Random metric measure spaces

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Tree-like metric spaces

- In population models, relationships between individuals are modelled with trees
- **Coalescent** trees: Genealogies of a constant size population
- Brownian **Continuum Random Tree**: limit object of a critical branching process
- Main question: what is a good way to **encode** trees? Especially in the case of infinite populations?
Tree-like metric spaces

- Example: Kingman coalescent:

- Coalescent trees are ultrametric

\[ r(u, v) \vee r(v, w) \geq r(u, w) \]
Tree-like metric spaces

- A tree is a metric space \((X, r)\):
  
  \[ r(u, v) = 2 \cdot \text{time to the common ancestor of } u, v \]

- In order to be able to 'pick' individuals from a population, consider a probability measure on \(X\).

- **Metric measure space**: \(X = (X, r, \mu)\) where \(r(., .)\): metric on \(X\), \(\mu\): probability measure on \(X\).

- **\(M\)**: the space of (isometry classes) of complete and separable metric measure spaces
Questions

- What does convergence of metric measure spaces mean?

- Is there a characterization of weak convergence of random metric measure spaces?
Philosophy

- Kolmogoroff, Aldous, ...
- $\mathcal{X}, \mathcal{X}_1, \mathcal{X}_2, \ldots$

\[ \mathcal{X}_n \xrightarrow{n \to \infty} \mathcal{X} \]

if all finite dimensional distributions converge

- Here:

Finite dimensional distributions = finite sampled subspaces
Polynomials

For $\phi : \mathbb{R}^2(n) \to \mathbb{R}$,

$$\Phi((X, r, \mu)) := \int \mu \otimes n(dx_1, ..., dx_n)\phi((r(x_i, x_j))_{1 \leq i < j \leq n})$$

polynomial of degree $n$

Examples: length, diameter of sample-subtree of $n$ points

Important fact: polynomials separate points

Gromov-weak topology: $\mathcal{X}_n \xrightarrow{n \to \infty} \mathcal{X}$ iff

$$\Phi(\mathcal{X}_n) \xrightarrow{n \to \infty} \Phi(\mathcal{X})$$ for all polynomials $\Phi$
Random Distance Distribution

Let $\mathcal{X} = (X, r, \mu)$. Every $x \in X$ defines the distribution of distances $\mu_x := r(x, .) \ast \mu$. Call

$$\hat{\mu} := (\mu_.) \ast \mu \in \mathcal{M}_1(\mathcal{M}_1(\mathbb{R}^+_+))$$

the random distance distribution of $\mathcal{X}$.

The distance distribution does not characterize $\mathcal{X}$. 

![Diagram showing examples of random distance distributions](attachment:diagram.png)

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Distance distribution and modulus of mass distribution

Let $\mathcal{X} = (X, r, \mu) \in \mathcal{M}$. 

- What do distances of two typical points in $\mathcal{X}$ look like?

\[ w_{\mathcal{X}} := r_*\mu \otimes^2, \text{ i.e. } w_{\mathcal{X}}(\cdot) := \mu \otimes^2 \{(x, x') : r(x, x') \in \cdot\} \]

- Which mass do thin points have?

\[ v_\delta(\mathcal{X}) := \inf \left\{ \varepsilon > 0 : \mu\{x \in X : \mu(B_\varepsilon(x)) \leq \delta\} \leq \varepsilon \right\}, \quad \delta > 0 \]
Distance distribution and modulus of mass distribution

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$$w_{\mathcal{X}} = \int \hat{\mu}(d\nu)\nu$$

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$$= \inf \left\{ \varepsilon > 0 : \hat{\mu}_{\mathcal{X}}\{\nu \in \mathcal{M}_1([0, \infty)) : \nu([0, \varepsilon]) \leq \delta \} \leq \varepsilon \right\}.$$
Polish

- **Theorem**: [Greven, P, Winter] The space $\mathbb{M}$, equipped with the Gromov-weak topology, is Polish.

- So, $\mathbb{M}$ is accessible to the notion of weak convergence.
Pre-compact sets

Theorem: [Greven, P, Winter] A sequence $\mathcal{X}_1, \mathcal{X}_2, \ldots$ is pre-compact iff:

- Distances do not explode:

  The family $\{w_{\mathcal{X}_1}, w_{\mathcal{X}_2}, w_{\mathcal{X}_3}, \ldots\}$ is tight

  and

- Thin points are uniformly rare:

  $$\lim_{\delta \to 0} \limsup_{n \to \infty} v_\delta(\mathcal{X}_n) = 0$$

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Counterexample 1

Indeed,

\[ \omega x_n = \frac{1}{2} \delta_0 + \frac{1}{2} \delta_n \text{ is not tight} \]
Counterexample II

Indeed,

\[ \nu_\delta(\mathcal{X}_n) = \begin{cases} 
0, & \text{for } 2^{-n} > \delta, \\
1, & \text{for } 2^{-n} \leq \delta, \text{ i.e. } n \geq \log_2(1/\delta), 
\end{cases} \]
Example III

- Restricted **doubling property** with doubling constant $C$:

  \[
  \mu_X(B_{2\varepsilon}(x)) \leq C \cdot \mu_X(B_{\varepsilon}(x)) \quad (x \in X, \varepsilon > 0)
  \]

- **Proposition:** Let $\mathcal{X}_1, \mathcal{X}_2, \ldots$ have the restricted doubling property with doubling constant $C$. The sequence is pre-compact iff $\{w_{\mathcal{X}_1}, w_{\mathcal{X}_2}, \ldots\}$ is tight.

- Let $\text{diam}(\mathcal{X}_n)$ be bounded. For $\varepsilon > 0$ and some large $N = N_\varepsilon$

  \[
  \mu(B_{\varepsilon}(x)) \geq \frac{1}{C} \mu(B_{2\varepsilon}(x)) \geq \ldots \geq \frac{1}{CN} \mu(B_{2N\varepsilon}(x)) = \frac{1}{CN}
  \]

  Set $\delta := \frac{1}{CN}$

  \[
  \mu\{x : \mu(B_{\varepsilon}(x)) > \delta\} = 1 > 1 - \varepsilon \quad \Rightarrow \quad \nu_\delta(\mathcal{X}_n) < \varepsilon
  \]
Random metric measure spaces

**Proposition:** A sequence $\mathbb{P}_1, \mathbb{P}_2, \ldots$ of distributions on $\mathbb{M}$ converges weakly iff:

- The sequence $\mathbb{P}_1, \mathbb{P}_2, \ldots$ is **tight**
- and
- All polynomials converge, i.e.

$$\mathbb{E}_1[\Phi], \mathbb{E}_2[\Phi], \ldots \text{ converges in } \mathbb{R}$$
Theorem: [Greven, P, Winter] A sequence $P_1, P_2, \ldots$ of distributions on $M$ is tight iff

$$(w_\cdot)_* P_1, (w_\cdot)_* P_2, \ldots$$

is tight in $M_1(\mathbb{R})$

and

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{E}_n \left[ \nu_\delta(X) \right] = 0$$
Example: Λ-coalescent measure trees

- Introduced by Pitman, Sagitov
- Start a process $\xi$ in the partition $\{\{1\}, \{2\}, \ldots\}$
- In $\xi$, from any $b$ blocks, $k$ merge at rate

$$\lambda_{b,k} := \int_0^1 \Lambda(dx) x^{k-2}(1-x)^{b-k}$$

for some non-negative, finite measure $\Lambda$ on $[0, 1]$

- $\Lambda = \delta_0$: Kingman coalescent
- Distribution on path space: $\mathbb{P}^\Lambda$. 
Example: $\Lambda$-coalescent measure trees

- Define the (completion of the) metric space

$$r^\xi(i,j) := \inf \{ t \geq 0 : i \sim_\xi(t) j \}.$$ 

- Define the random metric measure spaces

$$\mathbb{P}^{\Lambda,n} := (H_n)_* \mathbb{P}^{\Lambda} \quad \text{for} \quad H_n : \xi \mapsto (\mathbb{N}, r^\xi, \mu_n := \frac{1}{n} \sum_{i=1}^{n} \delta_i)$$
Example: Λ-coalescent measure trees

- **Theorem**: [Greven, P, Winter] The family \( \{P^\Lambda, n; n \in \mathbb{N}\} \) converges if and only if

\[
\int_0^1 \Lambda(dx) x^{-1} = \infty. \quad (*)
\]

- Let \( f = \{f(\pi) : \pi \in \xi(\varepsilon)\} \) be the ranked rearrangement of

\[
\tilde{f}(\pi) := \lim_{n \to \infty} \frac{1}{n} \# \{j \in \{1, \ldots, n\} : j \in \pi\}
\]

- [Pitman '99] \((*)\) is equivalent to the dust-free property

\[
\sum_{i} f(\pi_i) = 1 \iff \mathbb{P}^\Lambda\{\tilde{f}(\xi(\varepsilon)^1) = 0\} = 0
\]
Example: $\Lambda$-coalescent measure trees

- $\nu\chi$: Coalescence rate for any pair is $\lambda_{2,2} > 0$. So, expected time to coalescence bounded.
- $\nu_\delta$: Due to exchangeability

$$\mathbb{P}^{\Lambda,n}\{\nu_\delta(H_n(\xi)) \geq \varepsilon\} = \mathbb{P}^{\Lambda}\{\mu_n(B_{\varepsilon}(1)) \leq \delta\}.$$ 

Hence,

$$\lim_{\delta \to 0} \lim_{n \to \infty} \mathbb{P}^{\Lambda,n}\{\nu_\delta(H_n(\xi)) \geq \varepsilon\} = \mathbb{P}^{\Lambda}\{\tilde{f}((\xi(\varepsilon))^1) = 0\}.$$
Summary

- **Metric measure spaces** are useful in the context of (infinite) genealogical trees
- Metric measure spaces form a 'nice' space
- Pre-compactness results, as well as characterization of weak convergence can be given explicitly