SYMMETRIES IN LÉVY TERM STRUCTURE MODELS

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Abstract. Symmetry results between call and put options have been widely studied in equity markets. We provide similar symmetry results between caps and floors in a Heath–Jarrow–Morton, a LIBOR and a forward price model, driven by time-inhomogeneous Lévy processes. On the way, we review the basic properties of these models.

1. Introduction

The well-known caplet-floorlet parity relates caps and floors of the same strike and time to maturity; let $L(T,T)$ denote the LIBOR rate for the period $[T, T+\delta]$, then the values of a caplet and a floorlet with strike K and payoff $\delta(L(T,T)-K)^+$ and $\delta(K-L(T,T))^+$ respectively, are related via

$$C_0(K,T) = F_0(K,T) + B(0,T) - (1+\delta K)B(0,T+\delta),$$

where $B(0,T)$ denotes the price of a zero coupon bond maturing at T and $C_0(K,T)$ and $F_0(K,T)$ denote the present value of a caplet and a floorlet respectively, with strike rate K maturing at T.

The aim of this paper is to provide symmetries between caplets and floorlets with different strikes but the same time of maturity and moneyness, in term structure models driven by time-inhomogeneous Lévy processes. By 'moneyness' of a caplet (resp. floorlet), we mean the ratio of the initial forward LIBOR rate over the strike (resp. the reciprocal of this ratio).

The proofs are based on the choice of a suitable numéraire and the subsequent change of the probability measure; this method was pioneered by Geman et al. (1995). Three different approaches to modeling interest rates are considered: a Heath–Jarrow–Morton forward rate model, a model for the LIBOR, and a model for the forward price.

In equity markets there is a long list of articles discussing similar results, with driving processes of increasing generality; we refer to Carr (1994), Chesney and Gibson (1995), Carr and Chesney (1996), McDonald and Schroder (1998), Schroder (1999), Detemple (2001), Fajardo and Mordecki (2003) and Eberlein and Papapantoleon (2005), to mention just a part of the existing literature. Apart from providing better understanding of valuation formulas and simplifying computational work, such results are applied for statically hedging other – usually exotic – derivatives; see, e.g. Carr et al. (1998).

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Lévy processes have become standard tools in financial modeling; the recent volume of Kyprianou et al. (2005) contains an up-to-date account of applications of Lévy processes in finance. For more general considerations about Lévy processes one could refer to the books of Bertoin (1996), Sato (1999), and Applebaum (2004). The books of Schoutens (2003) and Cont and Tankov (2003) discuss Lévy processes with particular focus on their applications in finance.

Empirical evidence for the non-Gaussianity of daily returns from bond market data can be found in Raible (2000, chapter 5); the fit of the normal inverse Gaussian distribution to the same data is particularly good, further supporting the use of Lévy processes for modeling interest rates. Similar evidence appears in the risk-neutral world, i.e. from caplet implied volatility smiles and surfaces; we refer to Eberlein and Kluge (2005).

A valuation method for caps and floors – as well as for swaptions – in the Lévy Heath–Jarrow–Morton model is described in Eberlein and Kluge (2005). The method relies on a convolution representation of the option value and the use of Laplace transforms, beautifully derived in Raible (2000); see also the method based on Fourier transforms of Carr and Madan (1999). This method can be adapted to value caps and floors in the Lévy forward price model. Similar arguments, combined with an approximation (see also section 5), are used in Eberlein and Özkan (2005) for the valuation of caps and floors in the Lévy LIBOR model. A different approach to the one of Eberlein and Özkan, also relying on transform methods, is taken in Kluge (2005) for the valuation of caps, floors, and swaptions in the Lévy LIBOR model.

This paper is organized as follows: in section 2 we describe the driving time-inhomogeneous Lévy processes and in section 3 we review three different approaches to modeling interest rates based on Lévy processes; namely, an HJM forward rate model, a LIBOR model and a model for forward prices. In each of the subsequent sections a caplet-floorlet symmetry is provided for each of these models.

2. Time-inhomogeneous Lévy processes

Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a complete stochastic basis, where $\mathcal{F} = \mathcal{F}_{T^*}$ and the filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T^*]}$ satisfies the usual conditions; we assume that $T^* \in \mathbb{R}_+$ is a fixed time horizon. Following Eberlein, Jacod, and Raible (2005), we choose the driving process $L = (L_t)_{t\in[0,T^*]}$ as a time-inhomogeneous Lévy process, more specifically, as a *process* with *independent increments* and *absolutely continuous* characteristics, in the sequel abbreviated PIIAC. The law of L_t is described by the characteristic function

$$\mathbb{E}\left[e^{iuL_t}\right] = \exp \int_0^t \Big[ib_s u - \frac{c_s}{2}u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux)\lambda_s(\mathrm{d}x)\Big]\mathrm{d}s, \tag{2.1}$$

where $b_t \in \mathbb{R}$, $c_t \in \mathbb{R}_+$ and λ_t is a Lévy measure, i.e. satisfies $\lambda_t(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge |x|^2)\lambda_t(\mathrm{d}x) < \infty$, for all $t \in [0, T^*]$. The process L is càdlàg and satisfies Assumptions ($\mathbb{AC}$) and ($\mathbb{EM}$) given below; moreover, $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T^*]}$ is the filtration generated by $L = (L_t)_{t\in[0,T^*]}$.

Assumption ($\mathbb{AC}$). The triplets (b_t, c_t, λ_t) satisfy the following condition:

$$\int_0^{T^*} \Big[|b_t| + |c_t| + \int_{\mathbb{R}} (1 \wedge |x|^2)\lambda_t(\mathrm{d}x)\Big]\mathrm{d}t < \infty. \tag{2.2}$$

Assumption ($\mathbb{EM}$). There exist constants $M, \varepsilon > 0$ such that for every $u \in [-(1+\varepsilon)M, (1+\varepsilon)M]$

$$\int_0^{T^*} \int_{\{|x|>1\}} \exp(ux)\lambda_t(\mathrm{d}x)\mathrm{d}t < \infty. \tag{2.3}$$

Subject to these assumptions, L is a *special* semimartingale (therefore, no truncation function is needed) and its *triplet of semimartingale characteristics* (cf. Jacod and Shiryaev 2003, II.2.6) is given by

$$B_t = \int_0^t b_s \mathrm{d}s, \qquad C_t = \int_0^t c_s \mathrm{d}s, \quad \nu([0,t]\times A) = \int_0^t \int_A \lambda_s(\mathrm{d}x)\mathrm{d}s, \tag{2.4}$$

where $A \in \mathcal{B}(\mathbb{R})$. The triplet (b, c, λ) represents the *local characteristics* of L. The triplet of semimartingale characteristics (B, C, ν) completely describes the distribution of L. Moreover, L has the canonical decomposition (cf. Jacod and Shiryaev 2003, II.2.38 and Eberlein et al. 2005)

$$L_t = \int_0^t b_s \mathrm{d}s + \int_0^t \sqrt{c_s}\mathrm{d}W_s + \int_0^t \int_{\mathbb{R}} x(\mu^L - \nu)(\mathrm{d}s, \mathrm{d}x), \tag{2.5}$$

where μ^L is the random measure of jumps of the process L and W is a $\mathbb{P}$-standard Brownian motion.

We denote by θ_s the *cumulant* associated with the infinitely divisible distribution with Lévy triplet (b_s, c_s, λ_s), i.e. for $z \in [-(1+\varepsilon)M, (1+\varepsilon)M]$

$$\theta_s(z) := b_s z + \frac{c_s}{2}z^2 + \int_{\mathbb{R}} (e^{zx} - 1 - zx)\lambda_s(\mathrm{d}x). \tag{2.6}$$

In addition, we can extend θ_s to the complex domain $\mathbb{C}$, for $z \in \mathbb{C}$ with $\Re z \in [-(1+\varepsilon)M, (1+\varepsilon)M]$ and the characteristic function of L_t can be written as

$$\mathbb{E}\left[e^{iuL_t}\right] = \exp \int_0^t \theta_s(iu)\mathrm{d}s. \tag{2.7}$$

If L is a (time-homogeneous) Lévy process, then (b_s, c_s, λ_s) and thus also θ_s do not depend on s. In that case, θ equals the cumulant (log-moment generating function) of L_1.

Lemma 2.1. *Let L be a PIIAC satisfying assumption* ($\mathbb{EM}$) *and $f : \mathbb{R}_+ \to \mathbb{C}$ a continuous function such that $|\Re(f)| \le M$. Then*

$$\mathbb{E}\left[\exp \int_0^t f(s)\mathrm{d}L_s\right] = \exp \int_0^t \theta_s(f(s))\mathrm{d}s. \tag{2.8}$$

(The integrals are to be understood componentwise for real and imaginary part.)

Proof. The proof is similar to that of Lemma 3.1 in Eberlein and Raible (1999). □

Notation. We denote by $-\lambda_t$ the Lévy measure defined by

$$-\lambda_t([a,b]) := \lambda_t([-b,-a]) \tag{2.9}$$

for $a,b \in \mathbb{R}$, $a < b$, $t \in [0,T^*]$. Thus, $-\lambda_t$ is the mirror image of the original measure with respect to the vertical axis. For a compensator measure of the form $\nu(\mathrm{d}t,\mathrm{d}x) = \lambda_t(\mathrm{d}x)\mathrm{d}t$, we denote by $-\nu$ the measure defined as

$$-\nu(\mathrm{d}t,\mathrm{d}x) := -\lambda_t(\mathrm{d}x)\mathrm{d}t.$$

Furthermore, whenever we use the symbol "$-$" in front of a Lévy or a compensator measure, we will refer to measures defined as above.

Lemma 2.2. *Let L be a PIIAC with characteristic triplet (B,C,ν), satisfying assumption* ($\mathbb{EM}$)*. Then $L^\star := -L$ is again a PIIAC with characteristic triplet $(B^\star,C^\star,\nu^\star)$, where $B^\star = -B$, $C^\star = C$ and $\nu^\star = -\nu$.*

Proof. The proof can be found in Eberlein and Papapantoleon (2005). □

3. Lévy term structure models

In this section we shortly review three different models for the term structure of interest rates, based on time-inhomogeneous Lévy processes.

3.1. The Lévy HJM model.

A standard approach to modeling interest rates is that of Heath, Jarrow, and Morton (1992), where subject to modeling are either zero coupon bond prices or instantaneous continuously compounded forward rates. The Lévy HJM model has been introduced in Eberlein and Raible (1999) and extended to time-inhomogeneous Lévy processes in Eberlein et al. (2005) and Eberlein and Kluge (2005).

Assume that for every $T \in [0,T^*]$, there exists a zero coupon bond maturing at T traded in the market. Moreover, let $U \in [0,T^*]$.

Let $L = (L_t)_{t\in[0,T^*]}$ be a PIIAC on the stochastic basis $(\Omega,\mathcal{F},\mathbf{F},\mathbb{P})$ with semimartingale characteristics (B,C,ν) or local characteristics (b,c,λ).

The dynamics of instantaneous forward rates and zero coupon bond prices, in the time-inhomogeneous Lévy term structure model, has been derived in Eberlein and Kluge (2005); we refer the reader to this article for all the details.

The dynamics of the instantaneous continuously compounded forward rates for $T \in [0,T^*]$ is given by

$$f(t,T) = f(0,T) + \int_0^t \partial_2 A(s,T)\mathrm{d}s - \int_0^t \partial_2\Sigma(s,T)\mathrm{d}L_s, \quad 0 \le t \le T, \tag{3.1}$$

where ∂_2 denotes the derivative operator with respect to the second argument. The initial values $f(0,T)$ are deterministic, and bounded and measurable in T. Σ and A are deterministic real-valued functions defined on

$\Delta := \{(s,T) \in [0,T^*] \times [0,T^*]; s \leq T\}$, whose paths are continuously differentiable in the second variable. Moreover, they satisfy the following conditions

(B1): The volatility structure Σ is continuous in the first argument and bounded in the following way: for $(s,T) \in \Delta$ we have

$$0 \leq \Sigma(s,T) \leq \frac{M}{2},$$

where M is the constant from Assumption ($\mathbb{EM}$). Furthermore, $\Sigma(s,T) \neq 0$ for $s < T$ and $\Sigma(T,T) = 0$ for $T \in [0,T^*]$.

(B2): The drift coefficients $A(\cdot,T)$ are given by

$$A(s,T) = \theta_s(\Sigma(s,T)). \tag{3.2}$$

From Eberlein and Kluge (2006, (2.6)), we get that the time-T price of a zero coupon bond maturing at time U is

$$B(T,U) = \frac{B(0,U)}{B(0,T)} \exp\left(\int_0^T \Sigma(s,T,U)\mathrm{d}L_s - \int_0^T A(s,T,U)\mathrm{d}s \right), \tag{3.3}$$

where the following abbreviations are used:

$$\begin{aligned} \Sigma(s,T,U) &:= \Sigma(s,U) - \Sigma(s,T), \\ A(s,T,U) &:= A(s,U) - A(s,T). \end{aligned}$$

Similarly, using in Eberlein and Kluge (2006, (2.5)), we have for the money market account

$$B_T^M = \frac{1}{B(0,T)} \exp\left(\int_0^T A(s,T)\mathrm{d}s - \int_0^T \Sigma(s,T)\mathrm{d}L_s \right). \tag{3.4}$$

Remark 3.1. The drift condition (3.2) guarantees that bond prices discounted by the money market account are martingales; hence, $\mathbb{P}$ is a martingale measure. In addition, from Theorem 6.4 in Eberlein et al. (2005), we know that the martingale measure is *unique*.

3.2. **The Lévy LIBOR model.** The main pitfall of the HJM framework is the assumption of continuously compounded rates, while in real markets interest accrues according to a discrete grid, the tenor structure. LIBOR market models, that is, arbitrage–free term structure models on a discrete tenor, were constructed by Sandmann et al. (1995), Miltersen et al. (1997), Brace et al. (1997), and Jamshidian (1997). The Lévy LIBOR model was recently developed by Eberlein and Özkan (2005).

Let $0 = T_0 < T_1 < T_2 < \cdots < T_N < T_{N+1} = T^*$ denote a discrete tenor structure where $\delta = T_{i+1} - T_i$, $i = 0,1,\ldots,N$; since the model is constructed via backward induction, we denote $T_j^* = T^* - j\delta$ for $j = 0,1,\ldots,N$. We assume the following conditions are in force

(LR1): For any maturity T_i there exists a bounded, continuous, deterministic function $\lambda(\cdot,T_i) : [0,T_i] \to \mathbb{R}$, which represents the volatility of the forward LIBOR rate process $L(\cdot, T_i)$. Moreover,

$$\sum_{i=1}^{N} \left|\lambda(s, T_i)\right| \leq M,$$

for all $s \in [0, T^*]$, where M is the constant from Assumption $(\mathbb{EM})$ and $\lambda(s, T_i) = 0$ for all $s > T_i$.

(LR2): We assume a strictly positive and strictly decreasing initial term structure $B(0, T_i)$, $1 \leq i \leq N+1$. Consequently, the initial term structure of forward LIBOR rates is given, for $1 \leq i \leq N$, by

$$L(0, T_i) = \frac{1}{\delta}\Big(\frac{B(0, T_i)}{B(0, T_i + \delta)} - 1\Big).$$

As usual, we consider a complete stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P}_{T^*})$. Let $L = (L_t)_{t \in [0,T^*]}$ be a PIIAC on this stochastic basis with semimartingale characteristics $(0, C, \nu^{T^*})$ or local characteristics $(0, c, \lambda^{T^*})$.

The construction starts by postulating the dynamics of the forward LIBOR rate with the longest maturity $L(\cdot, T_1^*)$ under the terminal forward martingale measure $\mathbb{P}_{T^*}$ and proceeds by backward induction.

For an arbitrary T_j^*, $1 \leq j \leq N$, the dynamics of the forward LIBOR rate $L(\cdot, T_j^*)$ under the forward martingale measure $\mathbb{P}_{T_{j-1}^*}$, is

$$L(t, T_j^*) = L(0, T_j^*) \exp\left(\int_0^t b^L(s, T_j^*, T_{j-1}^*)\mathrm{d}s + \int_0^t \lambda(s, T_j^*)\mathrm{d}L_s^{T_{j-1}^*}\right), \quad (3.5)$$

where $L^{T_{j-1}^*}$ has the canonical decomposition

$$L_t^{T_{j-1}^*} = \int_0^t \sqrt{c_s}\mathrm{d}W_s^{T_{j-1}^*} + \int_0^t \int_{\mathbb{R}} x(\mu^L - \nu^{T_{j-1}^*})(\mathrm{d}s, \mathrm{d}x); \quad (3.6)$$

here, μ^L is the random measure of jumps of the process $L = L^{T^*}$, which is specified as a PIIAC under the terminal measure $\mathbb{P}_{T^*}$. The forward measure $\mathbb{P}_{T_{j-1}^*}$ is related to the terminal forward measure $\mathbb{P}_{T^*}$ via

$$\frac{\mathrm{d}\mathbb{P}_{T_{j-1}^*}}{\mathrm{d}\mathbb{P}_{T^*}} = \prod_{k=1}^{j-1} \frac{1 + \delta L(T_{j-1}^*, T_k^*)}{1 + \delta L(0, T_k^*)}.$$

Additionally, $W^{T_{j-1}^*}$ is a $\mathbb{P}_{T_{j-1}^*}$-Brownian motion which is related to the $\mathbb{P}_{T^*}$-Brownian motion via

$$\begin{aligned} W_t^{T_{j-1}^*} &= W_t^{T_{j-2}^*} - \int_0^t \alpha(s, T_{j-1}^*, T_{j-2}^*)\sqrt{c_s}\mathrm{d}s = \ldots \\ &= W_t^{T^*} - \int_0^t \left(\sum_{k=1}^{j-1} \alpha(s, T_k^*, T_{k-1}^*)\right)\sqrt{c_s}\mathrm{d}s, \end{aligned}$$

where

$$\alpha(t, T_k^*, T_{k-1}^*) = \frac{\delta L(t-, T_k^*)}{1 + \delta L(t-, T_k^*)}\lambda(t, T_k^*).$$

Similarly, $\nu^{T^*_{j-1}}$ is the $\mathbb{P}_{T^*_{j-1}}$-compensator of μ^L and is related to the $\mathbb{P}_{T^*}$-compensator of μ^L via

$$\begin{aligned}\nu^{T^*_{j-1}}(\mathrm{d}s,\mathrm{d}x) &= \beta(s,x,T^*_{j-1},T^*_{j-2})\nu^{T^*_{j-2}}(\mathrm{d}s,\mathrm{d}x) = \dots \\ &= \left(\prod_{k=1}^{j-1}\beta(s,x,T^*_k,T^*_{k-1})\right)\nu^{T^*}(\mathrm{d}s,\mathrm{d}x), \end{aligned} \tag{3.7}$$

where

$$\beta(t,x,T^*_k,T^*_{k-1}) = \frac{\delta L(t-,T^*_k)}{1+\delta L(t-,T^*_k)}\left(e^{\lambda(t,T^*_k)x}-1\right)+1. \tag{3.8}$$

In order to ensure that $L(\cdot,T^*_j)$ is a martingale under its corresponding forward martingale measure $\mathbb{P}_{T^*_{j-1}}$, we need to impose the following condition on the drift term of the forward LIBOR process

$$\begin{aligned} b^L(s,T^*_j,T^*_{j-1}) &= -\frac{1}{2}(\lambda(s,T^*_j))^2 c_s \\ &\quad - \int_{\mathbb{R}} \left(e^{\lambda(s,T^*_j)x}-1-\lambda(s,T^*_j)x\right)\lambda_s^{T^*_{j-1}}(\mathrm{d}x). \end{aligned} \tag{3.9}$$

Remark 3.2. Notice that the process $L^{T^*_{j-1}}$, driving the forward LIBOR rate $L(\cdot,T^*_j)$, and $L = L^{T^*}$ have the same martingale parts and differ only in the finite variation part (drift). An application of Girsanov's theorem for semimartingales yields that the $\mathbb{P}_{T^*_{j-1}}$-finite variation part of L is

$$\int_0^{\cdot} c_s \sum_{k=1}^{j-1}\alpha(s,T^*_k,T^*_{k-1})\mathrm{d}s + \int_0^{\cdot}\int_{\mathbb{R}} x\left(\prod_{k=1}^{j-1}\beta(s,x,T^*_k,T^*_{k-1})-1\right)\nu^{T^*}(\mathrm{d}s,\mathrm{d}x).$$

Remark 3.3. The process $L = L^{T^*}$ driving the most distant LIBOR rate $L(\cdot,T^*_1)$ is – by assumption – a time-inhomogeneous Lévy process. However, this is *not* the case for any of the processes $L^{T^*_{j-1}}$ driving the remaining LIBOR rates, because the *random* terms $\frac{\delta L(t-,T^*_j)}{1+\delta L(t-,T^*_j)}$ enter into the compensators $\nu^{T^*_{j-1}}$ during the construction; see equations (3.7) and (3.8).

3.3. **The Lévy forward price model.** A LIBOR-type model for the forward price using Lévy processes is also proposed in Eberlein and Özkan (2005, pp. 342–343). A detailed construction of the model is presented in Kluge (2005); there, it is also shown how this model can be embedded in the Lévy HJM model.

The advantage of this model is that the driving process remains a time-inhomogeneous Lévy process under each forward measure, hence it is particularly suitable for implementations. The downside is that negative LIBOR rates can occur, like in an HJM model.

Let $0 = T_0 < T_1 < T_2 < \cdots < T_N < T_{N+1} = T^*$ denote a discrete tenor structure where $\delta = T_{i+1}-T_i$, $i = 0,1,\dots,N$; the model is again constructed via backward induction, hence we denote $T^*_j = T^* - j\delta$ for $j = 0,1,\dots,N$. We assume the following conditions are in force

(FP1): For any maturity T_i there exists a bounded, continuous, deterministic function $\lambda(\cdot, T_i) : [0, T_i] \to \mathbb{R}$, which represents the volatility of the forward price process $F_B(\cdot, T_i, T_i + \delta)$. Moreover, we require that the volatility structure satisfies

$$\Big|\sum_{k=1}^{i} \lambda(\cdot, T_k)\Big| \le M, \qquad \forall\, i \in \{1, \dots, N\},$$

for all $s \in [0, T^*]$, where M is the constant from Assumption $(\mathbb{EM})$ and $\lambda(s, T_i) = 0$ for all $s > T_i$.

(FP2): We assume a strictly positive initial term structure $B(0, T_i)$, $1 \le i \le N + 1$. Consequently, the initial term structure of forward price processes is given, for $1 \le i \le N$, by

$$F_B(0, T_i, T_i + \delta) = \frac{B(0, T_i)}{B(0, T_i + \delta)}.$$

As usual, we consider a complete stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P}_{T^*})$. Let $L = (L_t)_{t\in[0,T^*]}$ be a PIIAC on this stochastic basis with semimartingale characteristics $(0, C, \nu^{T^*})$ or local characteristics $(0, c, \lambda^{T^*})$. The construction starts by defining the forward price process with the longest maturity $F_B(\cdot, T_1^*, T^*)$ under the terminal forward martingale measure $\mathbb{P}_{T^*}$ and then proceeds by backward induction.

We denote by $\mathbb{P}_{T^*_{j-1}}$ the forward measure associated with the maturity date T^*_{j-1}, $1 \le j \le N$. The forward price process $F_B(\cdot, T^*_j, T^*_{j-1})$ can be written, under the corresponding forward martingale measure $\mathbb{P}_{T^*_{j-1}}$, as

$$F_B(0, T^*_j, T^*_{j-1}) = F_B(0, T^*_j, T^*_{j-1}) \exp\left(\int_0^t b^L(s, T^*_j, T^*_{j-1})\mathrm{d}s + \int_0^t \lambda(s, T^*_j)\mathrm{d}L_s^{T^*_{j-1}}\right)$$

where

$$L_t^{T^*_{j-1}} = \int_0^t \sqrt{c_s}\mathrm{d}W_s^{T^*_{j-1}} + \int_0^t\int_{\mathbb{R}} x(\mu^L - \nu^{T^*_{j-1}})(\mathrm{d}s, \mathrm{d}x)$$

is a time-inhomogeneous Lévy process. Here μ^L is the random measure of jumps of the process $L = L^{T^*}$. $W^{T^*_{j-1}}$ is a $\mathbb{P}_{T^*_{j-1}}$-Brownian motion, which is related to a $\mathbb{P}_{T^*}$-Brownian motion via

$$\begin{aligned} W_t^{T^*_{j-1}} &= W_t^{T^*_{j-2}} - \int_0^t \lambda(s, T^*_{j-1})\sqrt{c_s}\mathrm{d}s = \dots \\ &= W_t^{T^*} - \int_0^t \left(\sum_{k=1}^{j-1} \lambda(s, T^*_k)\right)\sqrt{c_s}\mathrm{d}s. \end{aligned}$$

Additionally, $\nu^{T^*_{j-1}}$ is the $\mathbb{P}_{T^*_{j-1}}$-compensator of μ^L and this is related to the $\mathbb{P}_{T^*}$-compensator of μ^L via

$$\begin{aligned}\nu^{T^*_{j-1}}(\mathrm{d}s,\mathrm{d}x) &= \exp\Big(\lambda(s,T^*_{j-1})x\Big)\nu^{T^*_{j-2}}(\mathrm{d}s,\mathrm{d}x) = \dots\\ &= \exp\left(x\sum_{k=1}^{j-1}\lambda(s,T^*_k)\right)\nu^{T^*}(\mathrm{d}s,\mathrm{d}x).\end{aligned}$$

The forward process $F_B(\cdot,T^*_j,T^*_{j-1})$ has to be a martingale under the corresponding forward measure $\mathbb{P}_{T^*_{j-1}}$; therefore, we specify the drift term of the forward process to be

$$\begin{aligned}b^L(s,T^*_j,T^*_{j-1}) = &-\frac{1}{2}(\lambda(s,T^*_j))^2 c_s\\ &-\int_{\mathbb{R}}\Big(e^{\lambda(s,T^*_j)x}-1-\lambda(s,T^*_j)x\Big)\lambda_s^{T^*_{j-1}}(\mathrm{d}x).\end{aligned}\tag{3.10}$$

4. Symmetry in the Lévy HJM model

In this section we derive a symmetry relationship relating call and put options on zero coupon bonds. As a corollary of this result, we get a symmetry between caplets and floorlets in the Lévy HJM model.

For the symmetry result, we define the constant D via

$$\begin{aligned}D &:= \mathbb{E}\left[\frac{B(T,U)}{\left(B_T^M\right)^2}\right]\\ &= \mathbb{E}\Bigg[B(0,U)B(0,T)\exp\left(\int_0^T\big(\Sigma(s,U)+\Sigma(s,T)\big)\mathrm{d}L_s\right)\\ &\qquad\times\exp\left(\int_0^T-\big(A(s,U)+A(s,T)\big)\mathrm{d}s\right)\Bigg]\\ &= B(0,U)B(0,T)\exp\left(\int_0^T\theta_s\big(\overline{\Sigma}(s,T,U)\big)\mathrm{d}s\right)\\ &\qquad\times\exp\left(\int_0^T-\big(\theta_s(\Sigma(s,U))+\theta_s(\Sigma(s,T))\big)\mathrm{d}s\right),\end{aligned}$$

where we used the abbreviation $\overline{\Sigma}(s,T,U) := \Sigma(s,U)+\Sigma(s,T)$. In addition, we define a measure $\widetilde{\mathbb{P}}$ via the Radon–Nikodym derivative

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} := \frac{B(T,U)}{D\left(B_T^M\right)^2} = \frac{B(T,U)}{\left(B_T^M\right)^2\mathbb{E}\left[\frac{B(T,U)}{\left(B_T^M\right)^2}\right]} = \eta_T,\tag{4.1}$$

noting that $\mathbb{E}\left[\frac{B(T,U)}{D(B_T^M)^2}\right] = 1$ and that the two measures, $\mathbb{P}$ and $\widetilde{\mathbb{P}}$, are equivalent since η_T is strictly positive. The density process $(\eta_t)_{t\in[0,T]}$ related to this change of measure is given by the restriction of the Radon–Nikodym derivative to the σ-field $\mathcal{F}_t$, i.e. for $t \leq T$, we get

$$\begin{aligned}
\eta_t &= \mathbb{E}\left[\frac{B(T,U)}{D\left(B_T^M\right)^2}\middle|\mathcal{F}_t\right]\\
&= \mathbb{E}\left[\exp\left(\int_0^T \overline{\Sigma}(s,T,U)\mathrm{d}L_s - \int_0^T \theta_s\big(\overline{\Sigma}(s,T,U)\big)\mathrm{d}s\right)\middle|\mathcal{F}_t\right]\\
&= \exp\left(\int_0^t \overline{\Sigma}(s,T,U)\mathrm{d}L_s - \int_0^t \theta_s\big(\overline{\Sigma}(s,T,U)\big)\mathrm{d}s\right).
\end{aligned}$$

Remark 4.1. Notice that using (**B1**), we have that $\int_0^{\cdot} \Sigma(s,T)\mathrm{d}L_s$ is well defined. Moreover, from (**B1**) and ($\mathbb{EM}$) we get that $\int_0^{\cdot} \Sigma(s,T)\mathrm{d}L_s$ is *exponentially special* (cf. Kallsen and Shiryaev 2002, Definition 2.12). Applying Theorem 2.18 in Kallsen and Shiryaev (2002) we have that

$$\left(\exp\left[\int_0^t \overline{\Sigma}(s,T,U)\mathrm{d}L_s - \int_0^t \theta_s\big(\overline{\Sigma}(s,T,U)\big)\mathrm{d}s\right]\right)_{t\in[0,T]}$$

is a martingale and the last equality follows. Alternatively, this follows from Lemma 2.1 and Assumption (**B1**).

Now, rewriting the density process in the "usual" form

$$\begin{aligned}
\eta_t = \exp\Bigg(&\int_0^t \overline{\Sigma}(s,T,U)\sqrt{c_s}\mathrm{d}W_s - \int_0^t \left(\overline{\Sigma}(s,T,U)\right)^2\frac{c_s}{2}\mathrm{d}s\\
&+ \int_0^t\int_{\mathbb{R}} x\overline{\Sigma}(s,T,U)(\mu^L-\nu)(\mathrm{d}s,\mathrm{d}x)\\
&- \int_0^t\int_{\mathbb{R}} \left(e^{x\overline{\Sigma}(s,T,U)} - 1 - x\overline{\Sigma}(s,T,U)\right)\nu(\mathrm{d}s,\mathrm{d}x)\Bigg)
\end{aligned}$$

and using Theorem III.3.24 in Jacod and Shiryaev (2003), we get that the tuple (β, Y) of predictable processes that describes the change of measure is

$$\beta_s = \overline{\Sigma}(s,T,U) \quad \text{and} \quad Y(s,x) = e^{x\overline{\Sigma}(s,T,U)}. \tag{4.2}$$

Proposition 4.2. *The local characteristics of L under $\widetilde{\mathbb{P}}$ are*

$$\begin{cases}
\widetilde{b}_s &= b_s + \beta_s c_s + \int_{\mathbb{R}} x(Y(s,x)-1)\lambda_s(\mathrm{d}x)\\
\widetilde{c}_s &= c_s\\
\widetilde{\lambda}_s(\mathrm{d}x) &= Y(s,x)\lambda_s(\mathrm{d}x).
\end{cases} \tag{4.3}$$

Proof. This follows directly from Theorem III.3.24 in Jacod and Shiryaev (2003) and the tuple (β, Y). □

We denote the value of a call option, with maturity T and strike K, on a bond with maturity U, by

$$V_c\left(B(0,T);\frac{B(0,U)}{B(0,T)},K;C,\nu\right)=\mathbb{E}\left[\frac{1}{B_T^M}(B(T,U)-K)^+\right],$$

where $B(0,T)$ is the discount factor associated with the option's maturity date T and $B(0,U)/B(0,T)$ is the initial value of the forward price process $B(\cdot,U)/B(\cdot,T)$. The dynamics of B_T^M and $B(T,U)$ are given by equations (3.4) and (3.3) respectively and the drift terms A are determined by the two characteristics of the driving process (C,ν) and the volatility structures Σ, according to equation (3.2). Similar notation is used for the put option on a zero coupon bond.

Theorem 4.3. *Assume that bond prices are modeled according to the Lévy HJM model. Then, we can relate the value of a call and a put option on a bond via the following symmetry:*

$$V_c\left(B(0,T);\frac{B(0,U)}{B(0,T)},K;C,\nu\right)=V_p\left(B(0,T);K,\frac{B(0,U)}{B(0,T)};C,-f\nu\right)$$

where $f(s,x)=\exp\big((\Sigma(s,U)+\Sigma(s,T))x\big)$.

Proof. The price of a call option with maturity T and strike K, on a bond with maturity U, is given by

$$\begin{aligned}V_c&=\mathbb{E}\left[\frac{1}{B_T^M}(B(T,U)-K)^+\right]\\&=\mathbb{E}\left[\frac{KB(T,U)}{B_T^M}(K^{-1}-B(T,U)^{-1})^+\right]\\&=\mathbb{E}\left[\frac{B(T,U)}{D\left(B_T^M\right)^2}KDB_T^M(K^{-1}-B(T,U)^{-1})^+\right]\end{aligned}$$

and changing measure from $\mathbb{P}$ to $\widetilde{\mathbb{P}}$, we get that

$$V_c=\widetilde{\mathbb{E}}\left[KDB_T^M(K^{-1}-B(T,U)^{-1})^+\right].$$

This can be re-written as

$$\begin{aligned}V_c&=\widetilde{\mathbb{E}}\left[\frac{B(0,T)}{B(0,U)}DB_T^M\left(\frac{B(0,U)}{B(0,T)}-K\frac{B(0,U)}{B(0,T)}B(T,U)^{-1}\right)^+\right]\\&=\widetilde{\mathbb{E}}\left[\frac{1}{\widehat{B}_T^M}\left(\widehat{K}-\widehat{B}(T,U)\right)^+\right],\end{aligned}\tag{4.4}$$

for $(\widehat{B}_T^M)^{-1}:=\frac{B(0,T)}{B(0,U)}DB_T^M$, $\widehat{K}:=\frac{B(0,U)}{B(0,T)}$ and $\widehat{B}(T,U):=K\frac{B(0,U)}{B(0,T)}B(T,U)^{-1}$.

Firstly, we will calculate the dynamics of $\widehat{B}(T,U)$. We have

$$\widehat{B}(T,U)=K\exp\left(\int_0^T\big(\Sigma(s,T)-\Sigma(s,U)\big)\mathrm{d}L_s+\int_0^T\big(A(s,U)-A(s,T)\big)\mathrm{d}s\right).$$

Keeping in mind that the local characteristics of L under $\widetilde{\mathbb{P}}$ are given by Proposition 4.2, we define the time-inhomogeneous Lévy process $\widetilde{L}:=-L$.

The local characteristics of $\widetilde{L}$, using Lemma 2.2, are $(-\widetilde{b}, c, -\widetilde{\lambda})$. Then we get

$$\widehat{B}(T,U) = K \exp\left(\int_0^T \big(\Sigma(s,U) - \Sigma(s,T)\big) \mathrm{d}\widetilde{L}_s + \int_0^T \big(A(s,U) - A(s,T)\big) \mathrm{d}s \right).$$

Secondly, for the deterministic terms we have

$$\begin{aligned}
\exp\left(\int_0^T A(s,U) \mathrm{d}s \right) &= \mathbb{E}\left[\exp\Big(\int_0^T \Sigma(s,U) \mathrm{d}L_s \Big) \right] \\
&= \widetilde{\mathbb{E}}\left[\frac{D\left(B_T^M\right)^2}{B(T,U)} \exp\Big(\int_0^T \Sigma(s,U) \mathrm{d}L_s \Big) \right] \\
&= \widetilde{\mathbb{E}}\Bigg[\exp\Big(\int_0^T -\overline{\Sigma}(s,T,U) \mathrm{d}L_s \Big) \\
&\qquad \times \exp\Big(\int_0^T \Sigma(s,U) \mathrm{d}L_s \Big) \exp\Big(\int_0^T \theta_s\big(\overline{\Sigma}(s,T,U)\big) \mathrm{d}s \Big) \Bigg] \\
&= \widetilde{\mathbb{E}}\left[\exp\Big(\int_0^T -\Sigma(s,T) \mathrm{d}L_s \Big) \right] \exp\Big(\int_0^T \theta_s\big(\overline{\Sigma}(s,T,U)\big) \mathrm{d}s \Big) \\
&= \widetilde{\mathbb{E}}\left[\exp\Big(\int_0^T \Sigma(s,T) \mathrm{d}\widetilde{L}_s \Big) \right] \exp\Big(\int_0^T \theta_s\big(\overline{\Sigma}(s,T,U)\big) \mathrm{d}s \Big) \\
&= \exp\Big(\int_0^T \widetilde{A}(s,T) \mathrm{d}s \Big) \exp\Big(\int_0^T \theta_s\big(\overline{\Sigma}(s,T,U)\big) \mathrm{d}s \Big), \qquad (4.5)
\end{aligned}$$

where $\widetilde{A}(s,T) := \widetilde{\theta}_s(\Sigma(s,T))$ and $\widetilde{\theta}_s$ is the cumulant associated with the Lévy triplet $(-\widetilde{b}_s, c_s, -\widetilde{\lambda}_s)$. Similarly, for the other term we have

$$\begin{aligned}
\exp\left(\int_0^T A(s,T) \mathrm{d}s \right) &= \mathbb{E}\left[\exp\Big(\int_0^T \Sigma(s,T) \mathrm{d}L_s \Big) \right] \\
&= \exp\left(\int_0^T \widetilde{A}(s,U) \mathrm{d}s \right) \exp\left(\int_0^T \theta_s\big(\overline{\Sigma}(s,T,U)\big) \mathrm{d}s \right).
\end{aligned}$$

Therefore, the $\widetilde{\mathbb{P}}$-dynamics of $\widehat{B}(T,U)$ is

$$\widehat{B}(T,U) = K \exp\left(\int_0^T \big(\Sigma(s,U) - \Sigma(s,T)\big) \mathrm{d}\widetilde{L}_s + \int_0^T \big(\widetilde{A}(s,T) - \widetilde{A}(s,U)\big) \mathrm{d}s \right). \qquad (4.6)$$

Finally, for the term corresponding to the money-market account, we have

$$\begin{aligned}
\frac{1}{\widehat{B}_T^M} &= \frac{B(0,T)}{B(0,U)} B(0,U)B(0,T)\exp\left(\int_0^T \theta_s\big(\overline{\Sigma}(s,T,U)\big)\mathrm{d}s\right) \\
&\quad \times \exp\left(\int_0^T -\Big(\theta_s(\Sigma(s,U)) + \theta_s(\Sigma(s,T))\Big)\mathrm{d}s\right) \\
&\quad \times \frac{1}{B(0,T)}\exp\left(\int_0^T \theta_s(\Sigma(s,T))\mathrm{d}s - \int_0^T \Sigma(s,T)\mathrm{d}L_s\right) \\
&= B(0,T)\exp\left(\int_0^T \Sigma(s,T)\mathrm{d}\widetilde{L}_s\right)\exp\left(\int_0^T -A(s,U)\mathrm{d}s\right) \\
&\quad \times \exp\left(\int_0^T \theta_s\big(\overline{\Sigma}(s,T,U)\big)\mathrm{d}s\right)
\end{aligned}$$

and using equation (4.5), we get

$$\begin{aligned}
\frac{1}{\widehat{B}_T^M} &= B(0,T)\exp\left(\int_0^T \Sigma(s,T)\mathrm{d}\widetilde{L}_s\right)\exp\left(\int_0^T -\theta_s\big(\overline{\Sigma}(s,T,U)\big)\mathrm{d}s\right) \\
&\quad \times \exp\left(\int_0^T \theta_s\big(\overline{\Sigma}(s,T,U)\big)\mathrm{d}s\right)\exp\left(\int_0^T -\widetilde{A}(s,T)\mathrm{d}s\right) \\
&= B(0,T)\exp\left(\int_0^T \Sigma(s,T)\mathrm{d}\widetilde{L}_s - \int_0^T \widetilde{A}(s,T)\mathrm{d}s\right). \qquad (4.7)
\end{aligned}$$

In view of equations (4.4), (4.6) and (4.7), the desired result is proved. □

Remark 4.4. Notice that the change of measure from $\mathbb{P}$ to $\widetilde{\mathbb{P}}$ is not "structure-preserving" for time-homogeneous processes, e.g. Lévy processes. Therefore, even if we had modeled bond prices as exponentials of Lévy processes under $\mathbb{P}$, the process driving the bond prices under $\widetilde{\mathbb{P}}$ would have been a *time-inhomogeneous* Lévy process; the driving process would remain time-homogeneous only if the jump part vanished or in some pathetic cases (e.g. $\Sigma(\cdot,T) \equiv 0, \forall T \in [0,T^*]$). This is obvious from the structure of the function f in Theorem 4.3. A similar phenomenon does not occur when modeling equities with Lévy processes (compare e.g. Eberlein and Papantoleon 2005, Theorem 4.1).

Expressing the payoff of a caplet (resp. floorlet) as a put (resp. call) option on a zero coupon bond, cf. Appendix A, we get a symmetry directly relating the values of caplets and floorlets in the Lévy HJM model.

We denote the value of a floorlet with strike K maturing at time T_i that settles in arrears at T_{i+1}, by

$$\begin{aligned}V_{fl}\left(B(0,T_i);L(0,T_i),K;C,\nu\right) &= \mathbb{E}\left[\frac{1}{B^M_{T_{i+1}}}\delta\Big(K-L(T_i,T_i)\Big)^+\right]\\ &= (1+\delta K)\mathbb{E}\left[\frac{1}{B^M_{T_i}}\Big(B(T_i,T_{i+1})-\mathcal{K}\Big)^+\right],\end{aligned}$$

where $L(0,T_i)=\frac{1}{\delta}\big(\frac{B(0,T_i)}{B(0,T_{i+1})}-1\big)$ is the initial value of the forward LIBOR rate and the strike $\mathcal{K}:=1/(1+\delta K)$. Similar notation is used for a caplet.

Corollary 4.5. *Assume that bond prices are modeled according to the Lévy HJM model. Then, we can relate the value of a caplet and a floorlet via the following symmetry:*

$$V_{fl}\Big(B(0,T_i);L(0,T_i),K;C,\nu\Big)=\mathcal{C}V_{cl}\Big(B(0,T_i);K,L(0,T_i);C,-f\nu\Big)$$

where $\mathcal{C}:=\frac{1+\delta K}{1+\delta L(0,T_i)}$ *and* $f(s,x)=\exp\big((\Sigma(s,T_i)+\Sigma(s,T_{i+1}))x\big)$.

Proof. We simply use the result of Appendix A to express a floorlet as a call option on a zero coupon bond, then apply Theorem 4.3 and then the formula of Appendix A in the other direction, to express a put option on a zero coupon bond as a caplet. We get

$$\begin{aligned}V_{fl}\Big(B(0,T_i);L(0,T_i),K;C,\nu\Big) &= \mathcal{K}^{-1}V_c\Big(B(0,T_i);\frac{B(0,T_{i+1})}{B(0,T_i)},\mathcal{K};C,\nu\Big)\\ &= \mathcal{K}^{-1}V_p\Big(B(0,T_i);\mathcal{K},\frac{B(0,T_{i+1})}{B(0,T_i)};C,-f\nu\Big)\\ &= \mathcal{C}V_{cl}\Big(B(0,T_i);K,L(0,T_i);C,-f\nu\Big).\end{aligned}$$

□

5. Symmetry in the Lévy LIBOR model

We prove a symmetry relating the value of a caplet and a floorlet in the Lévy LIBOR model where, in addition, we employ the approximation of the random terms

$$\frac{\delta L(t-,T_j^*)}{1+\delta L(t-,T_j^*)}$$

by the deterministic initial values

$$\frac{\delta L(0,T_j^*)}{1+\delta L(0,T_j^*)}$$

proposed in Eberlein and Özkan (2005, pp. 342) (see also Schlögl 2002). In other words, we prove a caplet-floorlet symmetry if the process driving the LIBOR rate under its respective forward measure is a time-inhomogeneous Lévy process (see also Remark 3.3).

More specifically, we will approximate the *random* compensator $\nu^{T^*_{j-1}}$ of μ^L under the forward martingale measure $\mathbb{P}_{T^*_{j-1}}$ by the *non-random* compensator denoted $\nu^{\alpha,T^*_{j-1}}$, where

$$\nu^{\alpha,T^*_{j-1}}(\mathrm{d}s,\mathrm{d}x) := \prod_{k=1}^{j-1} \beta^{\alpha}(s,x,T^*_k,T^*_{k-1})\nu^{T^*}(\mathrm{d}s,\mathrm{d}x), \tag{5.1}$$

where

$$\beta^{\alpha}(s,x,T^*_k,T^*_{k-1}) := \frac{\delta L(0,T^*_k)}{1+\delta L(0,T^*_k)}\Big(e^{\lambda(s,T^*_k)x}-1\Big)+1; \tag{5.2}$$

compare with equations (3.7) and (3.8). Note that we do not apply any approximation for the continuous martingale part of the driving process.

Therefore, the approximation of the random compensator by the non-random one means that the process $L^{T^*_{j-1}}$, driving the LIBOR rate $L(\cdot,T^*_j)$ under the forward martingale measure $\mathbb{P}_{T^*_{j-1}}$, will be approximated by the time-inhomogeneous Lévy process $L^{\alpha,T^*_{j-1}}$ with triplet $(0,C,\nu^{\alpha,T^*_{j-1}})$, and the drift term $b^{\alpha,L}(\cdot,T^*_j,T^*_{j-1})$ is given by (3.9) for the non-random compensator $\nu^{\alpha,T^*_{j-1}}(\mathrm{d}s,\mathrm{d}x) =: \lambda_s^{\alpha,T^*_{j-1}}(\mathrm{d}x)\mathrm{d}s$.

Remark 5.1. The caplet-floorlet symmetry in this framework is a symmetry between two models that *approximate* the Lévy LIBOR model. This naturally implies an *approximate symmetry* between caplets and floorlets in the Lévy LIBOR model.

Remark 5.2. Notice that the caplet-floorlet symmetry is *exact* in the log-normal LIBOR model, i.e. when the jump component vanishes. It is also *exact* under the terminal martingale measure $\mathbb{P}_{T^*}$, because the driving process is – by assumption – a time-inhomogeneous Lévy process.

The payoff of a caplet with strike K, that is settled in arrears at time T^*_{j-1}, is $\delta(L(T^*_j,T^*_j)-K)^+$ and similarly, the payoff of a floorlet is $\delta(K-L(T^*_j,T^*_j))^+$.

Assuming that LIBOR rates are modeled according to the approximate Lévy LIBOR model, we denote the value of a caplet with strike K, by

$$V_c\left(L(0,T^*_j),K;C,\nu^{\alpha,T^*_{j-1}}\right) = B(0,T^*_{j-1})\,\mathbb{E}_{\mathbb{P}_{T^*_{j-1}}}\big[\delta(L(T^*_j,T^*_j)-K)^+\big]\,,$$

where $L(0,T^*_j)$ is the initial value of the (approximate) forward LIBOR process. Notice that the drift term is determined by the other two characteristics of the driving process $(C,\nu^{\alpha,T^*_{j-1}})$ and the volatility structure $\lambda(\cdot,T^*_j)$, according to equation (3.9). Moreover, the discount factor $B(0,T^*_{j-1})$ corresponds to the settlement date T^*_{j-1}. Similar notation is used for a floorlet.

Theorem 5.3. *Let the LIBOR rate be modeled according to the Lévy LIBOR model, using the approximation described above. We can relate the values of caplets and floorlets via the following symmetry*

$$V_c\left(L(0,T^*_j),K;C,\nu^{\alpha,T^*_{j-1}}\right) = V_f\left(K,L(0,T^*_j);C,-f\nu^{\alpha,T^*_{j-1}}\right)$$

where $f(s,x)=\exp(\lambda(s,T^*_j)x)$.

Proof. From the time-T_0 value of a caplet settled at time T_{j-1}^*, we get

$$\begin{aligned}
V_c &= B(0,T_{j-1}^*)\, \mathrm{E}_{\mathrm{P}_{T_{j-1}^*}}\big[\delta(L(T_j^*,T_j^*)-K)^+\big] \\
&= B(0,T_{j-1}^*)\, \mathrm{E}_{\mathrm{P}_{T_{j-1}^*}}\big[\delta K L(T_j^*,T_j^*)(K^{-1}-L(T_j^*,T_j^*)^{-1})^+\big] \\
&= B(0,T_{j-1}^*) K L(0,T_j^*) \\
&\quad \times \mathrm{E}_{\mathrm{P}_{T_{j-1}^*}}\left[\frac{L(T_j^*,T_j^*)}{L(0,T_j^*)}\delta(K^{-1}-L(T_j^*,T_j^*)^{-1})^+\right]. \qquad (5.3)
\end{aligned}$$

Define the measure $\widetilde{\mathrm{P}}_{T_{j-1}^*}$ via its Radon–Nikodym derivative

$$\frac{\mathrm{d}\widetilde{\mathrm{P}}_{T_{j-1}^*}}{\mathrm{d}\mathrm{P}_{T_{j-1}^*}} = \frac{L(T_j^*,T_j^*)}{L(0,T_j^*)} = \eta \qquad (5.4)$$

and the valuation problem (5.3), reduces to

$$V_c = B(0,T_{j-1}^*) K L(0,T_j^*)\, \mathrm{E}_{\widetilde{\mathrm{P}}_{T_{j-1}^*}}\big[\delta(K^{-1}-L(T_j^*,T_j^*)^{-1})^+\big]. \qquad (5.5)$$

The density process is given by the restriction of the Radon–Nikodym derivative to the σ-field $\mathcal{F}_t$, and because the LIBOR rate process is – by construction – a $\mathrm{P}_{T_{j-1}^*}$ martingale, we get

$$\begin{aligned}
\eta_t &= \mathrm{E}_{\mathrm{P}_{T_{j-1}^*}}\left[\frac{\mathrm{d}\widetilde{\mathrm{P}}_{T_{j-1}^*}}{\mathrm{d}\mathrm{P}_{T_{j-1}^*}}\bigg|\mathcal{F}_t\right] = \frac{L(t,T_j^*)}{L(0,T_j^*)} \\
&= \exp\Bigg(\int_0^t \lambda(s,T_j^*)c_s^{1/2}\mathrm{d}W_s^{T_{j-1}^*} - \frac{1}{2}\int_0^t \big(\lambda(s,T_j^*)\big)^2 c_s \mathrm{d}s \\
&\qquad + \int_0^t\int_{\mathbb{R}} x\lambda(s,T_j^*)(\mu^L-\nu^{\alpha,T_{j-1}^*})(\mathrm{d}s,\mathrm{d}x) \\
&\qquad - \int_0^t\int_{\mathbb{R}} \big(e^{\lambda(s,T_j^*)x}-1-\lambda(s,T_j^*)x\big)\nu^{\alpha,T_{j-1}^*}(\mathrm{d}s,\mathrm{d}x)\Bigg). \qquad (5.6)
\end{aligned}$$

Using Girsanov's theorem for semimartingales, cf. Jacod and Shiryaev (2003, Theorem III.3.24), it follows that the tuple of predictable processes which describes the change of measure is

$$\beta_s = \lambda(s,T_j^*) \quad \text{and} \quad Y(s,x) = e^{x\lambda(s,T_j^*)}. \qquad (5.7)$$

Additionally, we immediately recognize $\widetilde{W}_t^{T_{j-1}^*} = W_t^{T_{j-1}^*} - \int_0^t \lambda(s,T_j^*)c_s^{1/2}\mathrm{d}s$ as a $\widetilde{\mathrm{P}}_{T_{j-1}^*}$-Brownian motion and $\widetilde{\nu}^{\alpha,T_{j-1}^*}(\mathrm{d}t,\mathrm{d}x) = e^{x\lambda(t,T_j^*)}\nu^{\alpha,T_{j-1}^*}(\mathrm{d}t,\mathrm{d}x)$ as the (approximate) $\widetilde{\mathrm{P}}_{T_{j-1}^*}$-compensator of μ^L. Hence, the $\widetilde{\mathrm{P}}_{T_{j-1}^*}$ local characteristics of $L^{\alpha,T^*_{j-1}}$ are

$$\begin{cases} \widetilde{b}_s^{\alpha,T^*_{j-1}} &= \beta_s c_s + \int_{\mathbb{R}} x(Y(s,x)-1)\lambda_s^{\alpha,T^*_{j-1}}(\mathrm{d}x) \\ \widetilde{c}_s^{T^*_{j-1}} &= c_s \\ \widetilde{\lambda}_s^{\alpha,T^*_{j-1}}(\mathrm{d}x) &= Y(s,x)\lambda_s^{\alpha,T^*_{j-1}}(\mathrm{d}x) \end{cases} \tag{5.8}$$

and the canonical decomposition of $L^{\alpha,T^*_{j-1}}$ under $\widetilde{\mathbb{P}}_{T^*_{j-1}}$ is given by (2.5).

Let $L^{\mathcal{M},\alpha,T^*_{j-1}}$ be the martingale part of $L^{\alpha,T^*_{j-1}}$, i.e. $L^{\mathcal{M},\alpha,T^*_{j-1}}$ is a time-inhomogeneous Lévy process with local characteristics $(0,c,\widetilde{\lambda}^{\alpha,T^*_{j-1}})$.

Now, the dynamics of $L(t,T^*_j)^{-1}$ under $\widetilde{\mathbb{P}}_{T^*_{j-1}}$ is

$$\begin{aligned} L(t,T^*_j)^{-1} &= L(0,T^*_j)^{-1}\exp\left(-\int_0^t b^{\alpha,L}(s,T^*_j,T^*_{j-1})\mathrm{d}s - \int_0^t \lambda(s,T^*_j)\mathrm{d}L_s^{\alpha,T^*_{j-1}}\right) \\ &= L(0,T^*_j)^{-1}\exp\left(\int_0^t \widetilde{b}^{\alpha,L}(s,T^*_j,T^*_{j-1})\mathrm{d}s + \int_0^t \lambda(s,T^*_j)\mathrm{d}\widetilde{L}_s^{\alpha,T^*_{j-1}}\right) \\ &=: \widetilde{L}(t,T^*_j), \end{aligned} \tag{5.9}$$

where $\widetilde{L}^{\alpha,T^*_{j-1}} := -L^{\mathcal{M},\alpha,T^*_{j-1}}$ is the dual process of $L^{\mathcal{M},\alpha,T^*_{j-1}}$ and its triplet of local characteristics, using Lemma 2.2, is $(0,c,-\widetilde{\lambda}^{\alpha,T^*_{j-1}})$. Furthermore, we define the drift term $\widetilde{b}^{\alpha,L}(s,T^*_j,T^*_{j-1}) := -b^{\alpha,L}(s,T^*_j,T^*_{j-1}) - \lambda(s,T^*_j)\widetilde{b}_s^{\alpha,T^*_{j-1}}$.

The following simple calculation shows that the drift term $\widetilde{b}^{\alpha,L}(s,T^*_j,T^*_{j-1})$ corresponding to $\widetilde{L}(t,T^*_j)$, is of the same form as in (3.9). Keep in mind that $-\widetilde{\lambda}^{\alpha,T^*_{j-1}}$ is the mirror image of the Lévy measure $\widetilde{\lambda}^{\alpha,T^*_{j-1}}$.

$$\begin{aligned} \widetilde{b}^{\alpha,L}(s,T^*_j,T^*_{j-1}) &\overset{(3.9)}{\underset{(5.8)}{=}} -\frac{1}{2}\big(\lambda(s,T^*_j)\big)^2 c_s \\ &\qquad + \int_{\mathbb{R}} \left(e^{\lambda(s,T^*_j)x} - 1 - x\lambda(s,T^*_j)e^{\lambda(s,T^*_j)x}\right)\lambda_s^{\alpha,T^*_{j-1}}(\mathrm{d}x) \\ &\overset{(5.8)}{=} -\frac{1}{2}\big(\lambda(s,T^*_j)\big)^2 c_s \\ &\qquad - \int_{\mathbb{R}} \left(e^{-\lambda(s,T^*_j)x} - 1 + x\lambda(s,T^*_j)\right)\widetilde{\lambda}_s^{\alpha,T^*_{j-1}}(\mathrm{d}x). \end{aligned} \tag{5.10}$$

This concludes the proof, since

$$\begin{aligned} V_c &= B(0,T^*_{j-1})KL(0,T^*_j)\,\mathbb{E}_{\widetilde{\mathbb{P}}_{T^*_{j-1}}}\Big[\delta\big(K^{-1} - L(T^*_j,T^*_j)^{-1}\big)^+\Big] \\ &= B(0,T^*_{j-1})\,\mathbb{E}_{\widetilde{\mathbb{P}}_{T^*_{j-1}}}\Big[\delta\big(L(0,T^*_j) - \widehat{L}(T^*_j,T^*_j)\big)^+\Big], \end{aligned}$$

where $\widehat{L}(T^*_j,T^*_j) := KL(0,T^*_j)\widetilde{L}(T^*_j,T^*_j)$ and noting that the dynamics of $\widetilde{L}(\cdot,T^*_j)$ is given by (5.9) and (5.10) . □

6. Symmetry in the Lévy forward price model

In this section, we state a symmetry between call and put options on the forward price. Since a call option on the forward is equivalent to a caplet, see (A.1), this result can also be viewed as a symmetry between caplets and floorlets in the forward price model.

We denote the time-T_0 value of a call option on the forward price with strike $\mathcal{K}$, which is settled in arrears at time T^*_{j-1}, by

$$V_c\left(F^0_{T^*_j}, \mathcal{K}; C, \nu^{T^*_{j-1}}\right) = B(0, T^*_{j-1})\, \mathbb{E}_{\mathbf{P}_{T^*_{j-1}}}\left[(F_B(T^*_j, T^*_j, T^*_{j-1}) - \mathcal{K})^+\right]$$

where $F^0_{T^*_j} := F_B(0, T^*_j, T^*_{j-1})$. Note that the drift characteristic of the driving process is determined by the other two characteristics $(C, \nu^{T^*_{j-1}})$ and the volatility structure $\lambda(\cdot, T^*_j)$, using equation (3.10). Similar notation will be used for a put option on the forward price.

Theorem 6.1. *Assume that the forward process is modeled according to the Lévy forward price model. Then, we can relate the values of call and put options on the forward price via the following symmetry:*

$$V_c\left(F^0_{T^*_j}, \mathcal{K}; C, \nu^{T^*_{j-1}}\right) = V_p\left(\mathcal{K}, F^0_{T^*_j}; C, -f\nu^{T^*_{j-1}}\right)$$

where $f(s,x) = \exp(\lambda(s, T^*_j)x)$.

Proof. The proof is similar to that of Theorem 5.3 and therefore omitted. □

Appendix A. Transformations

We use the well-known relationships between the LIBOR, the forward and the bond price, to transform a caplet into a call option on the forward price or a put option on a bond. Similarly, a floorlet is transformed into a put option on the forward price or a call option on a bond.

Let $T_0 < T_1 < T_2 < \cdots < T_N < T_{N+1} = T^*$ denote a discrete tenor structure where $\delta = T_{i+1} - T_i$, $i = 0, 1, \ldots, N$. The time-T_{i+1} payoff of a caplet settled in arrears at time T_{i+1}, is

$$N\delta(L(T_i, T_i) - K)^+$$

where K is the strike rate and N is the notional amount.

Now, using the relationship between the LIBOR and the forward price, i.e. $F_B(T_i, T_i, T_{i+1}) = 1 + \delta L(T_i, T_i)$, we can rewrite the payoff of a caplet as a call option on the forward price. We have

$$\begin{aligned} N\delta(L(T_i, T_i) - K)^+ &= N\delta\left(\frac{F_B(T_i, T_i, T_{i+1}) - 1}{\delta} - K\right)^+ \\ &= N(F_B(T_i, T_i, T_{i+1}) - \mathcal{K})^+, \end{aligned} \tag{A.1}$$

where $\mathcal{K} = 1 + \delta K$.

Moreover, the payoff $N(F_B(T_i, T_i, T_{i+1}) - \mathcal{K})^+$ settled at time T_{i+1} is equal to the payoff $NB(T_i, T_{i+1})(F_B(T_i, T_i, T_{i+1}) - \mathcal{K})^+$, settled at time T_i. Using the relationship between forward and bond prices, i.e. $F_B(T_i, T_i, T_{i+1}) =$

$B(T_i, T_i)/B(T_i, T_{i+1})$, we have

$$\begin{aligned} NB(T_i, T_{i+1})(F_B(T_i, T_i, T_{i+1}) - \mathcal{K})^+ &= NB(T_i, T_{i+1}) \left(\frac{B(T_i, T_i)}{B(T_i, T_{i+1})} - \mathcal{K} \right)^+ \\ &= N(1 - \mathcal{K} B(T_i, T_{i+1}))^+ \\ &= \mathcal{N}(\mathcal{K} - B(T_i, T_{i+1}))^+, \end{aligned} \tag{A.2}$$

where $\mathcal{K} = \mathcal{K}^{-1}$ and $\mathcal{N} = N\mathcal{K}$.

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