jump Lévy process $X = (X_t)_{0 \le t \le T}$. Such a process is determined by a drift coefficient α and the Lévy measure $k(y)dy$ which describes the frequency of the jumps. For example in the case of a Variance Gamma process X the Lévy density k is of the form

$$k(y) = \frac{C}{|y|}(\exp(-G|y|)1_{\{y<0\}} + \exp(-M|y|)1_{\{y>0\}}) \tag{10}$$

with parameters C, G and M. The infinitesimal generator $\mathcal{L}$ of such a pure jump process is

$$\mathcal{L}u(x) = \alpha \frac{\partial u}{\partial x}(x) + \int_{\mathbb{R}} \left(u(x+y) - u(x) - \frac{\partial u}{\partial x}(x)y \right) k(y)dy \tag{11}$$

Figure 3 shows the Lévy density k for a Variance Gamma process with specific parameter values.

We want to value a financial contract which pays the amount $\phi(X_t)$ at time t. Denote by $u(x,t)$ its time zero value when $X_0 = x$. Assume that the interest rate r is constant and that we have chosen a risk neutral probability measure. The latter means that under this probability the underlying price process when discounted is a martingale. Then with E denoting the corresponding expectation operator we have

$$u(x,t) = E\big[e^{-rt}\phi(X_t) \mid X_0 = x\big] \tag{12}$$

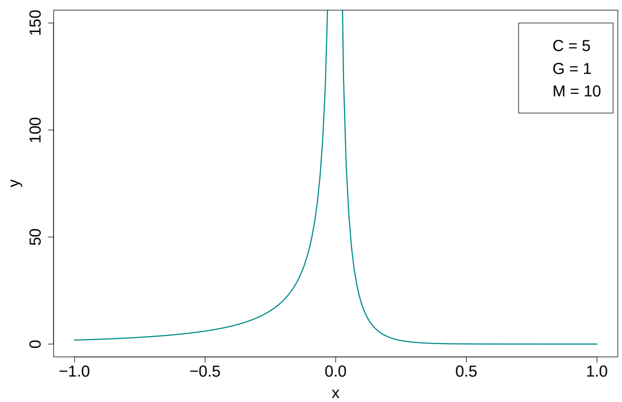

Figure 3 Variance gamma density

Assume that ϕ and the process which drives the model are such that the Feynman–Kac representation applies, then $u(x,t)$ is at the same time given by the solution of the PIDE

$$u_t = \mathcal{L}(u) - ru \tag{13}$$

with boundary condition $u(x,0) = \phi(x)$. We will consider two different approaches to distort equation (13) (see Eberlein, Madan, Pistorius, Schoutens and Yor (2014) [9]). For the first variant we assume that the Lévy density k satisfies $\int y^2 k(y)dy < \infty$, which is the case in a number of examples. Under this assumption

$$g(y) = \frac{y^2 k(y)}{\int y^2 k(y)dy} \tag{14}$$

is the density of a probability measure. Write the integral part of the operator $\mathcal{L}$ in equation (13) in the form

$$\int_{\mathbb{R}} \frac{u(x+y,t) - u(x,t) - u_x(x,t)y)}{y^2} \left(\int_{\mathbb{R}} y^2 k(y)dy \right) g(y)dy \tag{15}$$

or for short $\int_{\mathbb{R}} Y_{x,t}(y)g(y)dy$, where $Y_{x,t}$ is a real-valued random variable with distribution function

$$F_{Y_{x,t}}(v) = \int_{A(x,t,v)} g(y)dy \tag{16}$$

for $A(x,t,v) = \{y \mid Y_{x,t}(y) \leq v\}$. With this notation the integral part of the PIDE (13) has the form

$$\int_{\mathbb{R}} v dF_{Y_{x,t}}(v). \tag{17}$$

Given a concave distortion Ψ we will consider the distorted expectation

$$\int_{\mathbb{R}} v d\Psi(F_{Y_{x,t}}(v)). \tag{18}$$

In order to get a form in which this integral can be easily computed we decompose it into the negative and positive halfline and after using the integration by parts formula achieve the following representation where P^g indicates that probabilities are evaluated under the density g.

$$-\int_{-\infty}^{0} \Psi(P^g(Y_{x,t} \leq v))dv + \int_{0}^{\infty} (1 - \Psi(P^g(Y_{x,t} \leq v)))dv. \tag{19}$$

Define the new (distorted) operator

$$\mathcal{G}_{QV}u(x) = \alpha\frac{\partial u}{\partial x}(x) - \int_{-\infty}^{0} \Psi(P^g(Y_{x,t} \leq v))dv + \int_{0}^{\infty} (1 - \Psi(P^g(Y_{x,t} \leq 0)))dv \tag{20}$$

The bid value is the solution of the (distorted) PIDE

$$u_t = \mathcal{G}_{QV}(u) - ru. \tag{21}$$

The alternative approach replaces the density g defined above by a density h which is given by

$$h(y) = \frac{k(y)}{\int_{\{|y| \geq \epsilon\}} k(y)dy} \mathbb{1}_{\{|y| \geq \epsilon\}}. \tag{22}$$

In other words instead of exploiting a second moment property of the Lévy density k, we truncate k at the origin. Both approaches have essentially the effect that in a first step the very small jumps of the driving process are neglected. Proceeding with h instead of g one gets an alternative operator $\mathcal{G}_{NL}$ which can also be used to distort the PIDE (13). Figure 4 shows values of a sophisticated portfolio of derivatives as a function of the value of the underlying expressed in terms of moneyness. In the middle of the five value functions is the risk neutral value. The two lower lines are the bid values

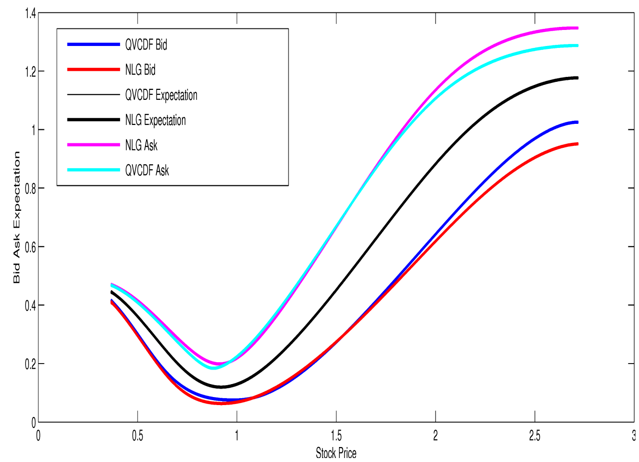

Figure 4

computed according to the two methods described above. The two upper lines are the corresponding ask values.

In Eberlein, Madan, Pistorius and Yor (2014) [10] two specific valuation situations are studied in detail. The first one considers the valuation of contracts with very long maturities. Typical examples are insurance contracts with an extremely long life time. In this context one could also consider the valuation of economic activities such as a company as a whole, which do not have a maturity at all. The second object which is studied concerns the two price valuation of insurance loss processes. Here the accumulated losses from now on into the distant future are investigated. In this context we develop in [10] another alternative to the two approaches which are outlined above. Instead of creating artificially probabilities given by the two densities g and h, one can as well directly distort a measure as long as the corresponding integral is finite.

References

[1] Artzner, P., F. Delbaen, J. M. Eber and D. Heath. Coherent measures of risk. *Mathematical Finance*, 9 (1999) 203–228

[2] Black, F. and M. Scholes. The pricing of options and corporate liabilities. *J. Political Econ.*, 81 (1973) 637–654

[3] Carr, P., H. Geman and D. Madan. Pricing and hedging in incomplete markets. *J. of Financial Economics*, 62 (2001) 131–167

[4] Cherny, A. and D. Madan. New measures for performance evaluation. *Rev. Financ. Stud.*, 22 (7) (2009) 2571–2606

[5] Eberlein, E. Fourier based valuation methods in mathematical finance. In *Quantitative Energy Finance*, F. Benth, V. Kholodnyi, and P. Laurence (eds.), Springer 2013, 85–114

[6] Eberlein, E. and K. Glau. Variational solutions of the pricing PIDEs for European options in Lévy models. *Applied Mathematical Finance*, 21 (5–6) (2014) 417–450

[7] Eberlein, E. and J. Jacod. On the range of options prices. *Finance and Stochastics*, 1 (1997) 131–140

[8] Eberlein, E. and W. Kluge. Exact pricing formulae for caps and swaptions in a Lévy term structure model. *J. of Computational Finance*, 9 (2) (2006) 99–125

[9] Eberlein, E., D. Madan, M. Pistorius, W. Schoutens, and M. Yor. Two price economies in continuous time. *Annals of Finance*, 10 (2014) 71–100

[10] Eberlein, E., D. Madan, M. Pistorius and M. Yor. Bid and ask prices as non-linear continuous time G-expectations based on distortions. *Mathematics and Financial Economics* 8 (3) (2014) 265–289

[11] Harrison, J. M. and D. M. Kreps. Martingales and arbitrage in multiperiod securities markets. *J. Econ. Theory*, 20 (1979) 381–408

[12] Harrison, J. M. and S. R. Pliska. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes Appl.*, 11 (1981) 215–260

[13] Madan, D. B. and A. Cherny. Markets as a counterparty: an introduction to conic finance. *Int. J. Theor. Appl. Finance*, 13 (8) (2010) 1149–1177