# CONSTRUCTION OF COST-EFFICIENT SELF-QUANTO CALLS AND PUTS IN EXPONENTIAL LÉVY MODELS

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**Abstract**

In this paper we derive explicit representations for cost-efficient puts and calls in financial markets which are driven by a Lévy process and where the pricing of derivatives is based on the Esscher martingale measure. Whereas the construction and evaluation of the efficient self-quanto call is a straightforward application of the general theory, the pricing of an efficient self-quanto put is more involved due to the lack of monotonicity of the standard payoff function. We show how to circumvent these difficulties and arrive at numerically tractable expressions. The potential savings of the cost-efficient strategies are illustrated in market models driven by NIG- and VG-processes using estimated parameters from German stock market data.

## 1. INTRODUCTION

The task of determining cost-efficient strategies is to construct resp. derive a payoff function which provides a predetermined payoff distribution at minimal costs. In other words, a cost-efficient strategy should provide the same chances of gaining or losing money as a given asset or derivative, but has a lower price than the latter one. This problem was first introduced by Dybvig (1988a,b) in the case of a discrete and arbitrage-free binomial model. Bernard and Boyle (2010), Bernard et al. (2014) give a solution of the efficient claim problem in a fairly general setting. They calculate in explicit form efficient strategies for several options in Black–Scholes markets.

In v. Hammerstein et al. (2014), their results are applied to certain classes of exponential Lévy models driven by Variance Gamma and Normal inverse Gaussian distributions. Under the assumption that the Esscher martingale measure is used for risk-neutral pricing, they investigate the impact of the risk-neutral Esscher parameter on the cost-efficient strategies and associated efficiency losses and derive concrete formulas for a variety of efficient options such as puts, calls, forwards, and spreads. Moreover, they consider the problem of hedging and provide explicit formulas for the deltas of cost-efficient calls and puts. Built on these results, we show in this paper how to obtain and price cost-efficient versions of self-quanto calls and puts and illustrate the theoretical results with a practical example using German stock market data.

The paper is structured as follows: Section 2 summarizes some basic definitions and results on cost-efficient payoffs in Lévy models. The self-quanto call and its efficient counterpart are discussed in Section 3, and formulas for the efficient self-quanto put are derived in Section 4. Explicit results based on real data from the German stock market are presented in Section 5, and Section 6 concludes.

## 2. GENERAL SETUP, BASIC NOTATION AND RESULTS

We assume to be given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ satisfying the usual conditions with finite trading horizon $[0,T], T \in \mathbb{R}_+$, on which the risky asset price process $(S_t)_{0\le t\le T}$ is defined and adapted to the filtration. Further, we suppose that there exists a constant risk-free interest rate $r$ and a risk-neutral measure $Q$ with $\frac{dQ}{dP}\big|_{\mathcal{F}_t} = Z_t$. A European style option with terminal payoff $X_T = h(S_T)$ for some payoff function $h$ then has the initial price (or cost)

$$c(X_T) = e^{-rT} E\big[Z_T X_T\big]$$

where we denote here and in the following with $E[\cdot] = E_P[\cdot]$ the expectation w.r.t. $P$.

**Definition 2.1 (Cost-efficient and most-expensive strategies)**

*a) A strategy (or payoff)* $\underline{X}_T \sim G$ *is called* cost-efficient *w.r.t. the payoff-distribution $G$ if any other strategy $X_T$ that generates the same payoff-distribution $G$ costs at least as much, i.e.*

$$c(\underline{X}_T) = e^{-rT} E[Z_T \underline{X}_T] = \min_{\{X_T \sim G\}} e^{-rT} E[Z_T X_T]. \tag{1}$$

*b) A strategy (or payoff)* $\overline{X}_T \sim G$ *is called* most-expensive *w.r.t. the payoff-distribution $G$ if any other strategy $X_T$ that generates the same payoff-distribution $G$ costs at most as much:*

$$c(\overline{X}_T) = e^{-rT} E[Z_T \overline{X}_T] = \max_{\{X_T \sim G\}} e^{-rT} E[Z_T X_T]. \tag{2}$$

*c) The* efficiency loss *of a strategy with payoff $X_T \sim G$ at maturity $T$ is defined as*

$$c(X_T) - c(\underline{X}_T).$$

Since the distribution $F_{Z_T}$of $Z_T$ and the payoff distribution $G$ have to be kept fixed, it can easily be seen that the problem of minimizing the cost is equivalent to finding a strategy $\underline{X}_T \sim G$ such that the covariance $\mathrm{Cov}(\underline{X}_T, Z_T)$ is minimized which can be achieved by constructing $\underline{X}_T$ in such a way that it is countermonotonic to $Z_T$. Analogously, the most-expensive payoff $\overline{X}_T$ has to be chosen comonotonic to $Z_T$. This general result was first obtained in Bernard and Boyle (2010).

To obtain a more explicit representation of cost-efficient resp. most-expensive payoffs, we further suppose that the asset price process $(S_t)_{0\leq t\leq T} = (S_0 e^{L_t})_{0\leq t\leq T}$ is of exponential Lévy type and that the risk-neutral measure $Q$ is the Esscher martingale measure. This approach is widespread and has been well established since the last two decades. Further information on the use of exponential Lévy processes in financial modeling can be found in the books of Schoutens (2003), Cont and Tankov (2004), and Rachev et al. (2011). For a more detailed description of Lévy processes themselves, we refer the reader to the book of Barndorff-Nielsen et al. (2001) and the monographs of Sato (1999), and Applebaum (2009). The Esscher transform of a probability measure has originally been introduced in actuarial sciences by Esscher (1932) and was first suggested as a useful tool for option pricing in the seminal paper of Gerber and Shiu (1994). A more precise analysis of the Esscher transform for exponential Lévy models is given in (Raible 2000, Chapter 1) and Hubalek and Sgarra (2006). For the Esscher martingale measure to be well-defined in our setting, the Lévy process $(L_t)_{t\geq 0}$ has to fulfill the

**Assumption 2.1** *The random variable $L_1$ is nondegenerate and possesses a moment generating function (mgf) $M_{L_1}(u) = E[e^{uL_1}]$ on some open interval $(a, b)$ with $a < 0 < b$ and $b - a > 1$.*

This condition is necessary (but not always sufficient) for the existence of the risk-neutral Esscher measure. Sufficient conditions were first given in (Raible 2000, Proposition 2.8).

**Definition 2.2** *We call an Esscher transform any change of $P$ to a locally equivalent measure $Q^\theta$ with a density process $Z_t^\theta = \frac{dQ^\theta}{dP}|_{\mathcal{F}_t}$ of the form*

$$Z_t^\theta = \frac{e^{\theta L_t}}{M_{L_t}(\theta)}, \tag{3}$$

*where $M_{L_t}$ is the mgf of $L_t$ as before, and $\theta \in (a, b)$.*

It can easily be shown that $(Z_t^\theta)_{t\geq 0}$ indeed is a density process for all $\theta \in (a, b)$, and $(L_t)_{t\geq 0}$ also is a Lévy process under $Q^\theta$ for all these $\theta$ (see, for example, (Raible 2000, Proposition 1.8)). However, there will be at most one parameter $\bar{\theta}$ for which the discounted asset price process $(e^{-rt}S_t)_{t\geq 0}$ is a martingale under the so-called *risk-neutral Esscher measure* or *Esscher martingale measure* $Q^{\bar{\theta}}$. This $\bar{\theta}$ has to solve the equation

$$e^r = \frac{M_{L_1}(\bar{\theta} + 1)}{M_{L_1}(\bar{\theta})}. \tag{4}$$

With these preliminaries, the general results of (Bernard et al. 2014, Proposition 3) can be reformulated in the present framework as follows (see (v. Hammerstein et al. 2014, Proposition 2.1)):

**Proposition 2.1** *Let $(L_t)_{t\geq 0}$ be a Lévy process with continuous distribution function $F_{L_T}$ at maturity $T > 0$, and assume that a solution $\bar{\theta}$ of (4) exists.*

*a) If $\bar{\theta} < 0$, then the cost-efficient payoff $\underline{X}_T$ and the most-expensive payoff $\overline{X}_T$ with distribution function $G$ are a.s. unique and are given by*

$$\underline{X}_T = G^{-1}(F_{L_T}(L_T)) \quad \textit{and} \quad \overline{X}_T = G^{-1}(1 - F_{L_T}(L_T)). \tag{5}$$

*Further, the following bounds for the cost of any strategy with terminal payoff $X_T \sim G$ hold:*

$$c(X_T) \geq E\big[e^{-rT} Z_T^{\bar\theta} \underline{X}_T\big] = \frac{1}{M_{L_T}(\bar\theta)} \int_0^1 e^{\bar\theta F_{L_T}^{-1}(1-y)-rT} G^{-1}(1-y)\, dy,$$

$$c(X_T) \leq E\big[e^{-rT} Z_T^{\bar\theta} \overline{X}_T\big] = \frac{1}{M_{L_T}(\bar\theta)} \int_0^1 e^{\bar\theta F_{L_T}^{-1}(1-y)-rT} G^{-1}(y)\, dy.$$

*b) If $\bar\theta > 0$, then the cost-efficient and the most-expensive payoffs are a.s. unique and given by*

$$\underline{X}_T = G^{-1}(1 - F_{L_T}(L_T)) \quad \textit{and} \quad \overline{X}_T = G^{-1}(F_{L_T}(L_T)). \tag{6}$$

*The bounds in a) hold true with $F_{L_T}^{-1}(1-y)$ replaced by $F_{L_T}^{-1}(y)$.*

From the previous proposition one can easily deduce the following characterization of cost-efficiency in exponential Lévy models where the notions increasing and decreasing have to be understood in the weak sense.

**Corollary 2.2** *Let $(L_t)_{t\geq 0}$ be a Lévy process with continuous distribution $F_{L_T}$ at maturity $T > 0$, and assume that a solution $\bar\theta$ of* (4) *exists.*

*a) If $\bar\theta < 0$, a payoff $X_T \sim G$ is cost-efficient if and only if it is increasing in $L_T$.*

*b) If $\bar\theta > 0$, a payoff $X_T \sim G$ is cost-efficient if and only if it is decreasing in $L_T$.*

*For the most-expensive strategy, the reverse holds true.*

Let us remark that the sign of the risk-neutral Esscher parameter $\bar\theta$ not only plays an essential role for the construction of cost-efficient strategies, but also characterizes the current market scenario. More specifically, a negative $\bar\theta < 0$ corresponds to a bullish market, and in case of $\bar\theta > 0$ we have a bearish market behaviour. A more detailed formulation and proof of this fact can be found in (v. Hammerstein et al. 2014, Proposition 2.2).

For the practical applications in Section 5 we shall consider two specific exponential Lévy models which we shortly describe in the following. Both are based on special sub- resp. limiting classes of the more general family of generalized hyperbolic (GH) distributions which was introduced in Barndorff-Nielsen (1977). A detailed description of uni- and multivariate GH distributions as well as their weak limits is provided in (v. Hammerstein 2011, Chapters 1 and 2).

**Normal inverse Gaussian model.** The Normal inverse Gaussian distribution (NIG) has been introduced to finance in Barndorff-Nielsen (1998). It can be obtained as a normal mean-variance mixture with an inverse Gaussian mixing distribution. This in particular entails that the infinite divisibility of the mixing inverse Gaussian distribution transfers to the NIG mixture distribution, thus there exists a Lévy process $(L_t)_{t\geq 0}$ with $\mathcal{L}(L_1) = NIG(\alpha, \beta, \delta, \mu)$. The density and mgf of an NIG distribution are given by

$$d_{NIG}(x) = \frac{\alpha\delta e^{\delta\sqrt{\alpha^2-\beta^2}}}{\pi} \frac{K_1\big(\alpha\sqrt{\delta^2 + (x-\mu)^2}\big)}{\sqrt{\delta^2 + (x-\mu)^2}}\, e^{\beta(x-\mu)}, \qquad M_{NIG}(u) = \frac{e^{u\mu+\delta\sqrt{\alpha^2-\beta^2}}}{e^{\delta\sqrt{\alpha^2-(\beta+u)^2}}}. \tag{7}$$

The parameter $\overline{\theta}$ of the risk-neutral Esscher martingale measure $Q^{\overline{\theta}}$, i.e., the solution of (4) (if it exists) is given by

$$\bar{\theta}_{NIG} = -\frac{1}{2} - \beta + \frac{r-\mu}{\delta}\sqrt{\frac{\alpha^2}{1+(\frac{r-\mu}{\delta})^2} - \frac{1}{4}}. \tag{8}$$

We have

$$d_{L_t}^{\bar{\theta}}(x) = \frac{e^{\bar{\theta}x}}{M_{NIG(\alpha,\beta,\delta t,\mu t)}(\bar{\theta})}\, d_{NIG(\alpha,\beta,\delta t,\mu t)}(x) = d_{NIG(\alpha,\beta+\bar{\theta},\delta t,\mu t)}(x) \tag{9}$$

which implies that $(L_t)_{t\geq 0}$ remains a NIG Lévy process under the risk-neutral Esscher measure $Q^{\bar{\theta}}$, but with skewness parameter $\beta$ replaced by $\beta+\bar{\theta}$.

**Variance Gamma model.** Similar to the NIG distributions, a Variance Gamma distribution (VG) can be represented as a normal mean-variance mixture with a mixing Gamma distribution. Symmetric VG distributions were first defined (with a different parametrization) in Madan and Seneta (1990), the general case with skewness was considered in Madan et al. (1998). Again, the infinite divisibility of the Gamma distribution transfers to the Variance Gamma distribution $VG(\lambda,\alpha,\beta,\mu)$ whose density and mgf are given by

$$d_{VG}(x) = \frac{(\alpha^2-\beta^2)^{\lambda}|x-\mu|^{\lambda-\frac{1}{2}}}{\sqrt{\pi}(2\alpha)^{\lambda-\frac{1}{2}}\Gamma(\lambda)}K_{\lambda}(\alpha|x-\mu|)e^{\beta(x-\mu)},\ M_{VG}(u) = e^{u\mu}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+u)^2}\right)^{\lambda}. \tag{10}$$

Here the condition $2\alpha > 1$ is sufficient to guarantee a unique solution $\bar{\theta}$ of equation (4) which is given by

$$\bar{\theta}_{VG} = \begin{cases} -\frac{1}{2}-\beta, & r=\mu, \\ -\frac{1}{1-e^{-\frac{r-\mu}{\lambda}}} - \beta + \operatorname{sign}(r-\mu)\sqrt{\frac{e^{-\frac{r-\mu}{\lambda}}}{\left(1-e^{-\frac{r-\mu}{\lambda}}\right)^2}+\alpha^2}, & r\neq\mu. \end{cases} \tag{11}$$

Similar as above, we have

$$d_{L_t}^{\bar{\theta}}(x) = \frac{e^{\bar{\theta}x}}{M_{VG(\lambda t,\alpha,\beta,\mu t)}(\bar{\theta})}\, d_{VG(\lambda t,\alpha,\beta,\mu t)}(x) = d_{VG(\lambda t,\alpha,\beta+\bar{\theta},\mu t)}(x), \tag{12}$$

hence under $Q^{\bar{\theta}}$ $(L_t)_{t\geq 0}$ again is a VG process, but with skewness parameter $\beta+\bar{\theta}$ instead of $\beta$.

## 3. STANDARD AND EFFICIENT SELF-QUANTO CALLS

A quanto option is a (typically European) option whose payoff is converted into a different currency or numeraire at maturity at a pre-specified rate, called the quanto-factor. In the special case of a self-quanto option the numeraire is the underlying asset price at maturity itself. The payoff of a long self-quanto call with maturity $T$ and strike price $K$ therefore is

$$X_T^{sqC} = S_T\cdot(S_T-K)_+ = S_0e^{L_T}(S_0e^{L_T}-K)_+$$

Applying the risk-neutral pricing rule, together with equation (4), we obtain the following formula for the time-0-price of a self-quanto call:

$$\begin{aligned} c(X_T^{sqC}) &= e^{-rT}E\big[Z_T^{\bar\theta}S_T\cdot(S_T-K)_+\big] \\ &= \frac{M_{L_T}(\bar\theta)}{M_{L_T}(\bar\theta+1)}\,E\left[\frac{e^{\bar\theta L_T}}{M_{L_T}(\bar\theta)}S_0e^{L_T}\big(S_0e^{L_T}-K\big)\mathbb{1}_{(\ln(K/S_0),\infty)}(L_T)\right] \\ &= S_0^2\frac{M_{L_T}(\bar\theta+2)}{M_{L_T}(\bar\theta+1)}\,E\big[Z_T^{\bar\theta+2}\mathbb{1}_{(\ln(K/S_0),\infty)}(L_T)\big]-KS_0\,E\big[Z_T^{\bar\theta+1}\mathbb{1}_{(\ln(K/S_0),\infty)}(L_T)\big] \end{aligned}$$

From equations (7) and (9) resp. (10) and (12) we can derive more explicit formulas for the NIG and VG models:

$$c(X_T^{sqC}) = \begin{cases} S_0^2\frac{e^{\mu T+\delta T\sqrt{\alpha^2-(\beta+\bar\theta+1)^2}}}{e^{\delta T\sqrt{\alpha^2-(\beta+\bar\theta+2)^2}}}\bar F_{NIG(\alpha,\beta+\bar\theta+2,\delta T,\mu T)}\big(\ln(K/S_0)\big)-KS_0\bar F_{NIG(\alpha,\beta+\bar\theta+1,\delta T,\mu T)}\big(\ln(K/S_0)\big) \\ S_0^2e^{\mu T}\left(\frac{\alpha^2-(\beta+\bar\theta+1)^2}{\alpha^2-(\beta+\bar\theta+2)^2}\right)^{\lambda T}\bar F_{VG(\lambda T,\alpha,\beta+\bar\theta+2,\mu T)}\big(\ln(K/S_0)\big)-KS_0\bar F_{VG(\lambda T,\alpha,\beta+\bar\theta+1,\mu T)}\big(\ln(K/S_0)\big) \end{cases}$$

where $\bar F(x) = 1-F(x)$ denotes the survival function of the corresponding distribution. For $0\le t\le T$, the time-$t$-price $c(X_{T,t}^{sqC})$ of the self-quanto call is obtained from the preceding formulas by replacing $S_0$ by $S_t$ and $T$ by $T-t$.

The payoff $X_T^{sqC}$ of a self-quanto call obviously is increasing in $L_T$ and therefore not cost-efficient if $\bar\theta>0$ by Corollary 2.2. According to Proposition 2.1 b), its efficient counterpart $\underline{X}_T^{sqC}$ is given by $G_{sqC}^{-1}(1-F_{L_T}(L_T))$. To derive the corresponding distribution function $G_{sqC}=F_{X_T^{sqC}}$, observe that the positive solution $S_T^*$ of the quadratic equation $S_T^2-KS_T=x$, $x>0$, is given by $S_T^*=\frac{K}{2}+\sqrt{\frac{K^2}{4}+x}$, hence

$$G_{sqC}(x)=P\big(X_T^{sqC}\le x\big)=\begin{cases}0 & \text{, if } x<0, \\ F_{L_T}\left(\ln\left(\frac{\frac{K}{2}+\sqrt{\frac{K^2}{4}+x}}{S_0}\right)\right) & \text{, if } x\ge 0.\end{cases}$$

The inverse then can easily be shown to equal

$$G_{sqC}^{-1}(y)=S_0e^{F_{L_T}^{-1}(y)}\big(S_0e^{F_{L_T}^{-1}(y)}-K\big)_+,\quad y\in(0,1),$$

consequently the cost-efficient strategy for a long self-quanto call in the case $\bar\theta>0$ is

$$\underline{X}_T^{sqC}=G_{sqC}^{-1}(1-F_{L_T}(L_T))=S_0e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))}\big(S_0e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))}-K\big)_+. \tag{13}$$

A comparison of the payoff functions $X_T^{sqC}$ and $\underline{X}_T^{sqC}$ of a standard resp. efficient self-quanto call on ThyssenKrupp with strike $K=16$ and maturity $T=22$ days can be found in Figure 1 below. The estimated NIG parameters for ThyssenKrupp used to calculate the efficient payoff profile can be found in Table 1 in Section 5.

Observe that in contrast to the standard payoff $X_T^{sqC}=h_{sqC}(S_T)=\tilde h_{sqC}(L_T)$, the payoff function $\underline{\tilde h}_{sqC}(L_T)$ of the efficient self-quanto call depends on the time to maturity because so do

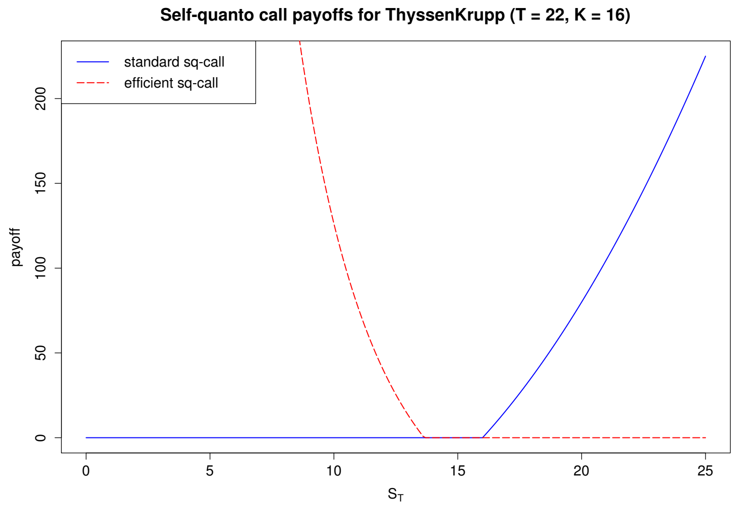


Figure 1: Payoff functions of a standard and efficient self-quanto call on ThyssenKrupp. The initial stock price is $S_0 = 15.25$, the closing price of ThyssenKrupp at July 1, 2013.

the distribution and quantile functions $F_{L_T}$ resp. $F_{L_T}^{-1}$. However, if an investor buys an efficient self-quanto call, its payoff profile is fixed at the purchase date and will not be altered afterwards. Once bought or sold, the payoff distribution of a cost-efficient contract only equals that of its classical counterpart at the (initial) trading date, but no longer in the remaining time to maturity. To calculate the price $c(\underline{X}_{T,t}^{sqC})$ of an efficient self-quanto call with a payoff function fixed at time 0 at some later point in time $t > 0$, one has to resort to the fact that $S_T = S_0 e^{L_T} \overset{d}{=} S_0 e^{L_t + L_{T-t}} = S_t e^{L_{T-t}}$ and thus replace $L_T = \ln(S_T/S_0)$ in (13) by $\ln(S_t e^{L_{T-t}}/S_0)$, that is,

$$\underline{X}_{T,t}^{sqC} = S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(\ln(S_t e^{L_{T-t}}/S_0)))} \left(S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(\ln(S_t e^{L_{T-t}}/S_0)))} - K\right)_+ .$$

The time-$t$-price of an efficient self-quanto call initiated at time 0 then can be calculated by

$$c(\underline{X}_{T,t}^{sqC}) = e^{-r(T-t)} S_0 \int_{-\infty}^{a} e^{F_{L_T}^{-1}(1-F_{L_T}(y+\ln(S_t/S_0)))} \left(S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(y+\ln(S_t/S_0)))} - K\right) d_{L_{T-t}}^{\bar{\theta}}(y)\, dy \tag{14}$$

where $a = F_{L_T}^{-1}(1 - F_{L_T}(\ln(K/S_0))) - \ln(S_t/S_0)$. If $t = 0$, one can alternatively use the general formula of Proposition 2.1, together with the representation of $G_{sqC}^{-1}$ given above.

## 4. STANDARD AND EFFICIENT SELF-QUANTO PUTS

The payoff of a long self-quanto put with maturity $T$ and strike price $K$ is

$$X_T^{sqP} = S_T \cdot (K - S_T)_+ = S_0 e^{L_T} (K - S_0 e^{L_T})_+$$

and similar as in the call case, we find the time-0-price of a self-quanto put to equal

$$c(X_T^{sqP}) = K S_0\, E\big[Z_T^{\bar{\theta}+1} \mathbb{1}_{(-\infty,\ln(K/S_0))}(L_T)\big] - S_0^2 \frac{M_{L_T}(\bar{\theta}+2)}{M_{L_T}(\bar{\theta}+1)}\, E\big[Z_T^{\bar{\theta}+2} \mathbb{1}_{(-\infty,\ln(K/S_0))}(L_T)\big]$$

which can be specialized in the NIG and VG models to

$$c(X_T^{sqP}) = \begin{cases} KS_0 F_{NIG(\alpha,\beta+\bar\theta+1,\delta T,\mu T)}\big(\ln(K/S_0)\big) - S_0^2 \frac{e^{\mu T+\delta T\sqrt{\alpha^2-(\beta+\bar\theta+1)^2}}}{e^{\delta T\sqrt{\alpha^2-(\beta+\bar\theta+2)^2}}} F_{NIG(\alpha,\beta+\bar\theta+2,\delta T,\mu T)}\big(\ln(K/S_0)\big) \\ KS_0 F_{VG(\lambda T,\alpha,\beta+\bar\theta+1,\mu T)}\big(\ln(K/S_0)\big) - S_0^2 e^{\mu T}\left(\frac{\alpha^2-(\beta+\bar\theta+1)^2}{\alpha^2-(\beta+\bar\theta+2)^2}\right)^{\lambda T} F_{VG(\lambda T,\alpha,\beta+\bar\theta+2,\mu T)}\big(\ln(K/S_0)\big) \end{cases}$$

Again, the time-$t$-price of the self-quanto put for $0 \le t \le T$ is obtained from the above equations by replacing $S_0$ by $S_t$ and $T$ by $T-t$.

The payoff function $X_T^{sqP} = h_{sqP}(S_T)$ of a self-quanto put is a parabola which is open from below and has the roots 0 and $K$ as well as a maximum at $S_T = \frac{K}{2}$. Hence, it is neither increasing nor decreasing in $S_T$ and therefore not in $L_T = \ln(S_T/S_0)$ either, so Corollary 2.2 implies that a self-quanto put can never be cost-efficient unless $\bar\theta = 0$.

The lack of monotonicity also makes the determination of the distribution function $G_{sqP}$ of the self-quanto put payoff and its inverse a little bit cumbersome. To derive them, first observe that the corresponding payoff function $\tilde h_{sqP}(x) = (S_0Ke^x - S_0^2e^{2x})\cdot \mathbb{1}_{(-\infty,\ln(K/S_0))}$ is strictly increasing on $(-\infty, \ln(K/(2S_0)))$ and strictly decreasing on $(\ln(K/(2S_0)), \ln(K/S_0))$, and has a maximum at $x = \ln(K/(2S_0))$ with value $\tilde h_{sqP}\big(\ln(K/(2S_0))\big) = \frac{K^2}{4}$. For $y \in (0, \ln(K/S_0))$ we have

$$\tilde h_{sqP}(x) = y \iff x = \ln\left(\frac{K+\sqrt{K^2-4y}}{2S_0}\right) \;\vee\; x = \ln\left(\frac{K-\sqrt{K^2-4y}}{2S_0}\right)$$

from which we obtain

$$\begin{aligned} G_{sqP}(x) &= P\big(\tilde h_{sqP}(L_T) \le x\big) \\ &= \begin{cases} 1 & \text{for } x \ge \frac{K^2}{4}, \\ F_{L_T}\left(\ln\left(\frac{K-\sqrt{K^2-4x}}{2S_0}\right)\right) + 1 - F_{L_T}\left(\ln\left(\frac{K+\sqrt{K^2-4x}}{2S_0}\right)\right) & \text{for } \frac{K^2}{4} > x > 0, \\ 1 - F_{L_T}\big(\ln(K/S_0)\big) & \text{for } x = 0, \\ 0 & \text{for } x < 0. \end{cases} \end{aligned}$$

The shape of the payoff function here leads to two summands in the representation of the payoff distribution $G_{sqP}$ on the interval $(0, \frac{K^2}{4})$, therefore its inverse $G_{sqP}^{-1}$ needed to construct the cost-efficient self-quanto put payoff $\underline{X}_T^{sqP}$ according to Proposition 2.1 can only be evaluated numerically (using some suitable root-finding algorithms), but not given in closed form.

If $\bar\theta < 0$, then we have $\underline{X}_T^{sqP} = G_{sqP}^{-1}(F_{L_T}(L_T)) = G_{sqP}^{-1}(F_{L_T}(\ln(S_T/S_0))$, and from the above representation of $G_{sqP}$ we conclude that $G_{sqP}^{-1}(F_{L_T}(\ln(S_T/S_0)) = 0$ if $S_T \le S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(\ln(K/S_0)))}$ resp. $L_T \le F_{L_T}^{-1}(1-F_{L_T}(\ln(K/S_0)))$. Otherwise, the payoff is positive and tends to $\frac{K^2}{4}$ if $S_T$ resp. $L_T$ tend to infinity.

If $\bar\theta > 0$, then $\underline{X}_T^{sqP} = G_{sqP}^{-1}(1 - F_{L_T}(L_T)) = G_{sqP}^{-1}(1 - F_{L_T}(\ln(S_T/S_0))$ which is zero if $S_T \ge K$ resp. $L_T \ge \ln(K/S_0)$ and tends to $\frac{K^2}{4}$ if $S_T \to 0$ resp. $L_T \to -\infty$. Hence, for $\bar\theta > 0$ the efficient self-quanto put payoff shows just the opposite behaviour as for $\bar\theta < 0$. This is in line with Corollary 2.2 which states, in other words, that a cost-efficient payoff must alter its monotonicity properties if the sign of the risk-neutral Esscher parameter $\bar\theta$ changes. The two different payoff

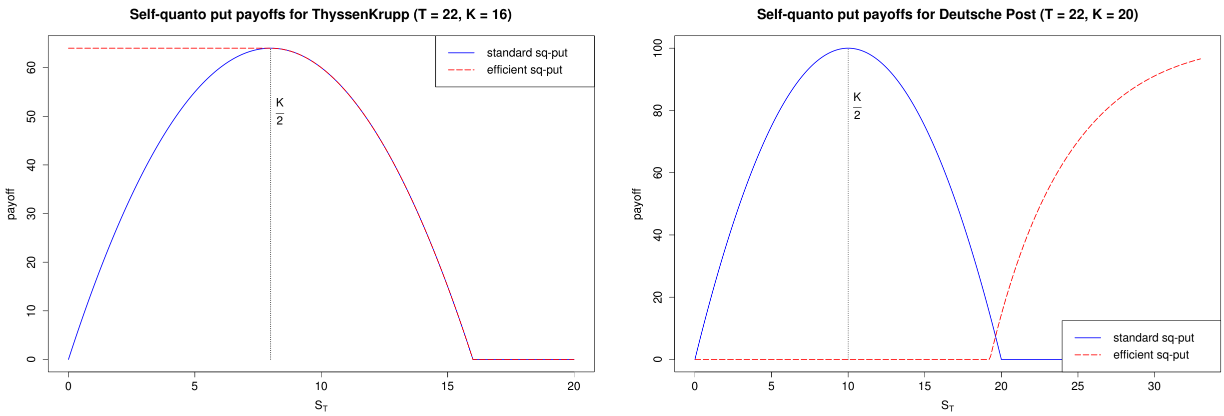


Figure 2: *Left:* Payoff functions of a standard and efficient self-quanto put on ThyssenKrupp ($\bar{\theta} > 0$). The initial stock price is $S_0 = 15.25$, the closing price of ThyssenKrupp at July 1, 2013. *Right:* Payoff functions of a standard and efficient self-quanto put on Deutsche Post ($\bar{\theta} < 0$). The initial stock price is $S_0 = 19.31$, the closing price of Deutsche Post at July 1, 2013.

profiles that can occur for an efficient self-quanto put are visualized in Figure 2 above. The estimated VG parameters for ThyssenKrupp and Deutsche Post that are used to calculate the efficient payoffs can be found in Table 1 in Section 5. As can be seen from the latter, the efficient payoff for ThyssenKrupp corresponds to the case $\bar{\theta} > 0$, whereas the efficient payoff for Deutsche Post has the typical shape for $\bar{\theta} < 0$.

For the time-$t$-price of an efficient self-quanto put that is issued at time $0$, one obtains, with the same reasoning as in Section 3,

$$\begin{aligned} c(\underline{X}_{T,t}^{sqP}) &= e^{-r(T-t)} E\big[Z_{T-t}^{\bar{\theta}} \underline{X}_{T,t}^{sqP}\big] \\ &= \begin{cases} e^{-r(T-t)} \displaystyle\int_{a_-}^{\infty} G_{sqP}^{-1}\big(F_{L_T}(y + \ln(S_t/S_0))\big) d_{L_{T-t}}^{\bar{\theta}}(y)\, dy =: c_t^{-}(S_t) & \text{if } \bar{\theta} < 0, \\ e^{-r(T-t)} \displaystyle\int_{-\infty}^{a_+} G_{sqP}^{-1}\big(1 - F_{L_T}(y + \ln(S_t/S_0))\big) d_{L_{T-t}}^{\bar{\theta}}(y)\, dy =: c_t^{+}(S_t) & \text{if } \bar{\theta} > 0, \end{cases} \end{aligned} \tag{15}$$

where $a_- = F_{L_T}^{-1}(1 - F_{L_T}(\ln(K/S_0))) - \ln(S_t/S_0)$ and $a_+ = \ln(K/S_t)$. Due to the necessary numerical determination of $G_{sqP}^{-1}(x)$, the integrals in (15) have to be truncated in practical applications to obtain sensible and stable results from a numerical evaluation. The inequalities

$$\begin{aligned} & e^{-r(T-t)} \int_{a_-}^{z_-} G_{sqP}^{-1}\big(F_{L_T}(y + \ln(S_t/S_0))\big) d_{L_{T-t}}^{\bar{\theta}}(y)\, dy \le c_t^{-}(S_t) \\ & \qquad \le e^{-r(T-t)} \int_{a_-}^{z_-} G_{sqP}^{-1}\big(F_{L_T}(y + \ln(S_t/S_0))\big) d_{L_{T-t}}^{\bar{\theta}}(y)\, dy + e^{-r(T-t)} \frac{K^2}{4} \bar{F}_{L_{T-t}}^{\bar{\theta}}(z_-), \end{aligned}$$

$$\begin{aligned} & e^{-r(T-t)} \int_{z_+}^{a_+} G_{sqP}^{-1}\big(1 - F_{L_T}(y + \ln(S_t/S_0))\big) d_{L_{T-t}}^{\bar{\theta}}(y)\, dy \le c_t^{+}(S_t) \\ & \qquad \le e^{-r(T-t)} \int_{z_+}^{a_+} G_{sqP}^{-1}\big(1 - F_{L_T}(y + \ln(S_t/S_0))\big) d_{L_{T-t}}^{\bar{\theta}}(y)\, dy + e^{-r(T-t)} \frac{K^2}{4} F_{L_{T-t}}^{\bar{\theta}}(z_+), \end{aligned}$$

which hold for all $z_- > a_-$ resp. $z_+ < a_+$ allow to well control the error caused by the truncation.

## 5. APPLICATION TO REAL MARKET DATA

In this section we want to apply the theoretical results obtained so far to some real data and parameters to get an impression how large the potential efficiency losses of the standard options can be. For our calculations, we use NIG and VG parameters estimated from two German stocks, ThyssenKrupp and Deutsche Post. We used data from a two-year period starting at June 1, 2011, and ending on June 28, 2013, to estimate the parameters from the log-returns of both stocks. The stock prices within the estimation period are shown in Figure 3, and the obtained parameters are summarized in Table 1. The interest rate used to calculate $\bar{\theta}$ is $r = 4.3838 \cdot 10^{-6}$ which corresponds to the continuously compounded 1-Month-Euribor rate of July 1, 2013.

Observe that the risk-neutral Esscher parameters $\bar{\theta}_{NIG}$ and $\bar{\theta}_{VG}$ are negative for Deutsche Post and positive for ThyssenKrupp, therefore a self-quanto call can only be improved for ThyssenKrupp, for Deutsche Post it already is cost-efficient. For the former, we calculate the prices of standard and efficient self-quanto calls with strike $K = 16$ which are issued on July 1, 2013, and mature on July 31, 2013, so the time $T$ to maturity is 22 trading days. The results are shown in Table 2. Apparently, the differences in prices and hence the efficiency losses are quite large, the standard self-quanto call costs almost twice as much as its efficient counterpart.

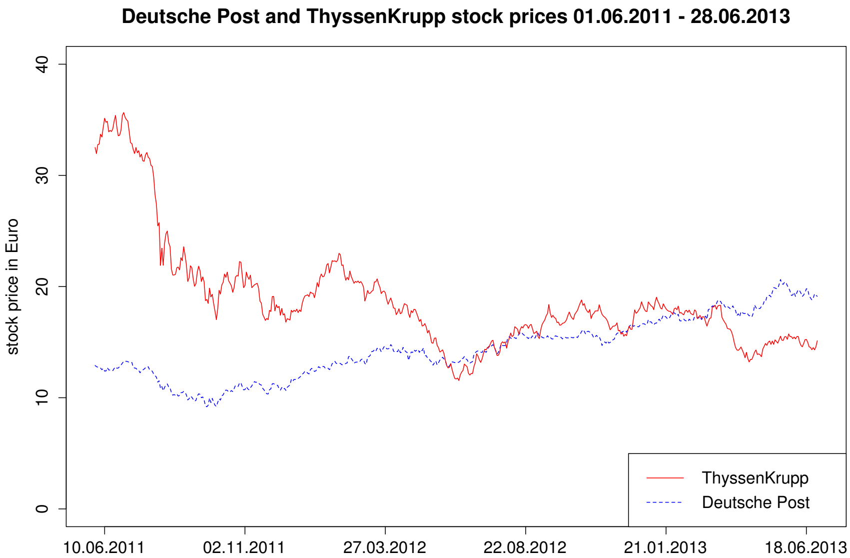


Figure 3: Daily closing prices of Deutsche Post and ThyssenKrupp used for parameter estimation.

| **Deutsche Post** | $\lambda$ | $\alpha$ | $\beta$ | $\delta$ | $\mu$ | $\bar{\theta}$ |
|---|---|---|---|---|---|---|
| NIG | -0.5 | 75.059 | 1.758 | 0.019 | 0.000306 | -3.4787 |
| VG | 1.942 | 126.266 | 3.719 | 0.0 | -0.000165 | -3.5220 |
| **ThyssenKrupp** | $\lambda$ | $\alpha$ | $\beta$ | $\delta$ | $\mu$ | $\bar{\theta}$ |
| NIG | -0.5 | 53.065 | -0.491 | 0.037 | -0.001101 | 1.5823 |
| VG | 2.659 | 87.894 | -0.613 | 0.0 | -0.001025 | 1.6080 |

Table 1: Estimated parameters from daily log-returns of Deutsche Post and Volkswagen for the NIG- and the VG-model.

| **ThyssenKrupp** | $c(X_T^{sqC})$ | $c(\underline{X}_T^{sqC})$ | **Efficiency loss in %** |
|---|---|---|---|
| NIG | 8.3288 | 4.2251 | 49.27 |
| VG | 8.2629 | 4.1609 | 49.64 |

Table 2: Comparison of the prices of a self-quanto call on ThyssenKrupp with strike $K = 16$ and $T = 22$, and its cost-efficient counterpart in the NIG and VG models. The initial stock price is $S_0 = 15.25$, the closing price of ThyssenKrupp on July 1, 2013.

In contrast to the self-quanto call, the standard self-quanto put is—at least theoretically—inefficient for both stocks since the risk-neutral Esscher parameter $\bar{\theta}$ is different from zero in all cases. In our example we assume that the standard and efficient self-quanto puts on ThyssenKrupp and Deutsche Post have the same issuance day and maturity date as the self-quanto calls above, and the strikes are again $K = 16$ for ThyssenKrupp and $K = 20$ for Deutsche Post. The obtained results are listed in Table 3. Whereas the efficiency losses for Deutsche Post are of comparable magnitude as in the call example, one surprisingly does not save anything by investing in the efficient self-quanto put on ThyssenKrupp.

This becomes clearer if we take a look back on the corresponding payoff function $\underline{X}_T^{sqP}$. Recall that the risk-neutral Esscher parameters for ThyssenKrupp are always positive, therefore the left plot of Figure 2 applies here. If $\bar{\theta} > 0$, then obviously $X_T^{sqP}$ and $\underline{X}_T^{sqP}$ are almost identical for $S_T \in (\frac{K}{2}, \infty)$ and only differ significantly if $S_T \in (0, \frac{K}{2})$. But if the risk-neutral probability $Q^{\bar{\theta}}(0 < S_T < \frac{K}{2})$ is very small, then it is intuitively evident that the prices $c(X_T^{sqP})$ and $c(\underline{X}_T^{sqP})$ should nearly coincide. This is the case here. The strike $K$ is very close to the initial stock price $S_0$, and the risk-neutral measure $Q^{\bar{\theta}}$ is more right-skewed than the real-word one $P$ (under the risk-neutral Esscher measure, only the skewness parameter $\beta$ of the NIG and VG distributions changes to $\beta + \bar{\theta}$), so under $Q^{\bar{\theta}}$ it becomes even more unlikely that $S_T < \frac{K}{2}$.

The evolution of the prices $c(X_{T,t}^{sqC})$, $c(\underline{X}_{T,t}^{sqC})$ of the standard and efficient self-quanto call on ThyssenKrupp as well as that of the prices $c(X_{T,t}^{sqP})$, $c(\underline{X}_{T,t}^{sqP})$ of the self-quanto puts on Deutsche Post in the NIG models during the lifetime of the options is shown in Figure 4. The prices of the efficient options always roughly move in the opposite direction of that of the standard options which reflects the reversed resp. altered monotonicity properties of the underlying payoff profiles.

| **ThyssenKrupp** | $c(X_T^{sqP})$ | $c(\underline{X}_T^{sqP})$ | **Efficiency loss in %** |
|---|---|---|---|
| NIG | 16.1541 | 16.1541 | 0.0 |
| VG | 16.1226 | 16.1226 | 0.0 |
| **Deutsche Post** | $c(X_T^{sqP})$ | $c(\underline{X}_T^{sqP})$ | **Efficiency loss in %** |
| NIG | 17.6912 | 10.2613 | 42.00 |
| VG | 17.6593 | 10.2152 | 42.15 |

Table 3: Comparison of the prices of standard and efficient self-quanto puts on ThyssenKrupp and Deutsche Post with strikes $K = 16$ resp. $K = 20$, and $T = 22$, in the NIG and VG models. The initial stock prices are $S_0 = 15.25$ for ThyssenKrupp and $S_0 = 19.31$ for Deutsche Post, which are the closing prices on July 1, 2013.

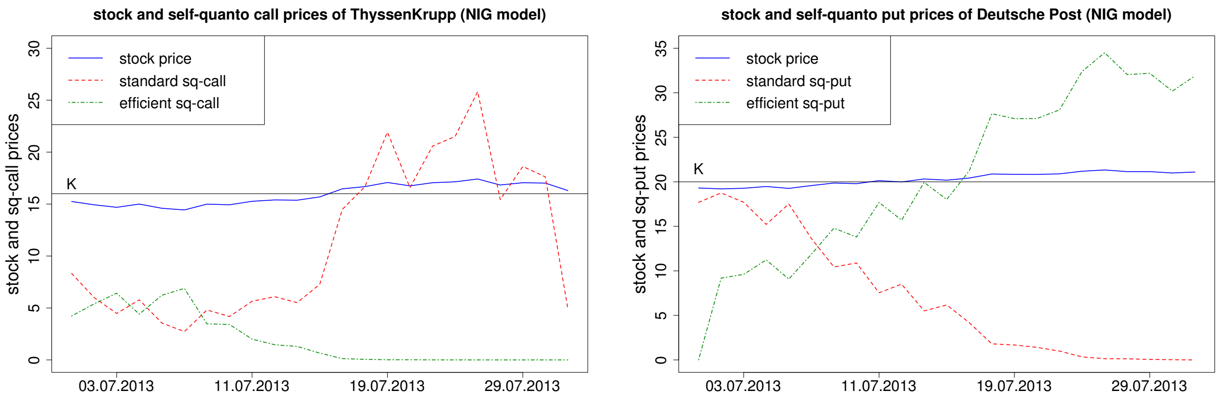


Figure 4: *Left:* Stock price of ThyssenKrupp from July 1, 2013, to July 31, 2013, and the prices $c(X_{T,t}^{sqC})$, $c(\underline{X}_{T,t}^{sqC})$ of the associated standard and efficient self-quanto calls. *Right:* Stock price of Deutsche Post from July 1, 2013, to July 31, 2013, and the prices $c(X_{T,t}^{sqP})$, $c(\underline{X}_{T,t}^{sqP})$ of the associated standard and efficient self-quanto puts.

## 6. SUMMARY AND CONCLUSION

We applied the concept of cost-efficiency to self-quanto puts and calls in exponential Lévy models where the risk-neutral measure is obtained by an Esscher transform. Whereas one can arrive—at least in principle—at closed-form solutions in the call case, things become more involved for the self-quanto put because of the lacking monotonicity properties of the corresponding payoff function. Nevertheless, the arising expressions and integrals remain numerically tractable and can be evaluated in an efficient and stable way which we demonstrated in a practical application using estimated parameters and real data from the German stock market. The observed efficiency losses are often quite large. However, the prices of the cost-efficient options are not always significantly lower than their classical counterparts. For efficient self-quanto puts that are issued at the money, the potential savings are negligible if the risk-neutral Esscher parameter is positive.

The evolution of the prices of standard and efficient options over time shows that they move in opposite directions: If the standard option expires worthless, its efficient counterpart typically ends up in the money, and vice versa. This should remind the reader that although cost-efficient options provide a cheaper way to participate in a certain payoff distribution, they are still speculative instruments which bear the risk of a total loss of one's investment.

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