Mathematics in Financial Risk Management

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Abstract

The paper gives an overview of mathematical models and methods used in financial risk management; the main area of application is credit risk. A brief introduction explains the mathematical issues arising in the risk management of a portfolio of loans. The paper continues with a formal overview of credit risk management models and discusses axiomatic approaches to risk measurement. We close with a section on dynamic credit risk models used in the pricing of credit derivatives. Mathematical techniques used stem from probability theory, statistics, convex analysis and stochastic process theory.

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1 Introduction

1.1 Financial Risk Management

Broadly speaking, risk management can be defined as a discipline for "Living with the possibility that future events may cause adverse effects" (Kloman 1999). In the context of risk management in financial institutions such as banks or insurance companies these adverse effects usually correspond to large losses on a portfolio of assets. Specific examples include: losses on a portfolio of market-traded securities such as stocks and bonds due to falling market prices (a so-called market risk event); losses on a pool of bonds or loans, caused by the default of some issuers or borrowers (credit risk); losses on a portfolio of insurance contracts due to the occurrence of large claims (insurance- or underwriting risk). An additional risk category is operational risk, which includes losses resulting from inadequate or failed internal processes, fraud or litigation.

In financial markets, there is in general no so-called "free lunch" or, in other words, no profit without risk. This is the reason why financial institutions actively take on risks. The role of financial risk management is to measure and manage these risks. Hence risk management can be seen as a core competence of an insurance company or a bank: by using its expertise and its capital, a financial institution can take on risks and manage them by various techniques such as diversification, hedging, or repackaging risks and transferring them back to markets, etc. While risk management has thus always been an integral part of the banking and insurance business, recent years have witnessed a large increase in the use of quantitative and mathematical

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techniques. Even more, regulators and supervisory authorities nowadays even require banks to use quantitative models as part of their risk management process.

Given the random nature of future events on financial markets, the field of stochastics (probability theory, statistics and the theory of stochastic processes) obviously plays an important role in quantitative risk management. In addition, techniques from convex analysis and optimization and numerical methods are frequently being used. In fact, part of the challenge in quantitative risk management stems from the fact that techniques from several existing quantitative disciplines are drawn together. The ideal skill-set of a quantitative risk manager includes concepts and techniques from such fields as mathematical finance and stochastic process theory, statistics, actuarial mathematics, econometrics and financial economics, combined of course with non-mathematical skills such as a sound understanding of financial markets and the ability to interact with colleagues with diverse training and background.

In this paper we give an introduction to some of the mathematical aspects of financial risk management. We have chosen the problem of measuring and managing the risks associated with a portfolio of bonds or loans as vehicle for our discussion. This choice is motivated by our common research interests; moreover, quantitative credit risk models are currently a hot topic in academia and industry.

1.2   Risk Management for a Loan Portfolio

The loss distribution.   Consider a portfolio of loans to m different counterparties, indexed by $i \in \{1, \ldots, m\}$. The standard way for measuring the risk in this portfolio is to look at the change in the portfolio-value over a fixed time horizon T such as one year (current time is $t = 0$). We start with a single loan with given exposure (size) e_i and maturity date (repayment date) bigger than T. The main risk is default risk, i.e. the risk that the borrower cannot repay the loan in full. Denote by $\tau_i > 0$ the random default time of borrower i and introduce the Bernoulli random variable

$$Y_i = 1_{\{\tau_i \leq T\}} := \begin{cases} 1, & \text{if } \tau_i \leq T, \\ 0, & \text{else .} \end{cases} \tag{1}$$

Assume that in case of default the borrower pays the lender the amount $(1 - \delta_i)e_i$, $\delta_i \in (0, 1]$ being the proportion of the exposure which is lost in default (the so-called *relative loss given default*). Abstracting from interest-rate payments the potential loss generated by loan i over the period $(0, T]$ is then given by $L_i = \delta_i e_i Y_i$. Denote by

$$\bar{p}_i := \mathbb{P}(Y_i = 1) = \mathbb{P}(\tau_i \leq T) \tag{2}$$

the *default probability* of counterparty i; $\bar{p}_i$ is by definition the probability that loan i causes a loss and plays therefore an important role in measuring the default risk of the loan.

The loss of the whole portfolio of m firms is then given by $L = \sum_{i=1}^{m} e_i \delta_i Y_i$. In realistic applications m can be quite large: loan portfolios of major commercial banks contain several million loans. The *portfolio loss distribution* is then determined by $F_L(l) = \mathbb{P}(L \leq l)$. Note that F_L depends on the multivariate distribution of the random vector $(Y_1, \ldots, Y_m)$ and not just on the individual default probabilities $\bar{p}_i, 1 \leq i \leq m$. In order to determine F_L we hence need a proper mathematical model for the joint distribution of $(Y_1, \ldots, Y_m)$; this issue is taken up in Section 2.2.

Dependence between defaults can have a large impact on the form of F_L and in particular on its right tail (the probability of large losses). This is illustrated in Figure 1, where we compare the loss distribution for a portfolio of 1000 firms that default independently (portfolio 1) with a more realistic portfolio of the same size where defaults are dependent (portfolio 2). In portfolio 2 defaults are weakly dependent in the sense that the correlation between default events

$(\text{corr}(Y_i, Y_j), \ i \neq j)$ is approximately $0.5\,\%$. In both cases the default probability is $\bar{p}_i \equiv 1\,\%$ so that on average we expect 10 defaults. We clearly see from Figure 1 that the loss distribution of portfolio 2 is skewed and that its right tail is substantially heavier than the right tail of the loss distribution of portfolio 1, illustrating the drastic impact of dependent defaults on credit loss distributions. There are in fact sound economic reasons for expecting dependence between defaults. To begin with, the financial health of a firm varies with randomly fluctuating macroeconomic factors such as changes in economic growth. Since different firms are affected by common macroeconomic factors, there is dependence between their defaults. Moreover, dependence between defaults is caused by direct economic links between firms such as a strong borrower-lender relationship or a small supplier for a larger production firm.
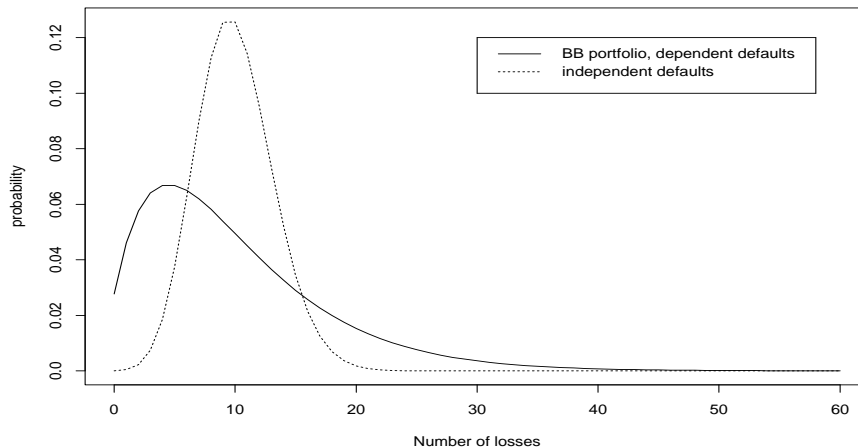
Figure 1. Comparison of the loss distribution of a homogeneous portfolio of 1000 loans with a default probability of $1\,\%$ assuming (i) independent defaults and (ii) a default correlation of $0.5\,\%$. We clearly see that the dependence between default generates a loss distribution with a heavier right tail.

Risk Measurement.   In practice, risk measures expressing the risk of a portfolio on a quantitative scale are needed for a variety of purposes. To begin with, financial institutions hold *risk capital* as buffer against unexpected losses in their portfolios. Regulators concerned with the solvency of financial institutions also have specific requirements on risk capital: under the current regulatory framework the amount of risk capital needed is related to the riskiness of the portfolio as measured via the risk measure Value-at-Risk (see (3) below for a definition). Moreover, risk measures are used by the management of a financial institution as a tool for limiting the amount of risk a subunit within the institution - such as a trading group - may take, and the profitability of a subunit is measured relative to the riskiness (appropriately measured) of its position.

Fix some risk management horizon T and denote by the random variable L the loss of a given portfolio over that horizon. Most modern risk measures are statistics of the distribution of L; such risk measures are frequently called *law-invariant risk measures* (Kusuoka 2001). The most popular law-invariant risk measure is *Value-at-Risk (VaR)*. Given some confidence level $\alpha \in (0, 1)$, say, $\alpha = 0.99$, the VaR of the portfolio at the confidence level α is defined by

$$\text{VaR}_\alpha(L) := \inf\{l \in \mathbb{R} \colon \mathbb{P}(L \leq l) \geq \alpha\}, \tag{3}$$

i.e. in statistical terms $\text{VaR}_\alpha(L)$ is simply the α-*quantile* of L. If L is integrable, an alternative

law-invariant risk measure is *Expected Shortfall* or *Average Value at Risk* given by

$$\text{ES}_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 \text{VaR}_u(L) du \,. \tag{4}$$

Instead of fixing a particular confidence level α, in (4) one averages VaR over all levels $u \geq \alpha$ and thus "looks further into the tail" of the loss distribution; in particular $\text{ES}_\alpha \geq \text{VaR}_\alpha$.

Of course, from a theoretical point of view it is not very satisfactory to introduce risk measures such as VaR or expected shortfall in a more or less ad hoc way. In Section 3 we therefore discuss axiomatic approaches to risk measurement and the related issue of risk-based performance measurement.

Securitization, credit derivatives, and dynamic credit risk models. Recent years have witnessed a rapid growth on the market for credit derivatives. These securities are primarily used for the management and the trading of credit risk. Credit derivatives have become popular, because they help financial firms to manage the credit risk on their books by selling parts of it to the wider financial sector. The payoff of most credit derivatives depends on the exact timing of defaults, so that dynamic (continuous-time) credit risk model are needed to study pricing and hedging of these products. The mathematical tools for analyzing credit derivatives hence stem from the field of stochastic process theory, in particular martingale theory and stochastic calculus. We discuss some of the current developments in Section 4.

Further reading. A short survey paper cannot do justice to all aspects of the vast and growing field of quantitative risk management. For further reading we refer to the books McNeil, Frey & Embrechts (2005) (for quantitative risk management in general), Bluhm, Overbeck & Wagner (2002) (for an introduction with strong focus on credit risk) or Crouhy, Galai & Mark (2001) (for institutional aspects of risk management); further references are provided in the text.

2 Credit Risk Management Models

In this section we discuss models for credit risk management. These models are typically static, meaning that the focus is the loss distribution over a fixed time period $[0, T]$ rather than the evolution of risk in time. This makes the mathematics underlying the models relatively simple (the key tools are random variables instead of stochastic processes) and permits us to discuss some key ideas in credit risk modelling in a non-technical setting. Note however, that the implementation of even these simple models poses substantial practical challenges: current approaches for parameter estimation and model validation are far from satisfactory. To a large extent this is due to the difficult data situation: credit loss data are collected on an annual or semi-annual basis so that a loss history for a loan portfolio ranging over 20 years contains at most 40 serially independent observations.

We begin with the issue of determining default probabilities for individual firms; portfolio models and related statistical questions are discussed in Sections 2.2 and 2.3.

2.1 Default probabilities

State variables. In order to determine the default probability $\bar{p}_i$ of a given firm i one typically introduces a *state variable* X_i measuring its credit quality. The link between state variable and default probability is then modelled by some function $p : \mathbb{R} \to [0, 1]$ so that $\bar{p}_i = p(X_i)$. This modelling suggests the following simple *moment estimator* for $p(\cdot)$: assume that there are N years of default data for a given portfolio available; denote by $m_t(x)$ the number of firms in year

t with X_i (roughly) equal to x and by $M_t(x)$ the number of those firms which have defaulted in year t. Then a simple estimator for $p(\cdot)$ is given by

$$\hat{p}(x) = \frac{1}{N} \sum_{t=1}^{N} \frac{M_t(x)}{m_t(x)}. \tag{5}$$

More sophisticated estimators can be developed in the context of a formal model for the joint distribution of default events in the portfolio; see Section 2.3 below.

Credit ratings. A popular state variable used in the so-called credit-migration models is the *credit rating* of a firm. Credit ratings for major companies or sovereigns are provided by rating agencies such as Moody's, Standard & Poor's (S&P) or Fitch. In the S&P rating system there are seven rating categories (AAA, AA, A, BBB, BB, B, CCC) with AAA being the highest and CCC the lowest rating of companies which have not defaulted; moreover, there is a default state. Moody's uses seven pre-default rating categories labelled Aaa, Aa, A, Baa, Ba, B, C, a finer alpha-numeric system is also in use. The rating system used by Fitch is similar to the S&P system. Rating agencies also provide so-called rating transition matrices; an example from Standard & Poor's is presented in Table 1. These matrices are determined from historical rating information; they give an estimate of the probability that a firm migrates from a given rating category to another category within a given year.

Initial	Rating at year-end (transition probabilities in %)							
rating	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	90.81	8.33	0.68	0.06	0.12	0.00	0.00	0.00
AA	0.70	90.65	7.79	0.64	0.06	0.14	0.02	0.00
A	0.09	2.27	91.05	5.52	0.74	0.26	0.01	0.06
BBB	0.02	0.33	5.95	86.93	5.30	1.17	1.12	0.18
BB	0.03	0.14	0.67	7.73	80.53	8.84	1.00	1.06
B	0.00	0.11	0.24	0.43	6.48	83.46	4.07	5.20
CCC	0.22	0.00	0.22	1.30	2.38	11.24	64.86	19.79

Table 1. Probabilities of migrating from one rating quality to another within 1 year expressed in %. Source: Standard & Poor's CreditWeek (15th April 1996).

In the simplest form of credit migration models it is assumed that the current credit rating of a firm completely determines the distribution of its future rating, or, in mathematical terms, that rating transitions follow a *Markov chain*. Under this assumption default probabilities can be read off from an estimated transition matrix. For instance, using the transition matrix presented in Table 1, the one-year default probability of a company whose current S&P credit rating is A is estimated to be 0.06 %, whereas the default probability of a CCC-rated company is estimated to be almost 20 %. While the Markovianity of rating transitions is convenient for financial modelling (see for instance (Jarrow, Lando & Turnbull 1997)), there is some doubt if the assumption can be maintained empirically; a good empirical study based on techniques from survival analysis is Lando & Skodeberg (2002). This tradeoff between tractability and realism is typical for the application of mathematical models in finance in general.

Firm-value models. Alternative state variables can be based on the *firm-value interpretation* of default. In this approach the asset-value of firm i is modelled as a nonnegative stochastic process $(V_{t,i})_{t \geq 0}$; liabilities are represented by some (deterministic) threshold D_i. In the simplest case the asset-value process is modelled as *geometric Brownian motion* so that $\ln V_{T,i}$ is normally

distributed. In line with economic intuition, it is assumed that default occurs if the asset value of the firm is to low to cover its liabilities. The precise modelling varies: in the simple Merton (1974) model the default indicator of firm i is defined by $Y_i := 1_{\{V_{T,i} \leq D_i\}}$, i.e. one checks the solvency of the firm only at the risk management horizon T. Somewhat closer to reality are perhaps the so-called *first-passage time* models (Black & Cox (1976), Longstaff & Schwartz (1995)), where

$$\tau_i := \inf\{t \geq 0 : V_{t,i} \leq D_i\}. \tag{6}$$

The name stems from the fact that in probability theory τ_i is known as first-passage time of the process $(V_{t,i})$ at the threshold D_i. There are by now many extensions of the simple model (6) such as unknown default thresholds or general jump-diffusion models for the asset value process; a good overview is given in Lando (2004).

A natural state-variable in this context is the so-called *distance to default* which is used in the popular KMV approach to modelling default probabilities; see for instance Crosbie & Bohn (2002). In this approach one puts

$$X_i := \frac{V_{0,i} - D_i}{\sigma_i V_{0,i}}, \tag{7}$$

where the *volatility* σ_i is defined to be the standard deviation of the logarithmic return $\ln V_{1,i} - \ln V_{0,i}$. The definition (7) can be motivated in the context of the Merton (1974)-model. In that model $(V_{1,i} - V_{0,i})/V_{0,i}$ is approximately $N(0, \sigma^2)$ distributed, so that (in practitioner language) "X_i gives the number of standard deviations the asset value is away from the default threshold". For more details on the KMV model we refer to McNeil et al. (2005), Section 8.2, or Bluhm et al. (2002), Sections 2 and 3.

2.2 Credit Portfolio Models

Now we return to the problem of modelling the joint distribution of the default indicator vector $\boldsymbol{Y} = (Y_1, \ldots, Y_m)$. There are two types of portfolio credit risk models, *threshold models* and *mixture models*.

Threshold models. These models can be viewed as multivariate extensions of the firm value models discussed in the previous subsection. Their defining attribute is the idea that default occurs for a company i when some *critical variable* X_i (such as the logarithmic asset value $\ln V_{T,i}$) lies below some deterministic threshold d_i (such as logarithmic liabilities $\ln D_i$) at the end of the time period $[0, T]$, i.e. we have $Y_i = 1_{\{X_i \leq d_i\}}$, $1 \leq i \leq m$. In this model class default dependence is caused by dependence of the components of the random vector $\boldsymbol{X} := (X_1, \ldots, X_m)$. In abstract terms the latter can be represented by the *copula* of $\boldsymbol{X}$. This mathematical concept is of relevance for the analysis and the modelling of dependent risk factors in general (Embrechts, McNeil & Straumann 2001) and therefore merits a brief digression.

Assume for simplicity that the marginal distributions $F_i(x) = \mathbb{P}(X_i \leq x)$ are continuous and strictly increasing. In that case the copula C of $\boldsymbol{X}$ can be defined as the distribution function of the random vector $\mathbf{U} := (F_1(X_1), \ldots, F_m(X_m))$. Note that $\mathbf{U}$ has uniform marginal distributions:

$$\mathbb{P}(U_i \leq u) = \mathbb{P}\big(X_i \leq F_i^{-1}(u)\big) = F_i(F_i^{-1}(u)) = u, \ u \in [0, 1].$$

C is by definition independent under strictly increasing transformations of the individual components of $\boldsymbol{X}$ and thus represents the dependence structure of this random vector. Moreover we have the following relation between the distribution function F of $\boldsymbol{X}$ and its copula C, known as *identity of Sklar:*

$$\begin{aligned} F(x_1, \ldots, x_m) := \mathbb{P}(X_1 \leq x_1, \ldots, X_m \leq x_m) &= \mathbb{P}(U_1 \leq F_1(x_1), \ldots, U_m \leq F_m(x_m)) \\ &= C(F_1(x_1), \ldots, F_m(x_m)), \end{aligned} \tag{8}$$

see McNeil et al. (2005), Section 5.1 for details and extensions. Relation (8) illustrates nicely how multivariate distributions are formed by coupling together marginal distributions and copulas. An example which is frequently being used is the so-called *Gauss copula* C_P^{Ga} defined as copula of a multivariate normally distributed random vector with correlation matrix P.

In threshold models for portfolio credit risk the copula of the critical-variable vector $\boldsymbol{X}$ governs the distribution of the default indicator vector $\boldsymbol{Y}$ in the following sense: given two models with critical variables $\boldsymbol{X}$ and $\widetilde{\boldsymbol{X}}$ and threshold vectors $\boldsymbol{d}$ and $\widetilde{\boldsymbol{d}}$. Then the corresponding default indicators $\boldsymbol{Y}$ and $\widetilde{\boldsymbol{Y}}$ have the same distribution if $\mathbb{P}(X_i \leq d_i) = \mathbb{P}(\tilde{X}_i \leq \tilde{d}_i)$ for all i (identical default probabilities) and if moreover $\boldsymbol{X}$ and $\widetilde{\boldsymbol{X}}$ have the same copula; see Section 8.3 of McNeil et al. (2005).

Credit portfolio models used in industry such as the popular KMV model (Kealhofer & Bohn 2001) typically use multivariate normal distributions with factor structure for the vector $\boldsymbol{X}$ (so-called Gauss-copula models). Formally, one puts

$$X_i = \sqrt{R_i}\sum_{j=1}^{l} \alpha_{ij}\Psi_j + \sqrt{1 - R_i}\,\epsilon_i,\ 1 \leq i \leq m; \tag{9}$$

Here $\boldsymbol{\Psi} = (\Psi_1, \ldots, \Psi_l)$ is an l-dimensional Gaussian random vector with $E(\Psi_i) = 0$ and $\mathrm{var}(\Psi_i) = 1$ representing country- and industry factors (so-called *systematic* factors); $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_m)$ is a vector with independent standard-normally distributed components representing firm-specific (*idiosyncratic*) risk; $\boldsymbol{\Psi}$ and $\boldsymbol{\epsilon}$ are independent; $0 \leq R_i \leq 1$ measures the part of the variance of X_i which is due to fluctuations of the systematic factors; the relative weights of the different factors are given by $\boldsymbol{\alpha} = (\alpha_{i,1}, \ldots, \alpha_{i,l})$ with $\sum_{j=1}^{l} \alpha_{ij} = 1$ for all i. From a practical point of view the factor structure is mainly introduced in order to reduce the dimensionality of the problem, so that in applications l is usually much smaller than m.

Bernoulli mixture models. In a mixture model the default risk of an obligor is assumed to depend on a set of common economic factors, such as macroeconomic variables, which are also modelled stochastically; given a realization of the factors, defaults of individual firms are assumed to be independent. Dependence between defaults thus stems from the dependence of individual default probabilities on the set of common factors. We start our analysis with a general definition.

Definition 2.1 (Bernoulli mixture model). Given some random vector $\boldsymbol{\Psi} = (\Psi_1, \ldots, \Psi_l)'$, the random vector $\boldsymbol{Y} = (Y_1, \ldots, Y_m)'$ follows a Bernoulli mixture model with factor vector $\boldsymbol{\Psi}$, if there are functions $p_i : \mathbb{R}^l \to [0, 1]$, $1 \leq i \leq m$, such that conditional on $\boldsymbol{\Psi}$ the default indicator $\boldsymbol{Y}$ is a vector of independent Bernoulli random variables with $\mathbb{P}(Y_i = 1|\boldsymbol{\Psi} = \boldsymbol{\psi}) = p_i(\boldsymbol{\psi})$.

For $\boldsymbol{y} = (y_1, \ldots, y_m)'$ in $\{0, 1\}^m$ we thus have that

$$\mathbb{P}(\boldsymbol{Y} = \boldsymbol{y} \mid \boldsymbol{\Psi} = \boldsymbol{\psi}) = \prod_{i=1}^{m} p_i(\boldsymbol{\psi})^{y_i}(1 - p_i(\boldsymbol{\psi}))^{1-y_i}, \tag{10}$$

and the unconditional distribution of the default indicator vector $\boldsymbol{Y}$ is obtained by integrating over the distribution of the factor vector $\boldsymbol{\Psi}$. In particular, the default probability of company i is given by $\bar{p}_i = \mathbb{P}(Y_i = 1) = E(p_i(\boldsymbol{\Psi}))$.

One-factor models. In many practical situations it is useful to consider a one-dimensional mixing variable Ψ and hence a one-factor model: one-factor models may be fitted statistically to default data without great difficulty (see Section 2.3 below); moreover, their behaviour for large portfolios is also particularly easy to understand, see for instance Section 8.4.3 of McNeil et al.

(2005). A simple one-factor model for a portfolio consisting of different homogeneous groups indexed by $r \in \{1, \ldots, k\}$ (representing for instance rating classes) would be to assume that

$$p_i(\Psi) = h(\mu_{r(i)} + \sigma\Psi)\,. \tag{11}$$

Here $h : \mathbb{R} \to (0,1)$ is a strictly increasing *link function*, such as $h(x) = \Phi(x)$, Φ the standard normal distribution function, or $h(x) = (1 + \exp(-x))^{-1}$ (the logistic distribution function); $r(i)$ gives the group membership of firm i; μ_r is a group-specific intercept term; $\sigma > 0$ is a scaling parameter and Ψ is standard normally distributed. Such a specification is commonly used in the class of *generalized linear mixed models* in statistics.

Inserting this specification in (10) we can find the conditional distribution of the default indicator vector. Suppose that there are m_r obligors in rating category r and write M_r for the number of defaults. The conditional distribution of the vector $\mathbf{M} = (M_1, \ldots, M_k)'$ is then given by

$$\mathbb{P}(\mathbf{M} = \boldsymbol{l} \mid \Psi = \psi) = \prod_{r=1}^{k} \binom{m_r}{l_r} \left(h(\mu_r + \sigma\psi)\right)^{l_r} \left(1 - h(\mu_r + \sigma\psi)\right)^{m_r - l_r}, \tag{12}$$

where $\boldsymbol{l} = (l_1, \ldots, l_k)'$.

Mapping of models. The threshold model (9) can be reformulated as a mixture model, cf. Bluhm et al. (2002), Section 2. This is a useful insight for a number of reasons. To begin with, Bernoulli mixture models are easy to simulate in Monte Carlo risk studies. Moreover, the mixture model format and the threshold model format give rise to different model-calibration strategies based on different types of data, so that a link between the model types is useful in view of the data problems arising in the statistical analysis of credit risk models.

Consider now a vector $\boldsymbol{X}$ of critical variables as in (9), default thresholds $d_1, \ldots, d_m$ and let $Y_i = 1_{\{X_i \le d_i\}}$. We have, using the independence of $\boldsymbol{\Psi}$ and $\boldsymbol{\epsilon}$ and the fact that $\epsilon_i \sim N(0,1)$,

$$\mathbb{P}(X_i \le d_i \mid \boldsymbol{\Psi} = \boldsymbol{\psi}) = \mathbb{P}\left(\epsilon_i \le \frac{d_i - \sqrt{R_i}\sum_{j=1}^{l}\alpha_{ij}\Psi_j}{\sqrt{1 - R_i}} \mid \boldsymbol{\Psi} = \boldsymbol{\psi}\right)$$

$$= \Phi\left(\frac{d_i - \sqrt{R_i}\sum_{j=1}^{l}\alpha_{ij}\psi_j}{\sqrt{1 - R_i}}\right) =: p_i(\boldsymbol{\psi})\,; \tag{13}$$

moreover, the independence of ϵ_i and ϵ_j, $i \ne j$, immediately implies that Y_i and Y_j are conditionally independent given the realisation of $\boldsymbol{\Psi}$. Note that since $X_i \sim N(0,1)$, the model can be calibrated to a set of unconditional default probabilities $\bar{p}_i$, $1 \le i \le m$, if we let $d_i = \Phi^{-1}(\bar{p}_i)$.

The above argument can be generalized to various other critical variable models with factor structure; see for instance Section 8.4.4 of McNeil et al. (2005).

2.3 Parameter estimation in credit portfolio models

Parameter estimation is an important issue in credit risk management. In threshold models one needs to determine the parameters of the factor representation (9). For this stock returns are typically used as proxy for the asset returns of a company; the factor model is then estimated by a mix of formal factor analysis and an ad-hoc assignment of factor weights based on economic arguments; see Kealhofer & Bohn (2001) for an example of this line of reasoning. In this section we describe alternative approaches which are based on the Bernoulli mixture format and historical default data. More specifically, we discuss the estimation of model parameters in the one-factor Bernoulli mixture model (11). Admittedly, model (11) is quite simplistic. However, given the present data situation, parameter estimation in Bernoulli mixture models based

solely on historical default information is only feasible for models with a low-dimensional factor structure.

We consider repeated cross-sectional data, i.e. observations of the default or non-default of groups of monitored companies in a number of time periods. This kind of data is readily available from rating agencies. Suppose as before that we have observations over N years and denote by $m_{t,r}$ the number of firms in year t and group r in our sample; $\hat{M}_{t,r}$ denotes the number of these firms which have actually defaulted and $\hat{\mathbf{M}}_t := (\hat{M}_{t,1}, \ldots, \hat{M}_{t,k})'$. In this simple model one neglects dependence of defaults over time (*serial* dependence) and assumes that the factor variables $(\Psi_t)_{t=1}^N$ for the different years are independent and standard normally distributed; moreover, in line with the mixture model formulation, we assume that defaults of individual firms are conditionally independent given $(\Psi_t)_{t=1}^N$. Using (12) and the independence of $(\Psi_t)_{t=1}^N$, we obtain the following form of the likelihood of the model parameters $\boldsymbol{\mu} := (\mu_1, \ldots, \mu_k)'$ and σ given the observed data $\hat{\mathbf{M}}_1, \ldots, \hat{\mathbf{M}}_N$:

$$L(\boldsymbol{\mu}, \sigma \mid \hat{\mathbf{M}}_1, \ldots, \hat{\mathbf{M}}_N) = \frac{1}{(2\pi)^{N/2}} \prod_{t=1}^N \int_{\mathbb{R}} \mathbb{P}\left(\mathbf{M} = \hat{\mathbf{M}}_t \mid \Psi = \psi, \boldsymbol{\mu}, \sigma\right) e^{-\psi^2/2} d\psi. \qquad (14)$$

The integrals in (14) are easily evaluated numerically, so that the model can be fitted using maximum likelihood estimation (MLE); see Frey & McNeil (2003) for details. Similar estimations based on moment matching techniques can be found in Bluhm et al. (2002), Section 2.7.

Since the factor Ψ_t is often interpreted as some measure of the state of the economy in year t, and since moreover business cycles tend to last over several years, it makes sense to assume some serial dependence of the time series $(\Psi_t)_{t=1}^N$ of factor variables. The simplest model would be a Markovian structure where the distribution of Ψ_t depends on the realization of Ψ_{t-1}. With this extension the model becomes a so-called *hidden Markov model* (Elliott & Moore 1995). For instance, McNeil & Wendin (2005) consider a model where $(\Psi_t)_{t=1}^N$ follows a so-called AR-1 process with dynamics

$$\Psi_t = \alpha \Psi_{t-1} + \varepsilon_t,$$

for $-1 < \alpha < 1$ and an iid sequence $(\varepsilon_t)_{t=1}^N$ of noise variables. Under this model assumption, the random variables $(\Psi_t)_{t=1}^N$ are not independent and the likelihood has a more complicated form, so that MLE is no longer feasible. McNeil & Wendin (2005) propose to use *Bayesian* approaches instead; as shown in their paper, Markov-Chain Monte Carlo (MCMC) methods (see for instance Robert & Casella (1999)) can be used to sample from the posterior distribution of the unknown model parameters.

3 Risk measures and capital allocation

3.1 Standard techniques for calculating and allocating risk capital

The development of the theoretical relationship between risk and expected return is built on two economic theories: portfolio theory and capital market theory (Markowitz (1952), Sharpe (1964), Lintner (1965)). Portfolio theory deals with the selection of portfolios that maximize expected returns consistent with individually acceptable levels of risk whereas capital market theory focuses on the relationship between security returns and risk. These theories also provide a natural framework for measuring profitability. The profitability analysis is commonly carried out by expressing the risk-return relationship as simple rational functions of risk- and return-components. The two basic variants of these so-called risk adjusted ratios are known as RORAC or RAROC, respectively; see Matten (2000) for details.

Techniques for measuring risk are a prerequisite for profitability analysis. In a bank, risk is usually quantified in terms of risk capital (or Economic Capital). The reason for the close

connection between risk and capital is the fact that the main purpose of the bank's capital is to protect the bank against extreme losses, i.e. capital which is invested in save and liquid assets should ensure solvency of the bank even in adverse economic scenarios. Hence, the actual capital requirements of a bank are determined by its risk profile.

From a bank's perspective, the investment of capital in riskless assets is not very attractive, since the return the bank can earn by investing in these assets is usually much lower than the return required by the shareholders of the bank. Therefore, in line with portfolio theory, risk is one of the components in the profitability analysis of the bank's business areas, portfolios and transactions. This task requires an allocation algorithm that splits the risk capital k of a portfolio X with subportfolios $X_1, \ldots, X_m$ into the *sub-portfolio contributions* $k_1, \ldots, k_m$ with $k = k_1 + \ldots + k_m$. The objective of this section is to review the main concepts for measuring and allocating risk capital.

In the classical portfolio theory, e.g. in the Capital Asset Pricing Model, the risk of a portfolio is measured by the variance (or volatility) of the portfolio distribution and risk capital is distributed proportional to covariances.[1] Techniques based on second moments are the natural choice for normally distributed portfolios. Loss distributions of credit portfolios, however, are asymmetric and heavy tailed. For these distributions second moments do not provide useful tail information and are therefore not suitable for measuring or allocating risk.

The current standard in credit portfolio modelling is to define the risk capital in terms of a quantile of the portfolio loss distribution, in financial lingo the Value-at-Risk (VaR) $\text{VaR}_\alpha(X)$ of the loss X of the portfolio at a specified confidence level α (see (3)). VaR has an intuitive economic interpretation, i.e. it specifies the capital needed to absorb losses with probability α, and has even achieved the high status of being written into industry regulations. However, VaR also has an obvious limitation as a risk measure: in general it is not subadditive. Subadditivity means that for two losses X and Y

$$\text{VaR}(X + Y) \leq \text{VaR}(X) + \text{VaR}(Y). \tag{15}$$

VaR is known to be subadditive for *elliptically distributed* random vectors (X, Y) (McNeil et al. 2005), and thus for this special case encourages diversification. For typical credit portfolios the assumption of an elliptical distribution cannot be maintained. Consequently diversification, which is commonly considered as a way to reduce risk, may increase Value-at-Risk. A specific example can be found in Section 6.1 of McNeil et al. (2005).

3.2 Coherent and convex risk measures

In recent years, the development of more appropriate risk measures has been one of the main topics in quantitative risk management. The starting point is the seminal paper Artzner et al. (1999). In this paper, an axiomatic approach to the quantification of risk is presented and a set of four axioms is proposed.

Definition 3.1 (Coherent risk measures). Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, L^∞ the space of all (almost surely) bounded random variables on Ω and V a subspace of the vector space L^∞. We will identify each portfolio X with its loss function, i.e. X is an element of V and $X(\omega)$ specifies the loss of X at a future date in state $\omega \in \Omega$. A risk measure ρ is a function from V to $\mathbb{R}$. It is called coherent if it is

monotonic:	$X \leq Y \ \Rightarrow \ \rho(X) \leq \rho(Y)$	$\forall X, Y \in V,$
translation invariant:	$\rho(X + a) \ = \ \rho(X) + a$	$\forall a \in \mathbb{R}, \ X \in V,$
positively homogeneous:	$\rho(aX) \ = \ a \cdot \rho(X)$	$\forall a \geq 0, \ X \in V,$
subadditive:	$\rho(X + Y) \ \leq \ \rho(X) + \rho(Y)$	$\forall X, Y \in V.$

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[1]The precise definition of this allocation scheme, called volatility allocation, is given in Section 3.6.

It seems to be accepted in the finance industry that the concept of a coherent risk measure provides a useful characterization of risk measures under fairly general conditions (see Artzner et al. (1997) for the motivation behind the choice of these axioms). A serious criticism to the necessity of the subadditivity and positive homogeneity can, however, be raised if liquidity risk is taken into account. This is the risk that the market cannot easily absorb the sell-off of large asset positions. In this situation, doubling the size of a position might more than double its risk. To take into account possible liquidity-driven violations to subadditivity and positive homogeneity, the concept of convex risk measures has been independently introduced in Heath & Ku (2004), Föllmer & Schied (2002) and Frittelli & Gianin (2002) by replacing the axioms on subadditivity and positive homogeneity by the weaker requirement of convexity.

Definition 3.2 (Convex risk measures). A translation invariant and monotonic risk measure $\rho : V \to \mathbb{R}$ is called **convex** if it has the property

$$\textbf{convex:} \quad \rho(aX + (1-a)Y) \leq a\rho(X) + (1-a)\rho(Y) \quad \forall X, Y \in V, \ a \in [0,1].$$

The debate on coherent versus convex risk measures is subject of current research and will not be covered in this survey article. We believe that coherent risk measures provide an appropriate axiomatic framework for most practical applications and will therefore focus on this concept. For the theory of convex risk measures we refer to the excellent exposition in Föllmer & Schied (2004).

Two other important areas of active research are not covered in this article: the theory of dynamic risk measures and the connection between risk measures, utility theory and portfolio choice. We refer the reader to the recent articles Cheridito et al. (2006) and Pirvu & Zitkovic (2006) and the literature surveys provided therein.

3.3 Representation theorems for coherent risk measures

A general technique for specifying coherent risk measures is given in Artzner et al. (1999).

Proposition 3.3. *Let $\mathcal{Q}$ be a set of absolutely continuous probability measures with respect to $\mathbb{P}$. The function*

$$\rho_{\mathcal{Q}}(X) := \sup\{\mathbf{E}_{\mathbb{Q}}(X) \mid \mathbb{Q} \in \mathcal{Q}\} \tag{16}$$

defines a coherent risk measure on L^{∞}.

Does every coherent risk measure have a representation of the form (16)? Artzner et al. (1999) have shown that this is indeed the case if the underlying probability space Ω is finite. For infinite Ω the situation is more complicated. It is shown in Theorem 2.3 in Delbaen (2002) that the representation of general coherent risk measures has to be based on the more general class of finitely additive probabilities. In order to represent a coherent risk measure ρ by standard, i.e. σ-additive, probability measures the coherent risk measure ρ has to satisfy an additional condition, the so-called Fatou property.

Definition 3.4 (Fatou property and monotonic convergence). Given a function $\rho : L^{\infty} \to \mathbb{R}$. Then ρ satisfies the *Fatou property*, if $\rho(X) \leq \liminf_{n \to \infty} \rho(X_n)$ for any uniformly bounded sequence $(X_n)_{n \geq 1}$ converging to X in probability; ρ satisfies the *monotonic convergence property*, if $\rho(X_n) \downarrow 0$ for any sequence $0 \leq X_n \leq 1$ such that $X_n \downarrow 0$.

For coherent risk measures the monotonic convergence property implies the Fatou property. Furthermore, the Fatou property (the monotonic convergence property) of ρ is equivalent to continuity of ρ from below (from above), see Föllmer & Schied (2004).

Theorem 3.5 (Representation of coherent risk measures). *Let ρ be a coherent risk measure. Then we have*

1. *ρ satisfies the Fatou property if and only if there exists an $L^1(\mathbb{P})$-closed, convex set $\mathcal{Q}$ of absolutely continuous probability measures on Ω with*

$$\rho(Y) = \sup\{E_\mathbb{Q}(Y) \mid \mathbb{Q} \in \mathcal{Q}\}. \tag{17}$$

2. *Assume that ρ can be represented in the form (17). Then ρ satisfies the monotonic convergence property if and only if for every $Y \in L^\infty$ there is a $\mathbb{Q}_Y \in \mathcal{Q}$ such that $\rho(Y)$ is exactly $E_{\mathbb{Q}_Y}(Y)$, i.e. $\rho(Y)$ is not only a supremum but also a maximum.*

The proof of the first part of the theorem given in Delbaen (2000, 2002) is mainly based on two theorems in functional analysis, the bipolar theorem and the Krein-Šmulian theorem. The proof of the second part uses James' characterization of weakly compact sets (Diestel 1975). The connection to dual representations of Fenchel-Legendre type is outlined in Föllmer & Schied (2004), see also Delbaen (2000, 2002) and Frittelli & Gianin (2002).

3.4 Expected shortfall

The most popular class of coherent risk measures is Expected Shortfall (see, for instance, Rockafellar & Uryasev (2000, 2001); Acerbi & Tasche (2002)). For an integrable random variable Y the Expected Shortfall at level α, denoted by ES_α, is the risk measure defined by

$$\mathrm{ES}_\alpha(Y) := (1-\alpha)^{-1} \int_\alpha^1 \mathrm{VaR}_u(Y) du.$$

It is easy to show that

$$\mathrm{ES}_\alpha(Y) = (1-\alpha)^{-1}\big\{E(Y\,\mathbf{1}_{\{Y>\mathrm{VaR}_\alpha(Y)\}}) + \mathrm{VaR}_\alpha(Y) \cdot \big(\mathbb{P}(Y \le \mathrm{VaR}_\alpha(Y)) - \alpha\big)\big\} \tag{18}$$

is an equivalent characterization of Expected Shortfall. Furthermore, ES_α is coherent (Acerbi & Tasche (2002)) and satisfies the monotonic convergence property. Hence, by Theorem 3.5, there exists a set $\mathcal{Q}$ of probability measures with

$$\mathrm{ES}_\alpha(Y) = \max\{E_\mathbb{Q}(Y) \mid \mathbb{Q} \in \mathcal{Q}\}. \tag{19}$$

This set consists of all absolutely continuous probability measures $\mathbb{Q}$ whose density $d\mathbb{Q}/d\mathbb{P}$ is $\mathbb{P}$-a.s. bounded by $1/(1-\alpha)$ (see, for example, Delbaen (2000)). Furthermore, it follows from (18) that for every $Y \in L^\infty$ the maximum in (19) is attained by the probability measure $\mathbb{Q}_Y$ given in terms of its density by

$$\frac{d\mathbb{Q}_Y}{d\mathbb{P}} := \frac{\mathbf{1}_{\{Y>\mathrm{VaR}_\alpha(Y)\}} + \beta_Y \mathbf{1}_{\{Y=\mathrm{VaR}_\alpha(Y)\}}}{1-\alpha}, \quad \text{with} \tag{20}$$

$$\beta_Y := \frac{\mathbb{P}(Y \le \mathrm{VaR}_\alpha(Y)) - \alpha}{\mathbb{P}(Y = \mathrm{VaR}_\alpha(Y))} \quad \text{if } \mathbb{P}(Y = \mathrm{VaR}_\alpha(Y)) > 0. \tag{21}$$

3.5 Spectral measures of risk

A particularly interesting subclass of coherent risk measures has been introduced in Kusuoka (2001), Acerbi (2002, 2004) and Tasche (2002). Spectral measures of risk can be defined by adding two axioms to the set of coherency axioms: law invariance and comonotonic additivity. Spectral risk measures are generalizations of Expected Shortfall. In fact, they can be defined as the convex hull of the Expected Shortfall measures. A third characterization provides a direct link

to risk aversion: spectral risk measures can be represented as integrals specified by appropriate risk aversion functions σ (see Theorem 3.7).

Recall that two real valued random variables X and Y are said to be *comonotonic* if there exist a real valued random variable Z and two non-decreasing functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $X = f(Z)$ and $Y = g(Z)$. A risk measure ρ will be called *law-invariant* if $\rho(X)$ depends only on the distribution of X. Note that VaR and Expected Shortfall are law-invariant. Furthermore, it has been recently shown in Jouini et al. (2006) that law-invariant convex risk measures have the Fatou property.

Definition 3.6 (Spectral risk measures). A coherent risk measure ρ is called a spectral risk measure if it is law-invariant and comonotonic additive, meaning that $\rho(X + Y) = \rho(X) + \rho(Y)$ for all comonotonic $X, Y \in V$.

Law invariance of a risk measure ρ is an essential property for practical applications: note that a risk measure can only be estimated from empirical loss data if it is law-invariant. Two comonotonic portfolios $X, Y \in V$ provide no diversification at all when added together. It is therefore a natural requirement that $\rho(X + Y)$ should equal the sum of $\rho(X)$ and $\rho(Y)$. If a risk measure is subadditive and comonotonic additive the upper bound $\rho(X) + \rho(Y)$ placed on $\rho(X + Y)$ by subadditivity is sharp as it can be actually attained in the case of comonotonic variables.

For a proof of the following theorem we refer to Kusuoka (2001), Acerbi (2002) and Tasche (2002). Generalizations can be found in Föllmer & Schied (2004) and Weber (2004).

Theorem 3.7 (Characterization of spectral risk measures). *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space with non-atomic $\mathbb{P}$, i.e. there exists a random variable that is uniformly distributed on (0,1). Then the following three conditions are equivalent for a risk measure ρ.*

1. *ρ is a spectral measure of risk.*

2. *ρ is in the convex hull of the Expected Shortfall measures.*

3. *ρ can be represented in the form*

$$\rho(X) = p \int_0^1 \mathrm{VaR}_u(X)\sigma(u)du + (1-p)\mathrm{VaR}_1(X)$$

where $p \in [0, 1]$ and σ is a non-decreasing density on $[0, 1]$, i.e. $\sigma \geq 0$ on $[0, 1]$, $\int_0^1 \sigma(u)du = 1$, and $\sigma(u_1) \leq \sigma(u_2)$ for $0 \leq u_1 \leq u_2 \leq 1$.

3.6   Capital Allocation

We now turn to the allocation of risk capital either to subportfolios or to business units. More formally, assume that a risk measure ρ has been fixed and let X be a portfolio which consists of subportfolios $X_1, \ldots, X_m$, i.e. $X = X_1 + \ldots + X_m$. The objective is to distribute the risk capital $k := \rho(X)$ of the portfolio X to its subportfolios, i.e. to compute risk contributions $k_1, \ldots, k_m$ of $X_1, \ldots, X_m$ with $k = k_1 + \ldots + k_m$.

Allocation techniques for risk capital are a prerequisite for portfolio management and performance measurement. In recent years, theoretical and practical aspects of different allocation schemes have been analyzed in a number of papers; see for instance Tasche (1999, 2002), Overbeck (2000), Delbaen (2000), Denault (2001), Hallerbach (2003). An allocation scheme proposed by several authors is the *allocation by the gradient* or *Euler principle*:[2] the capital allocated to

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[2]Recall Euler's well-known rule that states that if $f : S \to \mathbb{R}$ is positively homogeneous and differentiable at $x \in S \subseteq \mathbb{R}^n$, we have $f(x) = \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(x)$.

the subportfolio X_i of X is the derivative of the associated risk measure ρ at X in the direction of X_i (see (24) for a precise formalization). Tasche (1999) argues that allocation based on the Euler principle provides the right signals for performance measurement. Another justification for the Euler principle is given in Denault (2001) using cooperative game theory and the notion of "fairness". He shows that the Euler principle is the only fair allocation principle for a coherent risk measure. In the following we will review a simple axiomatization of capital allocation in Kalkbrener (2005). The main axioms are the property that the entire risk capital of a portfolio is allocated to its subportfolios and a diversification property that is closely linked to the subadditivity of the underlying risk measure. It turns out that in this framework the Euler principle is an immediate consequence of the proposed axioms.

The axiomatization is based on the assumption that the capital allocated to subportfolio X_i only depends on X_i and X but not on the decomposition of the remainder $X - X_i = \sum_{j \neq i} X_j$ of the portfolio. Hence, a capital allocation can be considered as a function Λ from $V \times V$ to $\mathbb{R}$. Its interpretation is, that $\Lambda(X, Y)$ represents the capital allocated to the portfolio X considered as a subportfolio of portfolio Y.

Definition 3.8 (Axiomatization of capital allocation). A function $\Lambda \colon V \times V \to \mathbb{R}$ is called a capital allocation with respect to a risk measure ρ if it satisfies the condition $\Lambda(X, X) = \rho(X)$ for all $X \in V$, i.e. if the capital allocated to X (considered as stand-alone portfolio) is the risk capital $\rho(X)$ of X.
The following requirements for a capital allocation Λ are proposed.

1. **Linearity.** For a given overall portfolio Z the capital allocated to a union of subportfolios is equal to the sum of the capital amounts allocated to the individual subportfolios. In particular, the risk capital of a portfolio equals the sum of the risk capital of its subportfolios. More formally, Λ is called linear if

$$\forall a, b \in \mathbb{R}, X, Y, Z \in V \quad \Lambda(aX + bY, Z) = a\Lambda(X, Z) + b\Lambda(Y, Z).$$

2. **Diversification.** The capital allocated to a subportfolio X of a larger portfolio Y never exceeds the risk capital of X considered as a stand-alone portfolio: Λ is called diversifying if

$$\forall X, Y \in V \quad \Lambda(X, Y) \leq \Lambda(X, X).$$

3. **Continuity.** A small increase in a position does only have a small effect on the risk capital allocated to that position: Λ is called continuous at $Y \in V$ if

$$\forall X \in V \quad \lim_{\epsilon \to 0} \Lambda(X, Y + \epsilon X) = \Lambda(X, Y).$$

Risk measures and capital allocation rules are closely related. First, given a capital allocation Λ the corresponding risk measure ρ is obviously given by the values of Λ on the diagonal, i.e. $\rho(X) = \Lambda(X, X)$. Conversely, for a positively homogeneous and subadditive risk measure ρ a corresponding capital allocation Λ_ρ can be constructed as follows: let V^* be the set of real linear functionals on V and for a given risk measure ρ consider the following subset

$$H_\rho := \{h \in V^* \mid h(X) \leq \rho(X) \text{ for all } X \in V\}.$$

It is an easy consequence of the Hahn-Banach Theorem that for a positively homogeneous and subadditive risk measure ρ

$$\rho(X) = \max\{h(X) \mid h \in H_\rho\} \tag{22}$$

for all $X \in V$. Hence for every $Y \in V$ there exists an $h_Y^\rho \in H_\rho$ with $h_Y^\rho(Y) = \rho(Y)$. This allows to define a capital allocation Λ_ρ by

$$\Lambda_\rho(X, Y) := h_Y^\rho(X). \tag{23}$$

The set H_ρ can be interpreted as a collection of (generalized) scenarios: the capital allocated to a subportfolio X of portfolio Y is simply the loss of X under scenario h_Y^ρ.

The following theorem (Theorem 4.2 in Kalkbrener (2005)) states the equivalence between positively homogeneous, subadditive (but not necessarily monotonic) risk measures and linear, diversifying capital allocations.

Theorem 3.9 (Existence of capital allocations). *Let $\rho\colon V \to \mathbb{R}$.*

a) *If there exists a linear, diversifying capital allocation Λ with associated risk measure ρ then ρ is positively homogeneous and subadditive.*

b) *If ρ is positively homogeneous and subadditive then Λ_ρ is a linear, diversifying capital allocation with associated risk measure ρ.*

If a linear, diversifying capital allocation Λ is moreover continuous at a portfolio $Y \in V$ it is uniquely determined by the directional derivative of its associated risk measure, as the next theorem (Theorem 4.3 in Kalkbrener (2005)) shows.

Theorem 3.10. *Let ρ be a positively homogeneous and sub-additive risk measure and $Y \in V$. Then the following three conditions are equivalent:*

a) *Λ_ρ is continuous at Y, i.e. for all $X \in V$ $\lim_{\epsilon \to 0} \Lambda_\rho(X, Y + \epsilon X) = \Lambda_\rho(X, Y)$.*

b) *The directional derivative*
$$\lim_{\epsilon \to 0} \frac{\rho(Y + \epsilon X) - \rho(Y)}{\epsilon} \tag{24}$$
exists for every $X \in V$.

c) *There exists a unique $h \in H_\rho$ with $h(Y) = \rho(Y)$.*

If these conditions are satisfied then $\Lambda_\rho(X, Y)$ equals (24) for all $X \in V$, i.e. Λ_ρ is given by the Euler principle.

Theorem 3.9 implies that in the general case, in particular for credit portfolios, there do not exist linear diversifying capital allocations for VaR since VaR is not subadditive. However, under regularity conditions (see, for example, Tasche (1999)), the directional derivative (24) exists for VaR_α and equals

$$E(X|Y = \text{VaR}_\alpha(Y)). \tag{25}$$

The volatility (or covariance) allocation, on the other hand, is linear and diversifying, as it is derived from the risk measure Standard Deviation using (23). More precisely, let c be a non-negative real number and define the risk measure ρ_c^{Std} and the capital allocation Λ_c^{Std} by

$$\rho_c^{Std}(X) := c \cdot \text{Std}(X) + E(X), \tag{26}$$

$$\Lambda_c^{Std}(X, Y) := \begin{cases} c \cdot \text{Cov}(X, Y)/\text{Std}(Y) + E(X) & \text{if } \text{Std}(Y) > 0, \\ E(X) & \text{if } \text{Std}(Y) = 0. \end{cases} \tag{27}$$

Then the risk measure ρ_c^{Std} is translation invariant, positively homogeneous and subadditive but not monotonic for $c > 0$. Λ_c^{Std} is a linear, diversifying capital allocation with respect to ρ_c^{Std}. If

Std$(Y) > 0$ then Λ_c^{Std} is continuous at Y and equals the directional derivative (24) by Theorem 3.10.

Expected Shortfall ES is a coherent risk measure and therefore positively homogeneous and subadditive. Hence, application of (23) to Expected Shortfall yields a linear, diversifying capital allocation with associated risk measure ES. The scenario function $h_Y^{ES}(X)$ for this risk measure is given by $E_{\mathbb{Q}_Y}(X)$, where the probability measure $\mathbb{Q}_Y$ is specified in (20). In summary,

$$\Lambda_\alpha^{ES}(X,Y) := E_{\mathbb{Q}_Y}(X) = \left(\int X \cdot \mathbf{1}_{\{Y > \mathrm{VaR}_\alpha(Y)\}} d\mathbb{P} + \beta_Y \int X \cdot \mathbf{1}_{\{Y = \mathrm{VaR}_\alpha(Y)\}} d\mathbb{P} \right) / (1 - \alpha)$$

is a linear, diversifying capital allocation with respect to ES_α. If

$$\mathbb{P}(Y > \mathrm{VaR}_\alpha(Y)) = 1 - \alpha \quad \text{or} \quad \mathbb{P}(Y \geq \mathrm{VaR}_\alpha(Y)) = 1 - \alpha \tag{28}$$

then Λ_α^{ES} is continuous at Y and equals the directional derivative (24). In particular, (28) holds if $\mathbb{P}(Y = \mathrm{VaR}_\alpha(Y)) = 0$; in that case $\Lambda_\alpha^{ES}(X,Y)$ takes the particularly intuitive form

$$\Lambda_\alpha^{ES}(X,Y) = E\left(X \mid Y > \mathrm{VaR}_\alpha(Y) \right).$$

The extension to spectral risk measures can be found in Overbeck (2004).

3.7   Case study: capital allocation in an investment banking portfolio

We will now analyze the practical consequences of different allocation schemes when applied to a realistic credit portfolio. The case study is based on a sample investment banking portfolio consisting of $m = 25000$ loans with an inhomogeneous exposure and default probability distribution. The average exposure size is 0.004% of the total exposure and the standard deviation of the exposure size is 0.026%. The portfolio expected loss is 0.72% and the unexpected loss, i.e. the standard deviation, is 0.87%. Default probabilities $\bar{p}_1, \ldots, \bar{p}_m$ of all companies are obtained from Deutsche Bank's rating system and vary between 0.02% and 27%. Default correlations are specified by a Bernoulli mixture model: for company i, the conditional default p_i has the form

$$p_i(\boldsymbol{\psi}) := \Phi\left(\frac{\Phi^{-1}(\bar{p}_i) - \sqrt{R_i} \sum_{j=1}^{96} \alpha_{ij} \psi_j}{\sqrt{1 - R_i}} \right). \tag{29}$$

where the 96 systematic factors $\boldsymbol{\Psi} = (\Psi_1, \ldots, \Psi_{96})$ follow a multi-dimensional normal distribution and represent different countries and industries; see (9) and (13).

The portfolio loss distribution L specified by this model does not have an analytic form. Monte Carlo simulation is therefore used for the calculation and allocation of risk capital. For this class of models, however, the Monte Carlo estimation of tail-focused risk measures like Value-at-Risk or Expected Shortfall is a demanding computational problem due to high statistical fluctuations. This stability problem is even more pronounced for Expected Shortfall contributions of individual transactions. Importance sampling is a variance reduction technique that has been successfully applied in credit portfolio models of this type. We refer to Glasserman &Li (2005), Kalkbrener et al. (2004) and Egloff et al. (2005) for details.

For the test portfolio we have calculated the risk measures $\mathrm{VaR}_{0.9998}(L)$, $\mathrm{ES}_{0.999}(L)$ and $\mathrm{ES}_{0.99}(L)$. The $\mathrm{VaR}_{0.9998}(L)$ is the risk measure used at Deutsche Bank for calculating Economic Capital, i.e. the capital requirement for absorbing unexpected losses over a one-year period with a high degree of certainty. The confidence level of 99.98% is derived from Deutsche Bank's target rating of AA+, which is associated with an annual default rate of 0.02%. The $\mathrm{ES}_{0.999}(L)$ has been chosen since it leads to a comparable amount of risk capital, while being based on a coherent risk measure. The $\mathrm{ES}_{0.99}(L)$ was calculated to study the impact of the confidence level α on the

properties of the Expected Shortfall measure. The application of these risk measures results in the following capital requirements (in percent of portfolio exposure):

$$\text{VaR}_{0.9998}(L) = 10.50\%, \qquad \text{ES}_{0.999}(L) = 9.43\%, \qquad \text{ES}_{0.99}(L) = 5.68\%.$$

In the next step the portfolio capital is distributed to the individual loans using different capital allocation algorithms. In credit portfolio models of the form (29) the application of the Euler principle to VaR_α leads to risk contributions for individual loans that are either 0 or the full exposure of the loan. This digital behaviour of the contribution (25) is due to the fact that $\{L = \text{VaR}_\alpha(L)\}$ is usually represented by a single combination of defaults and non-defaults of the m loans. We therefore do not distribute $\text{VaR}_{0.9998}(L)$ via the directional derivative (25) but follow the industry standard and use volatility contributions (27) instead. The $\text{ES}_{0.999}(L)$ and $\text{ES}_{0.99}(L)$ are allocated using Expected Shortfall contributions.

Figure 2 displays the 50 loans with the highest capital charge under Expected Shortfall allocation based on the 99.9% quantile. The relation of portfolio capital

$$\text{VaR}_{0.9998}(L) > \text{ES}_{0.999}(L) > \text{ES}_{0.99}(L)$$

also holds for each of these loans. However, the order of the capital consumption changes and the absolute differences in capital are significant: the highest capital consumption for Expected Shortfall is 93% of the exposure compared to almost 200% for covariances. In particular, under the covariance allocation the capital charge exceeds the overall exposure (the maximum possible loss) for almost all loans in this sub-sample. This demonstrates that the shortcomings of the covariance allocation, i.e. the fact that the underlying risk measure is not monotonic, are not purely theoretical but have implications for realistic credit portfolios.
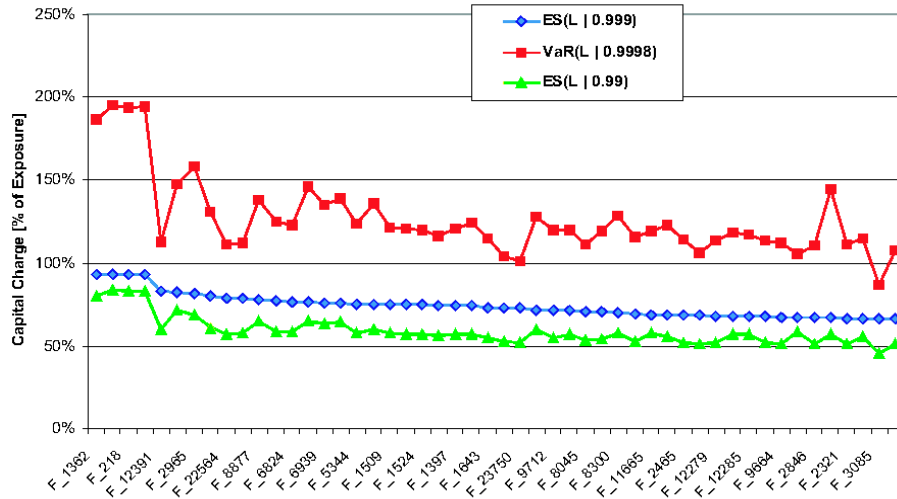
Figure 2. Comparison between Expected Shortfall and covariance capital allocation for loans with highest capital charges.

In contrast, Expected Shortfall contributions are usually higher than volatility contributions for investment-grade loans, i.e. for loans with a rating of BBB or above; see Kalkbrener et al. (2004) for details. This result illustrates that unrealistically high capital charges for poorly rated loans are avoided under Expected Shortfall allocation by distributing a higher proportion of the portfolio capital to highly rated loans.

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Expected shortfall contributions behave also very reasonably with respect to the second main risk driver in credit portfolios, namely *concentration risk.* This risk is caused by default correlations and name concentration. Expected Shortfall contributions measure the average contribution of individual loans to portfolio losses above a specified α-quantile. For a high α these losses are mainly driven by default correlations and name concentration and Expected Shortfall allocation therefore is - almost by definition - very sensitive to concentration risk. It is therefore not surprising that Expected Shortfall usually penalizes concentration risks more strongly than the covariance method. For instance, the 99.9% Expected Shortfall contribution at $R = 60\%$ is three times higher than at $R = 30\%$ for a typical AA+ rated loan in our portfolio whereas the volatility contribution of this loan not even doubles.[3] Overall, this case study strongly supports the view that Expected Shortfall contributions provide a reasonable methodology for allocating risk capital for credit portfolios.

4 Dynamic Credit Risk Models and Credit Derivatives

4.1 Overview

Credit derivatives. The volume in trading credit derivatives at the exchanges and directly between individual parties has increased enormously since the first of these products were introduced roughly fifteen years ago. The reason for this success is to a large extent due to the fact that they allow to transfer credit risk to a larger community of investors. Traditionally the risk arising from a loan contract could not be transferred and remained in the books of the lending institution until maturity. With credit derivatives the risk profile of a given portfolio of credits can be shaped according to specified limits. Concentrations of risk caused by geographic or industry sector factors can be removed. Also by selling a whole credit portfolio via a collateralized debt obligation (CDO) or a collateralized loan obligation (CLO), a financial institution can free part of its capital which can then be used for new business opportunities. Thus credit derivatives allow banks to use their capital more efficiently by acting more as a broker of risk than a taker of risk. Some important credit derivatives are introduced below; for further information we refer for instance to Schönbucher (2003) or Bluhm & Overbeck (2006).

Dynamic credit risk models. To analyse credit derivatives, static models which consider only a fixed future time horizon are no longer appropriate: the pay-off of most credit derivatives depends on the timing of credit events such as default or downgrading of a company; furthermore markets for certain credit products have become so liquid that investors can trade credit risk in a dynamic fashion. For these reasons dynamic (continuous time) models based on (sophisticated) tools from stochastic calculus are needed.

Dynamic credit risk models can be classified into *firm-value models*, as discussed briefly in Section 2.1, and *reduced-form models*: in this model class the precise mechanism leading to default is left unspecified; instead the default time of a firm is modelled as a nonnegative random variable, whose distribution typically depends on economic covariables. The approach is similar to the modelling philosophy underlying the Bernoulli mixture models introduced in Section 2.2. Reduced-form models are popular in practice, since they lead to tractable formulas for prices of credit derivatives. In particular, it is often possible to apply the well-developed pricing machinery for default-free term structure models to the analysis of defaultable securities; see for instance Lando (1998) or Duffie & Singleton (1999). Duffie & Lando (2001) provide a link between firm-value models and reduced-form models assuming that an investor has *incomplete information*; see also Blanchet-Scalliet & Jeanblanc (2004) or Frey & Runggaldier (2006) for a discussion

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[3]The R-parameter is the coupling of the loan to the systematic factors and therefore quantifies the correlation of the loan with the rest of the portfolio.

from a more theoretical viewpoint. For textbook treatments of dynamic credit risk models we refer to Bielecki & Rutkowski (2002), Bluhm et al. (2002), Duffie & Singleton (2003), Lando (2004), Schönbucher (2003) and Chapter 9 of McNeil et al. (2005). Currently a lot of research is devoted to the development of dynamic credit portfolio models. For reasons of space we cannot discuss this exciting field. An overview is given in Section 9.6 of McNeil et al. (2005), but the best way to get an impression of the current developments is to visit the excellent web-site www.default-risk.com.

Martingale modelling and credit spreads. The existence of a liquid market for credit products requires a specific modelling approach: pricing models for credit derivatives are set up under an *equivalent martingale measure* - an artificial probability measure turning discounted security prices into martingales (fair bets) - and model parameters are determined by equating model prices to prices actually observed on the market (model calibration). In this way it is ensured that the model does not permit any arbitrage (riskless profit) opportunities. Absence of arbitrage also immediately leads to the existence of *credit spreads*: the risk that a lender might loose part or all of his money due to default of a counterparty during the lifetime of a credit contract has to be compensated by an interest rate which is higher than the risk-free rate (the interest rate earned by default-free bonds). The difference between the risk-free rate and the rate one has to pay for a bond or loan subject to default risk is termed *spread*.

4.2 The Defaultable Lévy Libor Model

Among the many possible ways to quantify the dynamic evolution of credit spreads we outline in the following an approach which allows to capture the joint dynamics of risk-free interest rates and credit spreads; for details we refer to the original article Eberlein, Kluge & Schönbucher (2006). A number of instruments depend on both quantities so that modelling interest rates and credit spreads separately might lead to inconsistencies. Instead of describing the dynamics by a diffusion with continuous trajectories we will consider more powerful driving processes, namely *time-inhomogeneous Lévy processes*, also called processes with independent increments and absolutely continuous characteristics (PIIAC) (see Jacod & Shiryaev (2003)). This class of processes is rather flexible and in the context of credit risk even more appropriate than in equity models since credit risk-related information often arrives in such a way that it causes jumps in the underlying quantities: take for example the adjustment of the rating of a firm by one of the leading agencies. Models driven by Lévy processes capture such an abrupt movement more realistically than Brownian motion driven models which have continuous paths. In implementations typically generalized hyperbolic Lévy processes (see Eberlein (2001)) or any of its subclasses like hyperbolic or normal inverse Gaussian processes are used.

Let us consider a fixed time horizon T^* and a discrete tenor structure $T_0 < T_1 < \cdots < T_n = T^*$. T_k denotes the time points where certain periodic payments have to be made. As an example take quarterly or semiannual interest payments for a loan or a coupon-bearing bond over a period of 10 years. As underlying interest rate we consider the δ-forward Libor rates $L(t,T)$. The acronym Libor stands for London Interbank Offered Rate. $L(t,T_k)$ is the annualized interest rate which applies for a period of length $\delta_k = T_{k+1} - T_k$ starting at time point T_k as of time t. δ_k is typically 3 or 6 months. Formally $L(t,T_k)$ is defined by

$$L(t,T_k) = \frac{1}{\delta_k}\left(\frac{B(t,T_k)}{B(t,T_{k+1})} - 1\right) \tag{30}$$

where $B(t,T_k)$ denotes the price at time t of a zero coupon bond with maturity T_k. Zero coupon bond prices are also called *discount factors* since they represent the amount which due to interest earned increases to the face value 1 until maturity T_k, thus $B(T_k,T_k) = 1$. Actually the Libor

rate is not a risk-free rate since by definition it is the rate at which large internationally operating banks lend money to other large internationally operating banks. There is a very small default risk involved and consequently the Libor rate is slightly above the treasury rate. Since it is readily available it is convenient to take the Libor rate as the base rate. The corresponding rate for a contract which has a nonnegligible probability to default is the *defaultable forward Libor rate* $\overline{L}(t, T_k)$. Both rates are related by the equation

$$\overline{L}(t, T_k) = L(t, T_k) + S(t, T_k) \tag{31}$$

where $S(t, T_k)$ is the (positive) spread. Since $S(t, T_k)$ turns out not to be the quantity which will show up in valuation formulae for credit derivatives we will model instead the *forward default intensities* $H(t, T_k)$ given by

$$H(t, T_k) = \frac{S(t, T_k)}{1 + \delta_k L(t, T_k)}. \tag{32}$$

The term $\delta_k L(t, T_k)$ is small compared to 1, therefore, numerically $H(t, T_k)$ and $S(t, T_k)$ are quite close.

We start by specifying the dynamics of the most distant Libor rate by setting

$$L(t, T_{n-1}) = L(0, T_{n-1}) \exp\left(\int_0^t b^L(s, T_{n-1}) \, ds + \int_0^t \lambda(s, T_{n-1}) \, dL_s^{T^*} \right). \tag{33}$$

The fact that $L(\cdot, T_{n-1})$ is modeled as an exponential will guarantee its positivity. $\lambda(\cdot, T_{n-1})$ is a deterministic volatility structure and $L^{T^*} = (L_t^{T^*})$ is a time-inhomogeneous Lévy process which without loss of generality has the simple canonical representation

$$L_t^{T^*} = \int_0^t \sqrt{c_s} \, dW_s^{T^*} + \int_0^t \int_{\mathbb{R}} x(\mu - \nu^{T^*})(ds, dx). \tag{34}$$

The first term is a stochastic integral with respect to a standard Brownian motion W^{T^*} and represents the continuous Gaussian part, whereas the second integral, which is an integral with respect to the compensated random measure of jumps of L^{T^*}, is a purely discontinuous process. The drift term $b^L(\cdot, T_{n-1})$ will be chosen in such a way that $L(\cdot, T_{n-1})$ becomes a martingale under the *terminal forward measure* $\mathbb{P}_{T^*}$.[4]

Via a backward induction for each tenor time point T_k, *forward measures* $\mathbb{P}_{T_k}$ are derived. Although one could define each forward martingale measure $\mathbb{P}_{T_k}$ by giving explicitly its density relative to the spot martingale measure $\mathbb{P}$ – this is the usual martingale measure known from stock price models – the latter is not used in the context of Libor models. One starts with a probability measure $\mathbb{P}_{T^*}$ which is interpreted as the terminal forward measure and proceeds backwards in time by introducing successively the forward measures $\mathbb{P}_{T_k}$ via Radon–Nikodym derivatives

$$\frac{d\mathbb{P}_{T_k}}{d\mathbb{P}_{T_{k+1}}} = \frac{1 + \delta_k L(T_k, T_k)}{1 + \delta_k L(0, T_k)}.$$

Then, for each tenor time point T_k, under $\mathbb{P}_{T_{k+1}}$ the Libor rate $L(t, T_k)$ can be given in the following uniform form

$$L(t, T_k) = L(0, T_k) \exp\left(\int_0^t b^L(s, T_k) \, ds + \int_0^t \lambda(s, T_k) \, dL_s^{T_{k+1}} \right) \tag{35}$$

where also the driving processes $L^{T_{k+1}} = (L_t^{T_{k+1}})$ have to be derived from L^{T^*} during the backward induction. To implement this model one uses only mildly time-inhomogeneous Lévy processes,

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[4]P_{T^*} is the martingale measure corresponding to the numeraire $B(t, T^*)$, i.e. security prices expressed in units of $B(t, T^*)$ are P_{T^*}-martingales.

namely piecewise (time-homogeneous) Lévy processes. Typically three Lévy parameter sets – one for short, one for intermediate, and one for long maturities – are sufficient to calibrate the model to a volatility surface given by prices of interest rate derivatives such as caps, floors and swaptions. For some calibration results see Eberlein & Koval (2006), where the Lévy Libor model has been extended to a multicurrency setting.

The dynamics of the forward default intensities $H(\cdot, T_k)$ cannot be specified directly since it depends on the specification of the random time point at which a defaultable loan or bond actually defaults. There is a standard way to construct a random time for the default event. Let $\Gamma = (\Gamma_t)$ be a hazard process, that is an adapted, right-continuous, increasing process starting at 0 with $\lim_{t \to \infty} \Gamma_t = \infty$. Let η be a uniformly distributed random variable on the interval $[0,1]$, independent of the process $(\Gamma_t)_{t \geq 0}$, possibly defined on an extension of the underlying probability space. Then

$$\tau = \inf\{t > 0 \mid e^{-\Gamma_t} \leq \eta\} \tag{36}$$

defines a stopping time with respect to the 'right' filtration which can be used to indicate default. By choosing the hazard process Γ appropriately – only its values at the tenor time points T_k matter – one can now model the forward default intensities $H(t, T_k)$ in such a way that the dynamics is described in the same simple form (35) as given for the Libor rates, namely

$$
\begin{aligned}
H(t, T_k) = \ & H(0, T_k) \exp \left(\int_0^t b^H(s, T_k) \, ds + \int_0^t \sqrt{c_s} \gamma(s, T_k) \, dW_s^{T_{k+1}} \right. \\
& \left. + \int_0^t \int_{\mathbb{R}} \gamma(s, T_k) x (\mu - \nu^{T_{k+1}})(ds, dx) \right).
\end{aligned} \tag{37}
$$

Again this is done by a backward induction along the tenor time points and as in (35) the specific form as an exponential guarantees that the forward default intensities and thus the spreads $S(t, T_k)$ are positive.

Based on this joint model for interest and default rates we can now price defaultable instruments and credit derivatives. Let us start with a *defaultable coupon bond* with n coupons of a fixed amount c that are promised to be paid at the dates $T_1, \ldots, T_n$. In case default happens during the life time of the bond usually not everything is lost. There is a positive recovery. To incorporate this fact in the model, suitable recovery rules have to be fixed. The most appropriate scheme is the *recovery of par rule*. The assumption is then that if a coupon bond defaults in the time interval $(T_k, T_{k+1}]$, the recovery is given by a recovery rate $\pi \in [0, 1)$ times the sum of the notional amount, which we set equal to 1, and the interest accrued over the period $(T_k, T_{k+1}]$. The resulting amount is paid at time T_{k+1}. The promised interest payments for subsequent periods are lost.

Theorem 4.1 (Pricing of defaultable coupon bonds). *Under the recovery of par rule the arbitrage-free price at time $T_0 = 0$ of a defaultable bond with n coupons of amount c is*

$$B(0, c, n) = \overline{B}(0, T_n) + \sum_{k=0}^{n-1} \overline{B}(0, T_{k+1}) \left(c + \pi(1 + c) \delta_k E_{\overline{\mathbb{P}}_{T_{k+1}}} [H(T_k, T_k)] \right), \tag{38}$$

where $\overline{B}(0, T_k)$ are the pre-default prices of defaultable zero-coupon bonds with maturities T_k, which are known at time 0.

Note that the only random variables in this pricing formula are the forward default intensities. This is the reason why we aimed at describing the dynamics of $H(\cdot, T_k)$ in a relatively simple form. The expectations are taken with respect to the (restricted) defaultable forward measures $\overline{\mathbb{P}}_{T_{k+1}}$ for the dates T_k. These are the appropriate martingale measures in the defaultable world.

Their Radon–Nikodym densities with respect to the (default-free) forward measures $\mathbb{P}_{T_k}$ are given by

$$\frac{d\overline{\mathbb{P}}_{T_k}}{d\mathbb{P}_{T_k}} = \frac{B(0,T_k)}{\overline{B}(0,T_k)} e^{-\Gamma_{T_k}} = \frac{B(0,T_k)}{\overline{B}(0,T_k)} \prod_{i=0}^{k-1} \frac{1}{1 + \delta_i H(T_i,T_i)}. \tag{39}$$

Recall that $B(0,T_k)$ denotes the time-0 price of a default-free zero-coupon bond with maturity T_k. A formula similar to (38) can be obtained to price a *defaultable floating coupon bond* that pays an interest rate composed of the default-free Libor rate plus a constant spread x. Let us mention here that the change of measure technique is a key tool in interest rate and credit theory to obtain valuation formulae which are as simple as possible.

The most popular and heavily traded credit derivatives are *credit default swaps*. They can be used to insure defaultable financial instruments against default. In a credit default swap the protection buyer A pays periodically a fixed fee to the protection seller B until a prespecified credit event occurs or the final time point of the contract is reached. The credit event can be the default of a reference bond issued by a party C. The protection seller in turn will make a payment that covers the losses of A in case the credit event happens. Of course the credit event as well as the default payment have to be clearly specified. Let us consider a standard default swap with the maturity T_n where the credit event is defined to be the default of a certain fixed-coupon bond. According to the recovery scheme explained above, the default payment A will receive at time T_{k+1} if default happend in the period $(T_k,T_{k+1}]$ is $1 - \pi(1+c)$. The periodic fee s, the so-called *default swap rate*, is now determined in such a way that the initial value of the contract is zero. The time-0 value of the periodic fee payments is $s\left(\sum_{k=1}^{n} \overline{B}(0,T_{k-1})\right)$ since each fee payment of size s which has to be made at time T_{k-1} has to be discounted by the corresponding discount factor $\overline{B}(0,T_{k-1})$. Following the standard pricing principle for a contingent claim, some nontrivial analysis shows that the initial value of the payment A will receive in case of default is

$$\sum_{k=1}^{n}(1 - \pi(1+c))\overline{B}(0,T_k)\delta_{k-1}E_{\overline{\mathbb{P}}_{T_k}}[H(T_{k-1},T_{k-1})]. \tag{40}$$

Equating these two sums one gets the default swap rate

$$s = \frac{1 - \pi(1+c)}{\sum\limits_{k=1}^{n} \overline{B}(0,T_{k-1})} \sum_{k=1}^{n}\left(\overline{B}(0,T_k)\delta_{k-1}E_{\overline{\mathbb{P}}_{T_k}}[H(T_{k-1},T_{k-1})]\right). \tag{41}$$

The formula shows that again expectations of forward default intensities have to be evaluated under the corresponding defaultable forward measures. Another important class of credit derivatives which can be priced in this model framework are *credit default swaptions*. The holder of such an option has the right to enter a credit default swap at some prespecified time and swap rate. Credit default swaptions are typically *extension options* which are often imbedded in a credit default swap.

There is a very liquid market for credit default swaps. Therefore the current swap rates usually do not have to be determined by formula (41). Instead, credit default swaps are used as calibration instruments for the term structure of forward default intensities. In other words, given the currently quoted swap rates, (41) is used to extract the model parameters and then the so calibrated model can be used to price less liquid instruments for example in the OTC-market. Other derivatives which can be priced in this modelling framework are *total rate of return swaps*, *asset swaps*, *options on defaultable bonds*, and *credit spread options*.

References

Acerbi, C. (2002), 'Spectral measures of risk: a coherent representation of subjective risk aversion', *J. Banking Finance* **26**(7), 1505–1518.

Acerbi, C. (2004), Coherent representation of subjective risk-aversion, *in* G. Szegö, ed., 'Risk Measures for the 21st Century', Wiley, Chichester.

Acerbi, C. & Tasche, D. (2002), 'On the coherence of expected shortfall', *J. Banking Finance* **26**, 1487–1503.

Artzner, P., Delbaen, F., Eber, J. & Heath, D. (1997), 'Thinking coherently', *Risk* **10**(11), 68–71.

Artzner, P., Delbaen, F., Eber, J. & Heath, D. (1999), 'Coherent measures of risk', *Math. Finance* **9**, 203–228.

Bielecki, T. & Rutkowski, M. (2002), *Credit Risk: Modeling, Valuation, and Hedging*, Springer, Berlin.

Black, F. & Cox, J. (1976), 'Valuing corporate securities: some effects of bond indenture provisions', *J. Finance* **31**, 351–367.

Blanchet-Scalliet, C. & Jeanblanc, M. (2004), 'Hazard rate for credit risk and hedging defaultable contingent claims', *Finance and Stochastics* **8**, 145–159.

Bluhm, C. & Overbeck, L. (2006), *Structured Credit Portfolio Analysis Baskets & CDOs*, Chapman& Hall.

Bluhm, C., Overbeck, L. & Wagner, C. (2002), *An Introduction to Credit Risk Modeling*, CRC Press/Chapman & Hall.

Cheridito, P., Delbaen, F. & Kupper, M. (2006), 'Dynamic monetary risk measures for bounded discrete-time processes', *Electronic Journal of Probability* **11**, 57–106.

Crosbie, P. & Bohn, J. (2002), 'Modeling default risk', KMV working paper. Available from `http://www.kmv.com`.

Crouhy, M., Galai, D. & Mark, R. (2001), *Risk Management*, McGraw-Hill, New York.

Delbaen, F. (2000), 'Coherent risk measures', lecture notes, Cattedra Galiliana, Scuola Normale Superiore, Pisa.

Delbaen, F. (2002), Coherent risk measures on general probability spaces, *in* K. Sandmann & P. Schönbucher, eds, 'Advances in Finance and Stochastics', Springer, Berlin, pp. 1–37.

Denault, M. (2001), 'Coherent allocation of risk capital', *Journal of Risk* **4**(1).

Diestel, J. (1975), *Geometry of Banach Spaces - Selected Topics*, Vol. 485, Springer.

Duffie, D. & Lando, D. (2001), 'Term structure of credit risk with incomplete accounting observations', *Econometrica* **69**, 633–664.

Duffie, D. & Singleton, K. (1999), 'Modeling term structures of defaultable bonds', *Rev. Finan. Stud.* **12**, 687–720.

Duffie, D. & Singleton, K. (2003), *Credit Risk: Pricing, Measurement and Management*, Princeton University Press, Princeton and Oxford.

Eberlein, E. (2001), Application of generalized hyperbolic Lévy motions to finance, *in* O. Barndorff-Nielsen, T. Mikosch & S. Resnick, eds, 'Lévy Processes: Theory and Applications', Birkhäuser, Boston, pp. 319–337.

Eberlein, E., Kluge, W. & Schönbucher, P. (2006), 'The Lévy Libor model with default risk', *Journal of Credit Risk* **2**, 3–42.

Eberlein, E. & Koval, N. (2006), 'A cross-currency Lévy market model', *Quantitative Finance* **6**, 1–16.

Egloff, D., Leippold, M. & Dalbert, C. (2005), 'Optimal importance sampling for credit portfolios with stochastic approximations', working paper, Zürcher Kantonalbank, Zürich.

Elliott, R.J. and Aggoun, L. & Moore, J. (1995), *Hidden Markov Models: Estimation and Control*, Springer, New York.

Embrechts, P., McNeil, A. & Straumann, D. (2001), Correlation and dependency in risk management: properties and pitfalls, *in* M. Dempster & H. Moffatt, eds, 'Risk Management: Value at Risk and Beyond', Cambridge University Press, http://www.math.ethz.ch/∼mcneil, pp. 176–223.

Föllmer, H. and Schied, A. (2002), 'Convex measures of risk and trading constraints', *Finance and Stochastics* **6**, 429–447.

Föllmer, H. & Schied, A. (2004), *Stochastic Finance - An Introduction in Discrete Time*, 2nd edn, Walter de Gruyter, Berlin New York.

Frey, R. & McNeil, A. (2003), 'Dependent defaults in models of portfolio credit risk', *J. Risk* **6**(1), 59–92.

Frey, R. & Runggaldier, W. (2006), 'Credit risk and incomplete information: a nonlinear filtering approach', preprint, Universität Leipzig.

Frittelli, M. & Gianin, E. R. (2002), 'Putting order in risk measures', *J. Banking Finance* **26**, 1473–1486.

Glasserman, P. & Li, J. (2005), 'Importance sampling for portfolio credit risk', *Management Science* **51**, 1643–1656.

Hallerbach, W. (2003), 'Decomposing portfolio Value-at-Risk: A general analysis', *Journal of Risk* **5**(2), 1–18.

Heath, D. & Ku, H. (2004), 'Pareto equilibria with coherent measures of risk', *Mathematical Finance* **14**, 163–172.

Jacod, J. & Shiryaev, A. (2003), *Limit Theorems for Stochastic Processes*, 2nd edn, Springer Verlag, Berlin.

Jarrow, R., Lando, D. & Turnbull, S. (1997), 'A Markov model for the term structure of credit risk spreads', *Rev. Finan. Stud.* **10**, 481–523.

Jouini, E., Schachermayer, W. & Touzi, N. (2006), 'Law invariant risk measures have the Fatou property', preprint, Université Paris Dauphine.

Kalkbrener, M. (2005), 'An axiomatic approach to capital allocation', *Math. Finance* **15**, 425–437.

Kalkbrener, M., Lotter, H. & Overbeck, L. (2004), 'Sensible and efficient capital allocation for credit portfolios', *Risk* **17**(1), S19-S24.

Kealhofer, S. & Bohn, J. (2001), 'Portfolio management of default risk', KMV working paper. Available from http://www.kmv.com.

Kloman, H. (1999), Does risk matter?, *in* 'EARTH Matters, Special Edition on Risk Management', Lamont-Doherty Earth Observatory, Columbia University.

Kusuoka, S. (2001), 'On law invariant coherent risk measures', *Advances in Mathematical Economics* **3**, 83–95.

Lando, D. (1998), 'Cox processes and credit risky securities', *Rev. Derivatives Res.* **2**, 99–120.

Lando, D. (2004), *Credit Risk Modeling: Theory and Applications*, Princeton University Press, Princeton, New Jersey.

Lando, D. & Skodeberg, T. (2002), 'Analyzing rating transitions and rating drift with continuous observations', *Journal of Banking and Finance* **26**, 423–444.

Lintner, J. (1965), 'Security prices, risk and maximal gains from diversification', *J. Finance* **20**(4), 587–615.

Longstaff, F. & Schwartz, E. (1995), 'Valuing risky debt: A new approach', *J. Finance* **50**, 789–821.

Markowitz, H. (1952), 'Portfolio selection', *J. Finance* **7**, 77–91.

Matten, C. (2000), *Managing Bank Capital: Capital Allocation and Performance Measurement*, 2nd edn, Wiley, New York.

McNeil, A., Frey, R. & Embrechts, P. (2005), *Quantitative Risk Management: Concepts, Techniques and Tools*, Princeton University Press, Princeton, New Jersey.

McNeil, A. & Wendin, J. (2005), 'Bayesian inference for generalised linear mixed models of portfolio credit risk', preprint, ETH Zürich, forthcoming in *Journal of Empirical Finance.*

Merton, R. (1974), 'On the pricing of corporate debt: The risk structure of interest rates', *J. Finance* **29**, 449–470.

Overbeck, L. (2000), Allocation of economic capital in loan portfolios, *in* W. Härdle & G. Stahl, eds, 'Measuring Risk in Complex Stochastic Systems', Berlin.

Overbeck, L. (2004), Spectral Capital Allocation, *in* A. Dev , eds, 'Economic Capital, A Practitioner Guide', Risk Books, London.

Pirvu, T. & Zitkovic, G. (2006), 'Maximizing the growth rate under risk constraints', preprint, Dept. of Mathematics, University of Texas at Austin.

Robert, C. & Casella, G. (1999), *Monte Carlo Statistical Methods*, Springer, New York.

Rockafellar, R. T. & Uryasev, S. (2002), 'Conditional value-at-risk for general loss distributions', *J. Banking Finance* **26**, 1443–1471.

Rockafellar, R. & Uryasev, S. (2000), 'Optimization of conditional Value-at-Risk', *Journal of Risk* **2**, 21–42.

Schönbucher, P. (2003), *Credit Derivatives Pricing Models*, Wiley.

Sharpe, W. (1964), 'Capital asset prices: a theory of market equilibrium under conditions of risk', *J. Finance* **19**(3), 425–442.

Tasche, D. (1999), 'Risk contributions and performance measurement', preprint, Dept. of Mathematics, TU-München.

Tasche, D. (2002), 'Expected shortfall and beyond', *J. Banking Finance* **26**, 1519–1533.

Weber, S. (2004), 'Distribution invariant risk measures, entropy, and large deviations', preprint, Dept. of Mathematics, Humboldt Universität Berlin, forthcoming in Journal of Applied Probability.