Bounds for joint portfolios of dependent risks

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Abstract

In this paper, we survey, extend and improve several bounds for the distribution function and the tail probabilities of portfolios, where the dependence structure within the portfolio is completely unknown or only partially known. We present various methods for obtaining bounds based on rearrangements, duality theory, conditional moments and reduction techniques. In particular, we consider the case where only the simple marginal distributions are known, the general overlapping marginals case where certain joint distributions are known and the case of additional restrictions on the dependence structure, as, for example, the restriction to positive dependence. Some of the bounds pose considerable numerical challenge. We discuss the quality of the bounds and numerical aspects in some examples.

1 Introduction

For a risk vector $\vec{X} = (X_1, \dots, X_d)$, we consider the problem to find good (best possible) bounds for the distribution function and tail probability of the joint portfolio $S = \sum_{i=1}^d X_i$, when the marginal distribution functions F_i of X_i are known but the dependence structure between the components of $\vec{X}$ is either completely or partially unknown. The problem of obtaining bounds for the distribution of the sum of dependent risks has received considerable attention in the literature, since it has relevant applications. In quantitative risk management within banking and insurance, bounds for the distribution function or for the tail risk are needed to compute bounds on quantile-based risk measures for regulatory issues. Bounds for the distribution functions directly imply corresponding bounds for the Value-at-Risk of S, defined as

$$\mathrm{VaR}_\alpha(S) = \inf\{x \in \mathbb{R} : P(S \le x) \ge \alpha\}, \text{ for } \alpha \in [0,1].$$

For more details on this, we refer to the introduction in [EP][1] (2010a) and to [EP] (2010b).

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[1] Embrechts and Puccetti is abbreviated within this paper with [EP], Puccetti and Rüschendorf with [PR], Rüschendorf with [Rü].

The problem of deriving risk bounds has a long history and various techniques have been developed to solve these problems. Let

$$M(s) = \sup\left\{P(X_1 + \cdots + X_d \geq s) : F_{\vec{X}} \in \mathfrak{F}(F_1, \ldots, F_d)\right\}, \tag{1.1a}$$

$$m(s) = \inf\left\{P(X_1 + \cdots + X_d \geq s) : F_{\vec{X}} \in \mathfrak{F}(F_1, \ldots, F_d)\right\}, \tag{1.1b}$$

denote the sharp upper and lower *Fréchet bounds* over all possible dependence structures, that is over the *Fréchet class* $\mathfrak{F}(F_1, \ldots, F_d)$ of all joint distributions on $\mathbb{R}^d$ with given marginals $F_1, \ldots, F_d$. Knowledge of the sharp bounds M and m implies directly knowledge of sharp bounds for the Value-at-Risk, as

$$\mathrm{VaR}_\alpha(S) \leq M^{-1}(1-\alpha).$$

Equivalently to (1.1), we can also consider the modified problems

$$M^+(s) = \sup\left\{P(X_1 + \cdots + X_d > s) : F_{\vec{X}} \in \mathfrak{F}(F_1, \ldots, F_d)\right\},$$
$$m^+(s) = \inf\left\{P(X_1 + \cdots + X_d > s) : F_{\vec{X}} \in \mathfrak{F}(F_1, \ldots, F_d)\right\}.$$

Using that $P(S > s) = 1 - P(S \leq s)$, we can also consider sharp upper and lower bounds for the distribution function of the joint portfolio, that is

$$M^+(s) \leq P(S \leq s) \leq 1 - m^+(s).$$

In the following, we will derive bounds for any of the equivalent forms above.

Explicit results for the sharp Fréchet bounds (1.1) were derived independently for the case $d = 2$ in Makarov (1981) and [Rü] (1982):

$$\sup\{P(X_1 + X_2 \leq s) : X_i \sim F_i\} = \inf_{x \in \mathbb{R}} F_1(x-) + F_2(t-x), \tag{1.2a}$$

$$\inf\{P(X_1 + X_2 < s) : X_i \sim F_i\} = \sup_{x \in \mathbb{R}} F_1(x-) + F_2(t-x) - 1. \tag{1.2b}$$

In the above expressions, $F_1(x-)$ denotes the left-hand limit of F_1 in x. In some examples like discrete and continuous uniform distributions and Binomial distributions exact bounds were given, for the general case $d \geq 3$, in [Rü] (1982,1983a). The bounds in (1.2) have been extended to $d \geq 3$ (see Frank et al. (1987); Denuit et al. (1999); Embrechts et al. (2003); [Rü] (2005)) and yield the so called *standard bounds*. In contrast to the case $d = 2$, the standard bounds are no longer sharp. Essential improvements of the standard bounds were developed in a series of papers in [EP] (2006a,b,2010a). In these papers *dual bounds* are given for M and m which are based on a general dual representation derived from mass transportation theory. For $d \geq 3$ only in the recent paper Wang and Wang (2011) exact bounds were found in the homogeneous case that $F_1 = \cdots = F_d = F$ where F is a distribution on $[0,1]$ with monotonically non increasing (nondecreasing) density f.

In Section 2, we review and extend several results for the case of simple marginals i.e. for the case where the dependence structure of the portfolio $\vec{X}$ is completely unknown. Based on the relation to rearrangement problems, we describe the structure of couplings which yield the worst case dependence structure, that is which attains the Fréchet bounds for the distribution function, respectively for the tail risk. Then, we extend a recent simple way to calculate sharp bounds given in Wang and Wang (2011) for the homogeneous case with monotone densities to the case of general inhomogeneous marginals. We also give a derivation of the standard bounds based on duality arguments and sketch the proof for the general dual bounds. Apart from the homogeneous case, the dual bounds are not easy to calculate. We make some remarks and comments on the numerics and algorithms. In the monotone and homogeneous case, the dual bounds coincide with the sharp bounds given in Wang and Wang (2011) indicating that they are more generally of good quality.

Under the restriction of the dependence structure to positive dependence essential improvements of the bounds are given in Section 3.1. In Section 3.2, we consider the general case of higher dimensional marginals (overlapping marginals), where for some system $\mathscr{E} \subset 2^d$ the joint distribution F_J of $(X_j)_{j\in J}$ are known for any $J \in \mathscr{E}$. This is a different kind of restriction on the dependence structure. In particular, this class of dependence models includes the *series* case where one knows the distribution functions of $(X_i, X_{i+1}), 1 \le i \le d-1$, the *star-like* case where $\mathscr{E} = \{\{1, j\}, 2 \le j \le d\}$ or the case $\mathscr{E} = \{\{i, j\}, 1 \le i < j \le d\}$, where all the two-dimensional marginals are known. Under this extended knowledge of the joint dependence structure, one gets sharper bounds compared to the simple marginal case $\mathfrak{F}(F_1, \dots, F_d)$ of general possible dependence. In the case of star-like systems, one can give an exact reduction of the problem to the simple marginal case by the conditioning method described in [Rü] (1991). [EP] (2010a) used this reduction in combination with the standard bounds to describe improved bounds for the starlike case.

Furthermore, in [EP] (2010a), for the series case a reduction method to the simple marginal case was introduced and applied in examples. In this paper, we extend this reduction technique to the case of general overlapping marginal systems. This reduction allows us to give reasonable (good) bounds for general (overlapping) marginal structures. In a second step, we introduce an improved class of bounds by using an additional parameter describing different weights to the marginal components. In the final section of the paper, we describe several applications and numerical results on the comparison of the various bounds.

2 Bounds in the case of simple marginals

In this section, we consider the case when the risk vector $\vec{X} = (X_1, \dots, X_d)$ has given marginal distribution functions $F_1, \dots, F_d$ while its dependence structure is completely unknown. The aim is to obtain good bounds on the tail risk $P(S \ge s)$, resp. on the distribution function $P(S \le s)$, under the assumption that $F_{\vec{X}}$, the joint distribution of $\vec{X}$, lies in the Fréchet class $\mathfrak{F}(F_1, \dots, F_d)$. In risk management, it is typically of interest to have good upper bounds for the upper tail risk $P(S \ge s)$ of the portfolio for large values of s (as in insurance), or to have good lower bounds for the lower tail risk $P(S \le s)$ for s small, that is for tail losses of the portfolio (as in finance). In general, it will be difficult to evaluate $M(s)$ and $m(s)$ (or the corresponding bounds on the distributions) in explicit form. Thus, it is of interest to derive good upper bounds for $M(s)$ and lower bounds for $m(s)$. In the following, we review and extend several methods to obtain (good) bounds for the Fréchet problems for simple marginal systems. These methods include the rearrangement method, conditional moment bounds, the classical standard bounds and the dual bounds. Throughout the paper, for the sake of notational simplicity, we identify probability measures with the corresponding distribution functions.

2.1 The rearrangement method

The sharp Fréchet bounds M and m, defined in (1.1) describe sharp upper and lower bounds for the tail risk. The structure of the worst case dependence distributions, that is of the coupling which leads to the upper and lower bounds in (1.1) can be described with an account to rearrangement theory.

Let $F_i^{-1}(\alpha) = \inf\{x : F_i(x) \geq \alpha\}$ denote the generalized right-continuous inverse of the distribution F_i and let, for $A \subset [0,1]$, $F_i^{-1}|A$ be the restriction of F_i^{-1} to the set A. We write $f_i \sim_r F_i^{-1}|A$ to indicate that the function $f_i : A \to \mathbb{R}$ is a rearrangement of $F_i^{-1}|A$. We refer to [Rü] (1983b) for a basic introduction to the theory of rearrangements.

The optimal Fréchet bounds, then, can be equivalently formulated in terms of rearrangements (see Theorem 2 in [Rü] (1983b)):

$$M(s) = \sup\left\{P\left(\sum_{i=1}^{d} f_i(U) \geq s\right) : f_i \sim_r F_i^{-1}, 1 \leq i \leq d\right\}, \tag{2.1a}$$

$$m(s) = \inf\left\{P\left(\sum_{i=1}^{d} f_i(U) \geq s\right) : f_i \sim_r F_i^{-1}, 1 \leq i \leq d\right\}, \tag{2.1b}$$

where U is a random variable uniformly distributed on $(0,1)$, that is $P^U = \lambda$ is the Lebesgue measure on $(0,1)$. Based on (2.1), the structure of optimal couplings (i.e. worst dependence structure) for (1.1) is described in the following theorem; see also Proposition 3 in [Rü] (1982) and Theorem 4.1 in [PR] (2011).

Theorem 2.1 (Characterization of the worst case dependence structure) *For any marginal distributions $F_1, \dots, F_d$, we have that:*

a) $$M(s) = 1 - \inf\left\{\alpha \in [0,1] : \text{ there exist } f_i^\alpha \sim_r F_i^{-1}|[\alpha,1], 1 \leq i \leq d \text{ s.t. } \sum_{i=1}^{d} f_i^\alpha \geq s\right\},$$

$$m(s) = 1 - \sup\left\{\alpha \in [0,1] : \text{ there exist } f_i^\alpha \sim_r F_i^{-1}|[0,\alpha], 1 \leq i \leq d \text{ s.t. } \sum_{i=1}^{d} f_i^\alpha \leq s\right\}.$$

b) There exist optimal couplings $Y_i \sim F_i, 1 \leq i \leq d$ such that $P\left(\sum_{i=1}^d Y_i \geq s\right) = M(s)$ and

$$\left\{Y_i > F_i^{-1}(t_o)\right\} \subset \left\{\sum_{i=1}^{d} Y_i \geq s\right\} \subset \left\{Y_i \geq F_i^{-1}(t_o)\right\} \text{ a.s.,}$$

where $t_0 = M(s)$.

Remark 2.2 *(i) Part a) of Theorem 2.1 says that optimal couplings can be determined by finding the largest possible interval* $[\alpha,1]$ *such that on* $[\alpha,1]$ *there exist rearrangements of* $F_i^{-1}|[\alpha,1]$ *which sum up to a value larger than* s*. Similarly for the lower bound. This connection to rearrangements has been used in [PR] (2011) to develop a new and useful algorithm based on discrete rearrangements to approximate numerically the sharp Fréchet bounds* (1.1)*. The algorithm is shown to work well for general inhomogeneous portfolios up to dimension about* $d=30$*. We refer to [PR] (2011) for more details on the accuracy and speed of the rearrangement algorithm. In the case of Uniform and Binomial distributions, the rearrangement technique has been applied in [Rü] (1982; 1983a). It has also been used in Wang and Wang (2011) in the homogeneous case with monotone densities in* $[0,1]$.

(ii) *In the case of continuous distributions one gets, in part b) of Theorem 2.1, that*

$$\left\{\sum_{i=1}^d Y_i \geq s\right\} = \left\{Y_i \geq F_i^{-1}(t_o)\right\} \text{ a.s.}, \tag{2.3}$$

for all $1 \leq i \leq d$*. In the general case, one needs to use a randomization of the boundary* $\left\{Y_i = F_i^{-1}(t_o)\right\}$*. Equation* (2.3) *is intuitively obvious: one should use in each component only the largest part of the distribution.*

(iii) *Also the lower bound for* $m(s)$ *is attained in general. If we consider the slightly modified problem*

$$m^+(s) = \inf\left\{P(X_1+\cdots+X_d > s) : F_{\bar{X}} \in \mathfrak{F}(F_1,\ldots,F_d)\right\}, \tag{2.4}$$

then $\inf\{P(\sum_{i=1}^d Y_i) > s\} = 1-\sup\{P(\sum_{i=1}^d Y_i) \leq s\}$ *and an analogue to part b) holds. A simple compactness argument shows that the probability in* (2.4) *has to be defined differently than in* (1.1b) *in order to guarantee that the infimum is attained. For example, note that the problem*

$$\inf\left\{P(X_1+X_2 \geq 1) : X_i \sim U_{[0,1]}, 1 \leq i \leq 2\right\} = 0$$

and does not have a solution.

While the rearrangement method is connected with the construction of (optimal) couplings, we will discuss in the following section a method using only simple to calculate information on the marginal distributions.

2.2 The conditional moment method

A simple way to obtain bounds for the tail risk of an homogeneous portfolio $F_i = F, 1 \leq i \leq d$, was found recently in Wang and Wang (2011). In that paper, it was shown that the resulting bound is even sharp if the distribution F has a monotone density on $[0,1]$ satisfying a moderate moment condition. Moreover, this bound depends only on the first conditional moments of F and it is easy to calculate. In the following, we extend this conditional moment bound to general marginal distributions $F_1,\dots,F_d$.

Let $X_i \sim F_i$, $G_i = F_i^{-1}$ be the generalized inverse of F_i, $G = \sum_{i=1}^d G_i$ and assume that $\mu_i = \mathrm{E}[X_i]$ exists. For $a \in [0,1]$, define $\Psi(a)$ as the sum of the conditional first moments, that is

$$\Psi(a) = \frac{1}{1-a}\int_a^1 G(t)\,dt = \sum_{i=1}^d \mathrm{E}\left[X_i | X_i \geq G_i(a)\right].$$

The function Ψ is determined by the conditional first moments of the marginal distributions F_i. Obviously, Ψ is monotonically nondecreasing and $\Psi(0) = \mu = \sum_{i=1}^d \mu_i$.

Theorem 2.3 (Method of conditional moments) *Let $X_i \sim F_i$ have first moments μ_i, $1 \leq i \leq d$. Then, for $s \geq \mu$, we have*

$$M(s) \leq 1 - \Psi^-(s), \tag{2.5}$$

where $\Psi^-(s) = \sup\{t \in [0,1] : \Psi(t) \leq s\}$ is the left-continuous generalized inverse of Ψ.

Proof. With $X_i \sim F_i$ and $S = \sum_{i=1}^d X_i$, we have

$$\mu = \sum_{i=1}^d \mu_i = \mathrm{E}[S] \geq \mathrm{E}\left[S 1_{\{S<s\}}\right] + sP(S \geq s) \tag{2.6}$$

$$= \int_0^{P(S<s)} G(t)\,dt + sP(S \geq s) = \mu - \int_{P(S<s)}^1 G(t)\,dt + sP(S \geq s).$$

If $P(S \geq s) > 0$, this implies that $\Psi(P(S < s)) \geq s$ and thus $P(S < s) \geq \Psi^-(s)$. As a consequence, we obtain

$$P(S \geq s) \leq 1 - \Psi^-(s). \tag{2.7}$$

Remark 2.4 *(i) The conditional bound in* (2.5) *is sharp if and only if the estimate in* (2.6) *is an equality, that is if for the optimal coupling it holds true that $\{S \geq s\} = \{S = s\}$ a.s.. This means, by Theorem 2.1, that the corresponding optimal rearrangements f_i^α on $[\alpha,1]$ satisfy*

$$\sum_{i=1}^d f_i^\alpha(u) = s \text{ for all } u \in [\alpha,1],$$

with $1-\alpha = M(s)$. In Wang and Wang (2011), this property is called mixing *on the interval $[\alpha,1]$. In that paper, it is established that mixing holds true in the homogeneous case of monotone densities on $[0,1]$ under a moderate moment condition.*

(ii) *In a recent preprint, and independently of our paper, Wang et al. (2011) have established the conditional bound in* (2.5) *in the case of continuous marginal distributions. They used the equivalent form as lower bound for* $P(S<s) \geq \Psi^{-}(s)$ *in* (2.7)*. In their paper, there is also an extension of the sharpness result in Wang and Wang (2011) to the case of distributions with decreasing densities on an unbounded domain. For unbounded domains, the bound* (2.5) *typically fails to be sharp. To be a good bound it is indeed necessary that*

$$\sum_{i=1}^{d} E\big[X_i|X_i \geq G_i^{-1}(\alpha)\big] \approx s.$$

The construction of the optimal coupling in Wang et al. (2011) leading to the worst case dependence in the monotone case is therefore different from the construction in the case of bounded domains.

(iii) *The method to get upper bounds for* $M(s)$ *implies directly also a lower bound for* $P(S>s)$*. Denoted by* H *the conditional moment function associated to the random variable* $-X_i$*, we obtain*

$$P(S>s) = 1 - P((-S) \geq (-s)) \geq H^{-1}(-s).$$

Next, we give an extension of the method of conditional moments in Theorem 2.3 which gives good bounds also in the case of unbounded domains. For $s \geq \mu$ and $t \in [0,1]$, define the function H_t as

$$H_t(t_1) = \frac{1}{t_1 - t}\int_t^{t_1} G(u)\,du = \mathrm{E}[G(U_{[t,t_1]})], \text{ for } t_1 \geq t,$$

where $U_{[t,t_1]}$ denotes a random variable uniformly distributed on the interval $[t,t_1]$. The function H_t is monotonically nondecreasing in t_1 and monotonically non increasing in t. Let $H_t(1) \geq s$ and $G(t) \leq s$. This allows to define $t_1(t) = H_t^{-1}(s)$. If we assume continuity of the F_i, then we get

$$H_t(t_1(t)) = s.$$

Instead of continuity it is enough in the following to postulate existence of $t_1(t)$ such that the sum of the conditional expectations of the rearrangements on $[t, t_1(t)]$ is equal to s. Next, we define the optimal choice of such t's as

$$t_0 = t_0(s) = \inf\big\{t : (G_i|[t,t_1(t)]),\, 1 \leq i \leq d, \text{ are mixing}\big\},$$

that is t_0 is infimum of all the t's such that there exist rearrangements $f_i^t \sim_r G_i|[t,t_1(t)]$ which satisfy

$$\sum_{i=1}^{d} f_i^t = \mathrm{E}\left[\sum_{i=1}^{d} G_i|[t,t_1(t)]\right] = s. \tag{2.8}$$

Under the *mixing assumption* (2.8), there exist some $t \in [0,1]$ such that $t_1(t) > t$ and the random variables $(G_i|[t, t_1(t)])$ are mixing. Therefore, as indicated above, we get random variables $\tilde{V}_i \sim U_{[t,t_1(t)]}$ such that $\sum_{i=1}^d G_i(\tilde{V}_i) = s$. As a consequence, we also get some random variables $\tilde{\tilde{V}}_i \sim U_{[t_1(t),1]}$ with $\sum_{i=1}^d G_i(\tilde{\tilde{V}}_i) \geq s$. Finally, this implies the existence of random variables $V_i \sim U_{[t,1]}$ such that $\sum_{i=1}^d G_i(V_i) \geq s$. As a result, we can state the following theorem.

Theorem 2.5 (Extended conditional moment method) *Let $X_i \sim F_i, 1 \leq i \leq d$ be continuous and assume that the mixing condition* (2.8) *holds. Then, for $s \geq \mu$ we obtain the lower bound*

$$\inf\left\{P\left(\sum_{i=1}^d X_i < s\right) : X_i \sim F_i\right\} \geq t_0(s).$$

Remark 2.6 *The conditional mixing condition* (2.8) *is satisfied in the homogeneous case $F_i = F, 1 \leq i \leq d$ with density f, if f is decreasing on $[t, t_1(t)]$ and if, furthermore, the moderate moment condition*

$$s/n \geq G(t)(1 - 1/n) + 1/nG(t_1(t))$$

holds; see Corollary 2.9 in Wang and Wang (2011). In the particular case where F is concentrated on $[0,1]$ (or on a bounded domain) and $t_1(t_0) = 1$ our Theorem 2.3 implies that $t_0(s)$ is in fact a sharp bound under the moderate moment condition. This interesting sharp result for the monotone case is due to Wang and Wang (2011). An alternative case is obtained when the density of f is unimodal on some admissible interval $[t, t_1(t)]$. In this case, the mixing condition is satisfied, as described in Rüschendorf and Uckelmann (2002). Theorem 2.5 should give very good bounds also in the unbounded case which is confirmed by the results in Wang and Wang (2011). It also shows that the problem of establishing further mixing criteria is of interest.

2.3 Standard Bounds

Standard bounds are a natural generalization to arbitrary dimension d of the sharp bounds in (1.2) in the case $d = 2$. These bounds were derived in several ways in the literature, see Frank et al. (1987); Denuit et al. (1999); Embrechts et al. (2003); [Rü] (2005); and [EP] (2006b).

Theorem 2.7 (Standard Bounds) *Let $X_i \sim F_i, 1 \leq i \leq d$. Then, for any $s \in \mathbb{R}$, we have that*

$$\begin{aligned} &\max\left\{\sup_{\vec{u}\in\mathscr{U}(s)}\left\{F_1(x_1-) + \sum_{i=2}^d F_i(u_i)\right\} - (d-1), 0\right\} \\ \leq \quad & P\left(\sum_{i=1}^d X_i < s\right) \leq \min\left\{\inf_{\vec{u}\in\mathscr{U}(s)}\left\{\sum_{i=1}^d F_i(u_i-)\right\}, 1\right\}, \end{aligned} \tag{2.9}$$

where $\mathscr{U}(s) = \left\{\vec{u} = (u_1, \ldots, u_d) \in \mathbb{R}^d : \sum_{i=1}^d u_i = s\right\}$.

Proofs of (2.9), based on elementary probability, are given in the literature above. We will derive these bounds as a consequence of the following dual representation of $M(t)$ and $m(t)$; see Theorem 5 in [Rü] (1981).

Theorem 2.8 *The problems* (1.1a) *and* (1.1b) *have the following dual counterparts:*

$$M(s) = \inf\left\{ \sum_{i=1}^{d} \int g_i \, dF_i : g_i \text{ bounded}, 1 \le i \le d \text{ with } \sum_{i=1}^{d} g_i(x_i) \ge 1_{[s,+\infty)}\left(\sum_{i=1}^{d} X_i\right) \right\}, \tag{2.10a}$$

$$m(s) = \sup\left\{ \sum_{i=1}^{d} \int f_i \, dF_i : f_i \text{ bounded}, 1 \le i \le d \text{ with } \sum_{i=1}^{d} f_i(x_i) \le 1_{[s,+\infty)}\left(\sum_{i=1}^{d} X_i\right) \right\}. \tag{2.10b}$$

Proof of Theorem 2.7. First, we prove the $\ge$ in (2.9). Arbitrarily choose a vector $(u_1, \dots, u_d) \in \mathbb{R}^d$, such that $\sum_{i=1}^d u_i = s$. Then, define the functions $g_i : \mathbb{R} \to \mathbb{R}$, $\le i \le d$, as

$$g_1(x_1) = \begin{cases} 1, & \text{if } x_1 \ge u_1, \\ 0, & \text{otherwise,} \end{cases} \qquad g_i(x_i) = \begin{cases} 1, & \text{if } x_i > u_i, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } 2 \le i \le d.$$

We prove that the g_i's are an admissible choice in (2.10a). Since the g_i's are non-negative, it is sufficient to prove that, for any vector $(x_1, \dots, x_d) \in \mathbb{R}^d$ with $\sum_{i=1}^d x_i \ge s$, we have that $\sum_{i=1}^d g_i(x_i) \ge 1$. Suppose, instead, that $\sum_{i=1}^d g_i(x_i) < 1$. By definition of the g_i's, this implies that $g_i(x_i) = 0$ and, consequently, that $x_i \le u_i$, $1 \le i \le d$, with the first inequality ($i = 1$) being strict. Summing up all the latter inequalities, we obtain

$$\sum_{i=1}^{d} x_i < \sum_{i=1}^{d} u_i = s,$$

which proves admissibility of the g_i's. Observing that

$$\int_{\{x_1 \ge u_1\}} dF_1 = 1 - F_1(u_1-), \quad \int_{\{x_i > u_i\}} dF_i = 1 - F_i(u_i),$$

and taking the infimum over all $(u_1, \dots, u_d) \in \mathscr{U}(s)$, we obtain

$$P(S \ge s) \le \min\left\{ \inf_{\vec{u} \in \mathscr{U}(s)} \left\{ 1 - F_1(u_1-) + \sum_{i=2}^{d} (1 - F_i(u_i)) \right\}, 1 \right\},$$

which is equivalent to the left-hand side of (2.9), considering the trivial bound $P(S < s) \ge 0$.

The proof for the $\le$ in (2.9) is similar. Note that, analogously to (2.10b), we can write:

$$\begin{aligned} &\sup\{P(X_1 + \cdots + X_d < s) : X_i \sim F_i, 1 \le i \le d\} \\ &= \inf\left\{ \sum_{i=1}^{d} \int f_i dF_i : f_i \text{ bounded}, 1 \le i \le d \text{ with } \sum_{i=1}^{d} f_i(x_i) \ge 1_{(-\infty,s)}\left(\sum_{i=1}^{d} X_i\right) \right\}. \end{aligned} \tag{2.11}$$

Now choose a vector $(u_1,\dots,u_d) \in \mathscr{U}(s)$, and define the functions $f_i : B_i \to \mathbb{R}, 1 \le i \le d$, as

$$f_i = \begin{cases} 1, & \text{if } x_i < u_i, \\ 0, & \text{otherwise.} \end{cases}$$

Then, it is sufficient to prove that the f_i's are an admissible choice in (2.11), and produce the bound

$$P(S < s) \le \min\left\{ \inf_{\vec{u} \in \mathscr{U}(s)} \left\{ \sum_{i=1}^{d} F_i(u_i -) \right\}, 1 \right\}.$$

□

As mentioned in Section 1, in general the standard bounds are sharp only when $d = 2$. For the fast computation of standard bounds, a numerical method is described in Embrechts et al. (2003), while an analytical method is described in [EP] (2006a). We will not enter into the details of these methods, as standard bounds can be improved by using dual bounds. Note that the upper and lower standard bounds in Theorem 2.7 as well as in (1.2) are defined by slight modifications of the infimal and supremal convolution of the F_i.

2.4 Dual bounds

Dual bounds are a way to improve the standard bounds. This method was introduced in [EP] (2006a,b) and is based on the dual representation of the sharp Fréchet bounds (1.1) given in Theorem 2.8. In the homogeneous case $F_i = F, 1 \le i \le d$, this dual representation for $M(s)$ simplifies to

$$M(s) = \inf\left\{ n \int g\, dF : g \text{ bounded with } \sum_{i=1}^{d} g(x_i) \ge 1_{[s,+\infty)}\left(\sum_{i=1}^{d} X_i \right) \right\}. \tag{2.12}$$

Similarly, for $m(s)$; see Gaffke and Rüschendorf (1981) and [EP] (2006b, eq.(4.2)). While the dual representations in (2.10) are difficult to evaluate in general, they allow to establish good bounds obtained by choosing admissible piecewise linear dual functions in (2.10). The resulting bounds are called *dual bounds*.

Theorem 2.9 (Dual Bounds) *Let $X_i \sim F_i$ and $\overline{F}_i = 1 - F_i$ be the survival function of F_i. Then, for any $s \in \mathbb{R}$, we have*

$$M(s) \le \mathrm{D}(s) = \inf_{\vec{u} \in \overline{\mathscr{U}}(s)} \min\left\{ \frac{\sum_{i=1}^{d} \int_{u_i}^{s-\sum_{j\neq i} u_j} \overline{F}_i(t)\,dt}{s - \sum_{i=1}^{d} u_i}, 1 \right\}, \tag{2.13a}$$

$$m(s) \ge \mathrm{d}(s) = \sup_{\vec{u} \in \underline{\mathscr{U}}(s)} \max\left\{ \frac{\sum_{i=1}^{d} \int_{u_i}^{s-\sum_{j\neq i} u_j} \overline{F}_i(t)\,dt}{s - \sum_{i=1}^{d} u_i} - d + 1, 0 \right\}, \tag{2.13b}$$

where $\overline{\mathscr{U}}(s) = \left\{ \vec{u} \in \mathbb{R}^d : \sum_{i=1}^{d} u_i < s \right\}$ *and* $\underline{\mathscr{U}}(s) = \left\{ \vec{u} \in \mathbb{R}^d : \sum_{i=1}^{d} u_i > s \right\}$.

Proof. We give a sketch of the proof of (2.13a). A similar proof has been given in Theorem 3.2 in [EP] (2006a) for the case of bounds on $P(S<s)$. The bound (2.13a) is directly obtained by substituting in (2.10a) the admissible functions $g_i:\mathbb{R}\to\mathbb{R}, 1\le i\le d$, defined as

$$g_i(x_i)=\begin{cases}0, & \text{if } x_i\le u_i,\\ \frac{x_i-u_i}{s-\sum_{i=1}^d u_i}, & \text{if } u_i<x_i\le s-\sum_{j\neq i}u_j,\\ 1, & \text{otherwise,}\end{cases}$$

and taking the infimum over all $\vec{u}\in\overline{\mathscr{U}}(s)$. The proof for (2.13b) is analogous and based on (2.11). In order to obtain (2.13b), it is sufficient to prove that the functions $f_i, 1\le i\le d$, defined as

$$f_i(x_i)=\begin{cases}1, & \text{if } x_i\le s-\sum_{j\neq i}u_j,\\ \frac{u_i-x_i}{\sum_{i=1}^d u_i-s}, & \text{if } s-\sum_{j\neq i}u_j<x_i\le u_i,\\ 0, & \text{otherwise.}\end{cases}$$

are an admissible choice in (2.11), and to take the infimum over all $\vec{u}\in\underline{\mathscr{U}}(s)$. □

In the above proof, if we choose a vector $\vec{u}\in\mathscr{U}(s)$, that is $\sum_{i=1}^d u_i=s$, the piecewise-linear dual admissible choices becomes piecewise-constant and thus yields a standard bound. As a consequence, the dual bounds always improve the corresponding standard bounds. In the homogeneous case $F_i=F, 1\le i\le d$, the dual bounds (2.13) have a much simplified expression which we give in (5.3) below for the case of general overlapping marginals systems.

3 Restrictions on the dependence structure

In this section, we consider some types of restrictions on the dependence structure which reduce the Fréchet class of all possible dependence structure in Section 2 and thus leads to better bounds on the distribution function and on the tail probability of the aggregate risk when this additional information is available.

3.1 Restriction to positive dependent risk vectors

A natural condition which is available in several applications is the condition that the components of the risk vector $\vec{X}$ have some positive dependence structure. This positive dependence structure can be combined with the knowledge of the marginal distributions $F_1,\dots,F_d$ and leads to improved bounds for the distribution function, resp. the tail risk. A simple way to describe positive dependence is by the notion of *positive upper orthant dependence* (PUOD), resp. *positive lower orthant dependence* (PLOD). A random vector $\vec{X}$ is said to be PUOD if

$$\overline{F}_{\vec{X}}(\vec{x})=P\left(\vec{X}\ge\vec{x}\right)\ge\prod_{i=1}^d P(X_i\ge x_i)=\prod_{i=1}^d\overline{F}_i(x_i),\ \text{for all } \vec{x}\in\mathbb{R}^d. \tag{3.1}$$

$\vec{X}$ is said to be PLOD if

$$F_{\vec{X}}(\vec{x}) = P\left(\vec{X} \le \vec{x}\right) \ge \prod_{i=1}^{d} P(X_i \le x_i) = \prod_{i=1}^{d} F_i(x_i), \text{ for all } \vec{x} \in \mathbb{R}^d. \tag{3.2}$$

Moreover, $\vec{X}$ is called *positive quadrant dependent* (PQD) if X is both PUOD and PLOD. These notions belong to the earliest notions of positive dependence in probability. They were introduced in Lehmann (1966) and supplement and substitute the more classical notion of positive linear correlation.

Conditions (3.1) and (3.2) can be interpreted by saying that the distribution function $F_{\vec{X}}$, resp. the survival function $\overline{F}_{\vec{X}}$, is dominated by the corresponding distribution, resp. the corresponding tail function, of the product measure. We can consider them as partial ordering relations $\le_{\text{PUOD}}, \le_{\text{PLOD}}, \le_{\text{PQD}}$ on the class of probability measures on $\mathbb{R}^d$. In this sense, we can consider more general positive dependence bounds by assuming that $F_{\vec{X}} \le G$ where $\le$ is any of the above dependence orderings and G is any other distribution on $\mathbb{R}^d$. The following typically strong improvements of the risk bounds of the joint portfolio under the additional positive dependence restrictions have been given in several similar forms in the literature; see Williamson and Downs (1990); Embrechts et al. (2003); [Rü] (2005); [EP] (2006b).

Recall that $\mathscr{U}(s) = \left\{\vec{u} = (u_1, \dots, u_d) \in \mathbb{R}^d : \sum_{i=1}^d u_i = s\right\}$. For a d-dimensional distribution function G, we define the generalized G-infimal and G-supremal convolutions as

$$\bigwedge G(s) = \inf_{\vec{u} \in \mathscr{U}(s)} G(\vec{u}) \text{ and } \bigvee G(s) = \sup_{\vec{u} \in \mathscr{U}(s)} G(\vec{u}).$$

In case $G(\vec{u}) = \prod_{i=1}^d F_i(u_i)$ is the distribution of the product measure, we obtain the usual infimal and supremal convolutions

$$\bigwedge G(s) = \bigwedge\left(\prod_{i=1}^{d} F_i\right)(s), \quad \bigvee G(s) = \bigvee\left(\prod_{i=1}^{d} F_i\right)(s).$$

Theorem 3.1 (Positive dependence restriction) *Let $\vec{X}$ be a d-dimensional risk vector having marginals F_i, $1 \le i \le d$ and let G be a d-dimensional distribution function.*

a) If $G \le_{PLOD} F_{\vec{X}}$, then

$$P\left(\sum_{i=1}^{d} X_i \le s\right) \ge \bigvee G(s); \tag{3.3}$$

b) If $G \le_{PUOD} F_{\vec{X}}$, then

$$P\left(\sum_{i=1}^{d} X_i < s\right) \le 1 - \bigvee \overline{G}(s); \tag{3.4}$$

c) If $\vec{X}$ is PQD then

$$\bigvee\left(\prod_{i=1}^{d} F_i\right)(s) \le P\left(\sum_{i=1}^{d} X_i \le s\right), \quad P\left(\sum_{i=1}^{d} X_i < s\right) \le 1 - \bigvee\left(\prod_{i=1}^{d} \overline{F}_i\right)(s). \tag{3.5}$$

Proof.

a) If $G \leq_{\text{PLOD}} F_{\vec{X}}$, that is $G(\vec{x}) \leq F_{\vec{X}}(\vec{x})$ for all $\vec{x} \in \mathbb{R}^d$, then, for all $\vec{u} \in \mathscr{U}(s)$, we have

$$P\left(\sum_{i=1}^{d} X_i \leq s\right) \geq P(X_1 \leq u_1, \ldots, X_n \leq u_n) = F_{\vec{X}}(\vec{u}) \geq G(\vec{u}),$$

which implies (3.3).

b) If $G \leq_{\text{PUOD}} F_{\vec{X}}$, that is $\overline{F}_{\vec{X}}(\vec{x}) \geq \overline{G}(\vec{x})$ for all $\vec{x} \in \mathbb{R}^d$, then, for all $\vec{u} \in \mathscr{U}(s)$, we have

$$\begin{aligned} P\left(\sum_{i=1}^{d} X_i < s\right) &= 1 - P\left(\sum_{i=1}^{d} X_i \geq s\right) \\ &\leq 1 - P(X_1 \geq u_1, \ldots, X_n \geq u_n) = 1 - \overline{F}_{\vec{X}}(\vec{u}) \leq 1 - \overline{G}(\vec{u}), \end{aligned}$$

which implies (3.4).

c) is a consequence of a) and b) when $G = \prod F_i$ is the product measure. □

Remark 3.2 1. *Similarly to the standard bounds* (2.9)*, the bounds in Theorem 3.1 are in general sharp only when* $d = 2$*. However, they substantially improve the corresponding standard bounds* (2.9) *in the case of restriction to the positive dependence scenarios a), b) and c). It will be of considerable interest in the applications to exploit these substantially sharpened bounds, when positive dependence is present in the risk model.*

2. *As it is clear from the proof of Theorem 3.1, the functions* G *and* $\overline{G}$ *bounding the distribution* F *and, resp. the survival function* $\overline{F}$*, do not need in general to be distribution functions. It is enough to assume that* G *and* $\overline{G}$ *are nondecreasing functions. Using the Hoeffding-Fréchet bounds, we can generally assume that*

$$\max\left\{\sum_{i=1}^{d} F_i(x_i) - (d-1), 0\right\} \leq G(\vec{x}) \;\; \textit{and} \;\; \overline{G}(\vec{x}) \geq \max\left\{\sum_{i=1}^{d} \overline{F}_i(x_i) - (d-1), 0\right\}.$$

3. *In the case of trivial upper and lower bounds* $G, \overline{G}$ *the bounds in* (3.3), (3.4), (3.5) *reduce to the standard bounds in Theorem 2.7. In comparison to* (2.9)*, for the lower bound we consider here the* $\leq$ *case instead of the* $<$ *case there.*

3.2 General systems of marginals

In this section, we consider the case that not only the one dimensional marginal distributions of the risk vector are known, but also that for a class $\mathscr{E}$ of sets $J \subset \{1, \ldots, d\}$, we know the joint marginal distributions $F_J, J \in \mathscr{E}$. In this case, we get the generalized Fréchet class

$$\mathfrak{F}_{\mathscr{E}} = \mathfrak{F}(F_J, J \in \mathscr{E})$$

of all probability measures on $\mathbb{R}^d$ having marginals F_J on $\mathbb{R}^J$, for all $J \in \mathscr{E}$. W.l.o.g. we assume that $\bigcup_{J\in\mathscr{E}} J = \{1,\dots,d\}$. Thus, we have

$$\mathfrak{F}_{\mathscr{E}} \subset \mathfrak{F}(F_1,\dots,F_d),$$

that is $\mathfrak{F}_{\mathscr{E}}$ is a subclass of the class of all possible dependence structures. The knowledge of joint distributions restricts the class of possible dependence structures and thus leads to improved bounds for the distribution function or for the tail risk of the joint portfolio.

In order that the notion $\mathfrak{F}_{\mathscr{E}}$ makes sense we have to assume *consistency* of the marginal system $\mathfrak{F}_{\mathscr{E}}$, that is $J_1, J_2 \in \mathscr{E}, J_1 \cap J_2 \neq \emptyset$ implies that

$$\pi_{J_1\cap J_2} F_{J_1} = \pi_{J_1\cap J_2} F_{J_2},$$

where π_J are the projections on the components in J.

Consistency of $F_J, J \in \mathscr{E}$, is a necessary condition to guarantee that the Fréchet class $\mathfrak{F}_{\mathscr{E}}$ is non-empty. It has been shown by Vorob'ev (1962) and Kellerer (1964) that, when $\mathscr{E}$ is *regular* (it does not have *cycles*), consistency is enough to imply non emptiness. When $\mathscr{E}$ is *non-regular* as, for instance, $\mathscr{E} = \{\{1,2\},\{2,3\},\{1,3\}\}$, the Fréchet class $\mathfrak{F}_{\mathscr{E}}$ may be empty even with consistent marginals. Some relevant particular classes of marginal systems are:

- the *simple* system $\mathscr{E} = \{\{1\},\dots,\{d\}\}$,defining the regular Fréchet class
$$\mathfrak{F}(F_1,\dots,F_d),$$
- the *star-like* system $\mathscr{E}^\star = \{\{1, j\}, j = 2,\dots,d\}$, defining the regular Fréchet class
$$\mathfrak{F}(F_{12},F_{13},\dots,F_{1d}),$$
- the *series* system $\mathscr{E}^s = \{\{j, j+1\}, j = 1,\dots,d-1\}$, defining the regular Fréchet class
$$\mathfrak{F}(F_{12},F_{23},\dots,F_{d-1\,d}),$$
- the *pairwise* system $\mathscr{E}^p = \{\{i, j\}, 1 \le i < j \le d\}$, defining the non-regular Fréchet class
$$\mathfrak{F}(F_{ij}, 1 \le i < j \le d).$$

Our aim is to find good bounds for the tail risks

$$M_{\mathscr{E}}(s) = \sup\left\{P(X_1+\cdots+X_d \ge s) : F_{\vec{X}} \in \mathfrak{F}_{\mathscr{E}}\right\}, \tag{3.6a}$$

$$m_{\mathscr{E}}(s) = \inf\left\{P(X_1+\cdots+X_d \ge s) : F_{\vec{X}} \in \mathfrak{F}_{\mathscr{E}}\right\}, \tag{3.6b}$$

which improve the corresponding bounds $M(s)$ and $m(s)$ defined for the simple marginal system in (1.1).

A simple case is obtained when the marginals are multivariate but *non-overlapping*, that is $\mathscr{E} = \{J_1,\dots,J_n\}$ with $\cup_{r=1}^n J_r = \{1,\dots,d\}$ and $J_r \cap J_i = \emptyset$ for $r \neq i$. Then, for a risk vector $\vec{X}$ with $F_{\vec{X}} \in \mathfrak{F}_{\mathscr{E}}$, we define the random variables $Y_r = \sum_{i\in J_r} X_i$

and denote by H_r their distribution functions, for $1 \le r \le n$. Considering the corresponding Fréchet class $\mathfrak{H} = \mathfrak{F}(H_1, \dots, H_n)$, we get a complete reduction to the simple marginal case. Define the Fréchet problems

$$M_{\mathfrak{H}}(s) = \sup\left\{P(Y_1 + \cdots + Y_n \ge s) : F_{\vec{Y}} \in \mathfrak{F}(H_1, \dots, H_n)\right\},$$
$$m_{\mathfrak{H}}(s) = \inf\left\{P(Y_1 + \cdots + Y_n \ge s) : F_{\vec{Y}} \in \mathfrak{F}(H_1, \dots, H_n)\right\}.$$

Proposition 3.3 (Non-overlappingmultivariate marginals) *In the case of a non-overlapping, multivariate marginals class $\mathcal{E}$, we have that, for all $s \in \mathbb{R}$,*

$$M_{\mathcal{E}}(s) = M_{\mathfrak{H}}(s), m_{\mathcal{E}}(s) = m_{\mathfrak{H}}(s).$$

Proof. For the proof note that for $F_X \in \mathfrak{F}_{\mathcal{E}}$ we have $F_{Y_r} = H_r$ and thus $F_Y \in \mathfrak{H}$ where $Y = (Y_1, \dots, Y_r)$. This implies that $M_{\mathcal{E}}(s) \le M_{\mathfrak{H}}(s)$. Conversely, if $Y = (Y_1, \dots, Y_r)$ is any vector with d.f. $F_y \in \mathfrak{H}$, then by a classical result on stochastic equations there exist X_{J_i} with $X_{J_i} \sim F_{J_i}$ and $\sum_{j \in J_r} X_j = Y_i$ a.s. $1 \le i \le r$. This implies the converse inequality $M_{\mathfrak{H}}(s) \le M_{\mathcal{E}}(s)$. The case of $m_{\mathcal{E}}(s)$ is similar. □

For some classes of marginal systems $\mathcal{E}$, the 'conditioning method' was introduced in [Rü] (1991); see also Joe (1997). This method is useful to obtain improved (sharp) bounds in the case of the basic series system $\mathcal{E}_3^s$ and in the case of the star-like system $\mathcal{E}^\star$.

Let $F_{i|x_1}$ denote the conditional distribution of X_i given $X_1 = x_1$ and define

$$M_{2,\dots,d|x_1}(s) = \sup\{P(X_2 + \cdots + X_d \ge s) : X_i \sim F_{i|x_1}, 2 \le i \le d\},$$
$$m_{2,\dots,d|x_1}(s) = \inf\{P(X_2 + \cdots + X_d \ge s) : X_i \sim F_{i|x_1}, 2 \le i \le d\}$$

the Fréchet bounds for the $(d-1)$- dimensional simple marginal system where the marginals are given by the conditional distributions $F_{i|x_1}, 2 \le i \le d$. The following result in [Rü] (1991) states that a complete reduction of the star-like system to the simple marginal system is justified.

Theorem 3.4 (Star-like system) *Let $\vec{X}$ a risk vector with distribution $F_{\vec{X}} \in \mathfrak{F}_{\mathcal{E}^\star}$. Then, for any $s \in \mathbb{R}$, we have that*

$$M_{\mathcal{E}^\star}(s) = \int M_{2,\dots,d|x_1}(s - x_1)\, dF_1(x_1), \tag{3.7a}$$

$$m_{\mathcal{E}^\star}(s) = \int m_{2,\dots,d|x_1}(s - x_1)\, dF_1(x_1). \tag{3.7b}$$

The above conditional bounds have been combined with the standard bounds defined in (2.9) for the simple marginal system $\mathfrak{F}(F_{2|x_1}, \dots, F_{d|x_1})$ in [EP] (2010a) and applied to some examples. Based on the fact that the standard bounds are sharp for two-dimensional vectors, it has been shown in [Rü] (1991) that the conditional bounds (3.7) imply explicit sharp bounds for $d = 3$.

For the series system $\mathcal{E}^s = \mathcal{E}_3^s$ of marginals a reduction of the generalized Fréchet problems $M_{\mathcal{E}^s}$ and $m_{\mathcal{E}^s}$ to a Fréchet problem with simple marginals was given in [EP]

(2010a). In contrast to the conditioning method for the star-like case, this reduction does not lead to sharp bounds but to reasonable good bounds. In the following, we extend this reduction method to general (regular or non-regular) marginal systems. In consequence, we get reasonable good bounds for any system of marginal information whether overlapping or non-overlapping.

Let $\mathscr{E} = \{J_1,\dots,J_n\}$ be any marginal system with corresponding Fréchet class $\mathfrak{F}_{\mathscr{E}} = \mathfrak{F}(F_J, J \in \mathscr{E})$. Let $\eta_i = \#\{J_r \in \mathscr{E} : i \in J_r\}$ denote the number of the sets $J \in \mathscr{E}$ in which the index i appears. We associate to the risk vector $\vec{X}$ with $\mathfrak{F}_{\vec{X}} \in \mathfrak{F}_{\mathscr{E}}$ the random variables Y_r defined by

$$Y_r = \sum_{i \in J_r} \frac{X_i}{\eta_i},\ r = 1,\dots,n. \tag{3.8}$$

Let H_r be the distribution of Y_r, and let $\mathfrak{H}$ denote the simple Fréchet class $\mathfrak{H} = \mathfrak{F}(H_1,\dots,H_n)$. Recall that

$$\begin{aligned} M_{\mathfrak{H}} &= \sup\left\{P(Y_1 + \cdots + Y_n \geq s) : F_{\vec{Y}} \in \mathfrak{F}(H_1,\dots,H_n)\right\}, \\ m_{\mathfrak{H}} &= \inf\left\{P(Y_1 + \cdots + Y_n \geq s) : F_{\vec{Y}} \in \mathfrak{F}(H_1,\dots,H_n)\right\}. \end{aligned}$$

Then, we obtain the following reduction result by reducing the Fréchet problems $M_{\mathscr{E}}, m_{\mathscr{E}}$ to the simple Fréchet class problems $M_{\mathfrak{H}}, m_{\mathfrak{H}}$.

Theorem 3.5 (Reduction to simple Fréchet classes) *Let $\mathscr{E} = \{J_1,\dots,J_n\}$ be an arbitrary consistent marginal system with corresponding Fréchet class $\mathfrak{F}_{\mathscr{E}} = \mathfrak{F}(F_J, J \in \mathscr{E})$. Then, for any $s \in \mathbb{R}$, we have that*

$$M_{\mathscr{E}}(s) \leq M_{\mathfrak{H}}(s),\ \textit{and}\ m_{\mathscr{E}}(s) \geq m_{\mathfrak{H}}(s). \tag{3.9}$$

Proof. Since the distribution $F_{\vec{X}}$ of the vector $\vec{X}$ belongs to $\mathfrak{F}_{\mathscr{E}} = \mathfrak{F}(F_J, J \in \mathscr{E})$, the distribution $H_{\vec{Y}}$ of $\vec{Y}$ belongs to the simple Fréchet class $\mathfrak{F}(H_1,\dots,H_n)$. Using that

$$\sum_{i=1}^{d} X_i = \sum_{r=1}^{n} \sum_{i \in J_r} \frac{X_i}{\eta_i} = \sum_{r=1}^{n} Y_r, \tag{3.10}$$

we obtain

$$\begin{aligned} M_{\mathscr{E}}(s) &= \sup\left\{P(X_1 + \cdots + X_d \geq s) : F_{\vec{X}} \in \mathfrak{F}_{\mathscr{E}}\right\} \\ &= \sup\left\{P(Y_1 + \cdots + Y_n \geq s) : F_{\vec{X}} \in \mathfrak{F}_{\mathscr{E}}\right\}. \\ &\leq \sup\left\{P(Y_1 + \cdots + Y_n \geq s) : F_{\vec{Y}} \in \mathfrak{H}\right\} = M_{\mathfrak{H}}(s), \\ m_{\mathscr{E}}(s) &= \inf\left\{P(X_1 + \cdots + X_d \geq s) : F_{\vec{X}} \in \mathfrak{F}_{\mathscr{E}}\right\} \\ &= \inf\left\{P(Y_1 + \cdots + Y_n \geq s) : F_{\vec{X}} \in \mathfrak{F}_{\mathscr{E}}\right\}, \\ &\geq \inf\left\{P(Y_1 + \cdots + Y_n \geq s) : F_{\vec{Y}} \in \mathfrak{H}\right\} = m_{\mathfrak{H}}(s). \end{aligned}$$

□

As a consequence of the standard and dual bounds given in Section 2 , we now obtain the following standard and dual bounds for general marginal systems.

Theorem 3.6 (Reduced standard bounds) *Let $F_J, J \in \mathscr{E}$, with $\mathscr{E} = \{J_1,\dots,J_n\}$, be a consistent system of marginals generating a nonempty Fréchet class $\mathfrak{F}_{\mathscr{E}}$. Let $\vec{X} = (X_1,\dots,X_d)$ be a random vector having distribution function $F_{\vec{X}}$ in $\mathfrak{F}_{\mathscr{E}}$. For a given threshold $s \in \mathbb{R}$, define the set $\mathscr{U}(s) \subset \mathbb{R}^n$ as*

$$\mathscr{U}(s) = \left\{ \vec{u} = (u_1,\dots,u_n) \in \mathbb{R}^n : \sum_{r=1}^{n} u_r = s \right\}.$$

Then, we have

$$M_{\mathscr{E}}(s) \le M_{\mathfrak{H}}(s) \le \mathrm{S}_{\mathfrak{H}}(s) = \inf_{\vec{u}\in\mathscr{U}(s)} \left\{ \min\left\{ \overline{H}_1(u_1^-) + \sum_{r=2}^{n} \overline{H}_r(u_r), 1 \right\} \right\}, \tag{3.11a}$$

$$m_{\mathscr{E}}(s) \ge m_{\mathfrak{H}}(s) \ge \mathrm{s}_{\mathfrak{H}}(s) = \sup_{\vec{u}\in\mathscr{U}(s)} \left\{ \max\left\{ \sum_{r=1}^{n} \overline{H}_r(u_r^-) - n + 1, 0 \right\} \right\}. \tag{3.11b}$$

Theorem 3.7 (Reduced dual bounds) *Under the same assumptions of Theorem 3.6, we have*

$$M_{\mathscr{E}}(s) \le M_{\mathfrak{H}}(s) \le \mathrm{D}_{\mathfrak{H}}(s) = \inf_{\vec{u}\in\overline{\mathscr{U}}(s)} \min\left\{ \frac{\sum_{r=1}^{n} \int_{u_r}^{s-\sum_{j\neq r} u_j} \overline{H}_r(t)\,dt}{s - \sum_{r=1}^{n} u_r}, 1 \right\}, \tag{3.12a}$$

$$m_{\mathscr{E}}(s) \ge m_{\mathfrak{H}}(s) \ge \mathrm{d}_{\mathfrak{H}}(s) = \sup_{\vec{u}\in\underline{\mathscr{U}}(s)} \max\left\{ \frac{\sum_{r=1}^{n} \int_{u_r}^{s-\sum_{j\neq r} u_j} \overline{H}_r(t)\,dt}{s - \sum_{r=1}^{n} u_r} - n + 1, 0 \right\}, \tag{3.12b}$$

where $\overline{\mathscr{U}}(s) = \left\{ \vec{u} \in \mathbb{R}^n : \sum_{r=1}^{n} u_r < s \right\}$ and $\underline{\mathscr{U}}(s) = \left\{ \vec{u} \in \mathbb{R}^n : \sum_{r=1}^{n} u_r > s \right\}$.

In case $\mathscr{E} = \{\{1\},\dots,\{d\}\}$ is the simple system of marginals, in the above theorems we have $n = d$ and $H_i = F_i, 1 \le i \le d$. Thus, the reduced bounds (3.11) and (3.11) are equivalent forms of the standard, resp. dual bounds given in Section 2. Moreover, since the standard bounds are particular cases of the dual ones, we have that

$$\mathrm{D}_{\mathfrak{H}}(s) \le \mathrm{S}_{\mathfrak{H}}(s) \text{ and } \mathrm{d}_{\mathfrak{H}}(s) \ge \mathrm{s}_{\mathfrak{H}}(s), \text{ for all } s \in \mathbb{R}.$$

4 Improving the reduced bounds by generalized weighting schemes

In this section, we improve the bounds introduced in Section 3.2 for an overlapping system of marginals. It is possible that the two inequalities in (3.9) are strict, implying that

$$\mathrm{s}_{\mathfrak{H}}(s) \le \mathrm{d}_{\mathfrak{H}}(s) \le m_{\mathfrak{H}}(s) < m_{\mathscr{E}}(s), \tag{4.1}$$

$$M_{\mathscr{E}}(s) < M_{\mathfrak{H}}(s) \le \mathrm{D}_{\mathfrak{H}}(s) \le \mathrm{S}_{\mathfrak{H}}(s). \tag{4.2}$$

In order to reduce the gap between the sharp bounds $m_{\mathscr{E}}(s)$ and $M_{\mathscr{E}}(s)$ and their reduced standard and dual counterparts, one can parameterize the Fréchet class $\mathfrak{H}$ as follows. Let $\vec{\alpha} = \{\alpha_i^r \in [0,1], i = 1,\dots,d, r = 1,\dots,n\}$ be a set of weighting parameters such that

$$\alpha_i^r > 0 \text{ if and only if } i \in J_r, \text{ and } \sum_{r=1}^{n} \alpha_i^r = 1. \tag{4.3}$$

The main idea is to choose for the reduction to simple marginal classes an optimal system of weights putting different weights on the components within their groups J_i. Then, for $1 \le r \le n$, define the random variables $Y_r^{\vec{\alpha}}$ as

$$Y_r^{\vec{\alpha}} = \sum_{i=1}^{d} \alpha_i^r X_i.$$

Hence, we have:

$$\sum_{r=1}^{n} Y_r^{\vec{\alpha}} = \sum_{r=1}^{n}\sum_{i=1}^{d} \alpha_i^r X_i = \sum_{i=1}^{d}\sum_{r=1}^{n} \alpha_i^r X_i = \sum_{i=1}^{d} X_i \sum_{r=1}^{n} \alpha_i^r = \sum_{i=1}^{d} X_i. \tag{4.4}$$

We denote by $H_r^{\vec{\alpha}}$ the distribution of the random variable $Y_r^{\vec{\alpha}}$. Since $\alpha_i^r > 0$ if and only if $i \in J_r$, the random variable $Y_r^{\vec{\alpha}}$ depends only on the X_i whose index i appears within the same J_r. Considering also that $F_{\vec{X}} \in \mathfrak{F}(F_J, J \in \mathscr{E})$, the marginals of the random vector $\vec{Y}^{\vec{\alpha}} = (Y_1^{\vec{\alpha}},\dots,Y_n^{\vec{\alpha}})$ turn out to be fixed. Therefore, the joint distribution $H_{\vec{Y}}^{\vec{\alpha}}$ of $\vec{Y}^{\vec{\alpha}}$ belongs to the simple Fréchet class $\mathfrak{H}^{\vec{\alpha}} = \mathfrak{F}(H_1^{\vec{\alpha}},\dots,H_n^{\vec{\alpha}})$ and we obtain a reduction result analogous to Theorem 3.5.

Theorem 4.1 (Weighting scheme bounds) *Let $\mathscr{E} = \{J_1,\dots,J_n\}$ be an arbitrary consistent marginal system with corresponding Fréchet class $\mathfrak{F}_{\mathscr{E}} = \mathfrak{F}(F_J, J \in \mathscr{E})$. For $\vec{\alpha}$ satisfying conditions* (4.3)*, define the simple Fréchet bounds*

$$M_{\mathfrak{H}}^{\vec{\alpha}}(s) = \sup\left\{P(Y_1^{\vec{\alpha}} + \cdots + Y_n^{\vec{\alpha}} \ge s) : F_{\vec{Y}} \in \mathfrak{H}^{\vec{\alpha}}\right\},$$
$$m_{\mathfrak{H}}^{\vec{\alpha}}(s) = \inf\left\{P(Y_1^{\vec{\alpha}} + \cdots + Y_n^{\vec{\alpha}} \ge s) : F_{\vec{Y}} \in \mathfrak{H}^{\vec{\alpha}}\right\}.$$

Then, for any $s \in \mathbb{R}$, we have

$$M_{\mathscr{E}}(s) \le M_{\mathfrak{H}}^{\vec{\alpha}}(s) \text{ and } m_{\mathscr{E}}(s) \ge m_{\mathscr{E}_n}^{\vec{\alpha}}(s). \tag{4.5}$$

By applying Theorems 3.6 and 3.7 to the Fréchet class $\mathfrak{H}^{\vec{\alpha}}$, we define the reduced standard bounds $\mathrm{S}_{\mathfrak{H}}^{\vec{\alpha}}(s)$, $\mathrm{s}_{\mathfrak{H}}^{\vec{\alpha}}(s)$, and the reduced dual bounds $\mathrm{D}_{\mathfrak{H}}^{\vec{\alpha}}(s)$, $\mathrm{d}_{\mathfrak{H}}^{\vec{\alpha}}(s)$. These bounds are obtained by replacing, in equations (3.11) and (3.12), the marginals H_r's by the parameterized marginals $H_r^{\vec{\alpha}}$. Note that, choosing $\alpha_i^r = 1/\eta_i$, $i \in J_r, 1 \le r \le n$, these parameterized reduced standard and dual bounds coincide with the ones given in (3.11) and (3.12).

To summarize, for any fixed set of parameters $\alpha_i^r \in [0,1]$ satisfying (4.3), the following inequalities hold:

$$\mathrm{s}_{\mathfrak{H}}^{\vec{\alpha}}(s) \le \mathrm{d}_{\mathfrak{H}}^{\vec{\alpha}}(s) \le m_{\mathscr{E}}(s),$$
$$M_{\mathscr{E}}(s) \le \mathrm{D}_{\mathfrak{H}}^{\vec{\alpha}}(s) \le \mathrm{S}_{\mathfrak{H}}^{\vec{\alpha}}(s).$$

At this point, one can look for the $\vec{\alpha}$ yielding the best-possible parameterized dual bound $\mathrm{d}^*(s)$ and $\mathrm{D}^*(s)$, defined as

$$\mathrm{d}^*(s) = \sup_{\vec{\alpha}\in\mathfrak{A}} \mathrm{d}_{\mathfrak{H}}^{\vec{\alpha}}(s) \quad \text{and} \quad \mathrm{D}^*(s) = \inf_{\vec{\alpha}\in\mathfrak{A}} \mathrm{D}_{\mathfrak{H}}^{\vec{\alpha}}(s). \tag{4.6}$$

where $\mathfrak{A} \subset \mathbb{R}^n \times \mathbb{R}^d$ denotes the set of admissible choice of the parameters $\vec{\alpha}$. In the following section, we give some examples where we evaluate the bounds in (4.6).

5 Applications and numerical results

In this section, we compute numerically the standard bounds (3.11) and the dual bounds (3.12) for systems of homogeneous and inhomogeneous marginals. First, we give some remarks concerning the optimization problems which define the bounds.

In the following, we assume that each random component X_i of the vector $\vec{X}$ is a.s. bounded from below by some constant L', that is $F_i(L') = 0$, for all $i = 1,\dots,d$. In quantitative risk management, for example, the vector $\vec{X}$ typically represents a portfolio of non-negative random losses, that is $L = 0$. As a result, we can reduce the feasible set in the calculation of the standard bounds (3.11).

Lemma 5.1 *Under the assumption of Theorem 3.6, assume that each random component X_i of the vector $\vec{X}$ is a.s. bounded from below by L', for some real $L' < s$. Hence, the random variables Y_r, as defined in* (3.8)*, are bounded from below by L and, in the computation of the standard bounds* (3.11)*, we can reduce to considering the feasible set*

$$\mathscr{U}(s) = \left\{ \vec{u} = (u_1,\dots,u_n) \in [L, s-(n-1)L]^n : \sum_{r=1}^{n} u_r = s \right\}. \tag{5.1}$$

Proof. It is sufficient to notice that the argument of the inf in (3.11a) is always equal to 1 when $u_r < L$, for some $r = 1,\dots,n$. Then, we can restrict to considering $u_r \geq L$ and, from $\sum_{r=1}^{n} u_r = s$, $u_r \leq s-(n-1)L, r = 1,\dots,n$. Analogous restrictions follow for (3.11b) by observing that the argument of the sup in (3.11b) is decreasing when $u_r < L$, for some $r = 1,\dots,n$. □

As standard bounds are particular cases of dual bounds, the solutions to (3.12) are to be searched in the interior of the domains $\overline{\mathscr{U}}(s)$ and $\underline{\mathscr{U}}(s)$. As a consequence, the dual bounds are attained at a stationary point in the interior of these domains.

5.1 A first application: homogeneous marginals

When the random variables $Y_r \sim H_r$ defined in (3.8) are all identically distributed and non-negative, it is possible to reduce the d-dimensional optimization problem (3.11) and (3.12) to one-dimensional problems. Assume, then, that $H_r = H, 1 \leq r \leq n$ and that $L = 0$. Substituting $L = 0$ in the simplified expression of $\mathscr{U}(s)$ obtained in Lemma 5.1, using the homogeneity of the marginals and the symmetry of the problem, a list of candidates to be solutions of (3.11a) is given by the n vectors

$\vec{u}^k = (u_1^k, \dots, u_n^k) \in \mathbb{R}^n, k = 1, \dots, n$ where

$$u_r^k = s/k, \text{ for } 1 \le r \le k,$$
$$u_r^k = 0, \text{ for } k+1 \le r \le n.$$

For instance, we have $\vec{u}^1 = (s, 0, \dots, 0), \vec{u}^2 = (s/2, s/2, \dots, 0)$, $\vec{u}^n = (s/n, s/n, \dots, s/n)$. If we restrict to considering only the vectors $\vec{u}^k$, the inequalities (3.11) are maintained, that is we have

$$\mathrm{S}_{\mathfrak{H}}(s) \le \mathrm{S}'_{\mathfrak{H}}(s) = \min_{k=1,\dots,n} \left\{ \min\left\{ \sum_{r=1}^n \overline{H}(u_r^k), 1 \right\} \right\} = \min\left\{ n\overline{H}(s/n), 1 \right\}, \tag{5.2a}$$

$$\mathrm{s}_{\mathfrak{H}}(s) \ge \mathrm{s}'_{\mathfrak{H}}(s) = \max_{k=1,\dots,n} \left\{ \max\left\{ \sum_{r=1}^n \overline{H}(u_r^k) - n + 1, 0 \right\} \right\} = \max_{k=1,\dots,n} \left\{ k\overline{H}(s/k) - n + 1, 0 \right\}. \tag{5.2b}$$

Within this section, we will compute the simplified reduced standard bounds $\mathrm{S}'_{\mathfrak{H}}(s)$ and $\mathrm{s}'_{\mathfrak{H}}(s)$. In some specific cases of non-negative homogeneous marginals, we have that $\mathrm{S}'_{\mathfrak{H}}(s) = \mathrm{S}_{\mathfrak{H}}(s)$ and $\mathrm{s}'_{\mathfrak{H}}(s) = \mathrm{s}_{\mathfrak{H}}(s)$; see Theorem 3.1 and Lemma 3.1 in [EP] (2006a).

With respect to the computation of dual bounds in the case of homogeneous marginals, using (2.12) we can reduce the list of candidates to be solutions of (3.12a) to all the vectors $\vec{u} = (u, \dots, u)$ with $u < s/n$. Analogously, for (3.12b). Similarly to the case of standard bounds, we obtain the much simplified bounds, given by an optimization problem in one real parameter u instead of d parameters in the inhomogeneous case

$$\mathrm{D}_{\mathfrak{H}}(s) \le \mathrm{D}'_{\mathfrak{H}}(s) = \inf_{u < s/n} \min\left\{ \frac{n\int_u^{s-(n-1)u} \overline{H}(t)dt}{s - nu}, 1 \right\}, \tag{5.3a}$$

$$\mathrm{d}_{\mathfrak{H}}(s) \ge \mathrm{d}'_{\mathfrak{H}}(s) = \sup_{u > s/n} \max\left\{ \frac{n\int_u^{s-(n-1)u} \overline{H}(t)dt}{s - nu} - n + 1, 0 \right\}. \tag{5.3b}$$

In the following applications, we compute the reduced dual bounds $\mathrm{D}'_{\mathfrak{H}}(s)$ and $\mathrm{d}'_{\mathfrak{H}}(s)$.

As a first applications, we choose the star-like system $\mathscr{E}^\star = \{\{1,2\}, \dots, \{1,d\}\}$, for which $n = d - 1$. This system has been studied in [EP] (2010a), where the standard bounds (3.7) are computed based on the conditional representation of $m_{\mathscr{E}^\star}$ and $M_{\mathscr{E}^\star}$. We will show that, for the same system of marginals, the upper dual bounds (5.3a) are better than these standard bounds. We choose the same marginals as in [EP] (2010a), that is we assume the bivariate distributions $F_{1r}, r = 2, \dots, d$ to be identical and generated by coupling two Pareto marginals having tail parameter $\theta > 0$ by a Frank copula with parameter $\delta \ne 0$. Under these assumptions, the $(d-1)$ random variables Y_r have all distribution

$$H(x) = P(Y_r \le x) = P\left(\frac{X_1}{n} + X_i \le x \right), i = 2, \dots, d.$$

Here, we have that

$$H(x) = \int_0^{nx} P(X_i \le x - x_1/n | X_1 = x_1)\, dF_1(x_1) = \int_0^{nx} F_{i|x_1}(x - x_1/n)\, dF_1(x_1),$$

where we denote by $F_{i|x_1}(u) = F_i(u - x_1/n)$ the distribution of the random variable $(X_i|X_1 = x_1)$, for $2 \le i \le d$. Within this framework, the conditional distribution function $F_{i|x_1}$ is available in closed form and it is identical to the function $G_{x_1}(x)$ given in [EP] (2010a, p. 181).

In Figure 1, we plot the reduced upper standard bounds $\mathrm{S}'_{\mathfrak{H}}$ (see (5.2a)), the standard bounds deriving from (3.7) and the reduced dual bounds $\mathrm{D}'_{\mathfrak{H}}$ (see (5.3a)) on $M_{\mathscr{E}_d^\star}$ for a random vector of $d = 4$ Pareto(2)-distributed risks under the star-like marginal system described above. The parameter of the Frank copula is set to $\delta = 2$. The dual bounds improve any standard bounds at any real threshold s, coherently with the fact that, for any threshold s, we have

$$M_{\mathscr{E}^\star}(s) \le M_{\mathfrak{H}}(s) \le \mathrm{D}'_{\mathfrak{H}}(s) \le \mathrm{S}'_{\mathfrak{H}}(s).$$

In Figure 1, we also provide numerical ranges for the bounds $M_{\mathfrak{H}}(s)$, at some thresholds s of interest. These ranges have been calculated by applying the rearrangement algorithm introduced in [PR] (2011) for the simple Fréchet class $\mathfrak{H}$. This algorithm, is more accurate than the linear program algorithms in Section 5 of [EP] (2010a). The numerical ranges show that the reduced dual bounds (5.3) are accurate. The calculation of dual bounds in a homogeneous setting is independent from the dimension d of the risk vector, and computationally much easier than the rearrangement algorithm. These results suggest to use (reduced) dual bounds for homogeneous settings.

5.2 A second application: Inhomogeneous star-like system

In this section, we compute numerically reduced upper dual bounds (4.6) in a computationally manageable example, where the distributions H_r of the random variables Y_r are different. We keep the assumption that the random vector $\vec{X}$ is a.s. bounded from below.

Having different marginals, we can not reduce the dimensionality of the optimization problems (3.11) and (3.12)and multiple local optima can occur. Both (3.11) and (3.12) become *Global Optimization* (GO) problems. This, of course, calls for more advanced optimization techniques. For the application to follow, we restrict to using the GO routines present in MATLAB, but we are aware that several further search algorithms are available. For an introductory review, we refer to Section 4.3 in [EP] (2006a).

As an example, we take again the star-like system $\mathscr{E}_d^\star = \{\{1,2\},\dots,\{1,d\}\}$, and use the parameterization of the corresponding Fréchet class, as described in Section 4. This parameterization introduces inhomogeneous marginals. For a vector $\vec{\alpha} = (\alpha_2,\dots,\alpha_d)$ such that

$$\alpha_r \ge 0, r = 2,\dots,d \text{ and } \sum_{r=2}^{d} \alpha_r = 1, \tag{5.4}$$

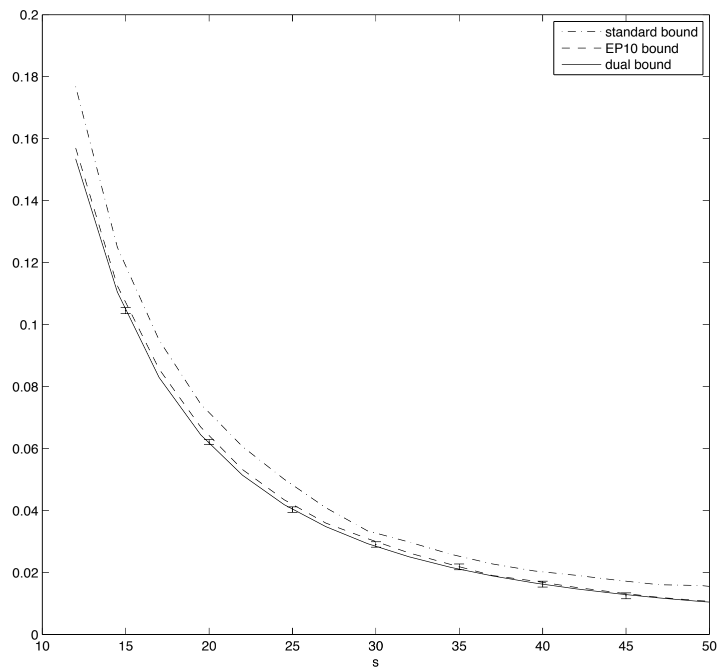

Figure 1: Standard bounds $\mathrm{S}'_{\mathfrak{H}}$ (see (5.2a)), *EP10 bound* (3.7) and dual bounds $\mathrm{D}'_{\mathfrak{H}}$ (see (5.3a)) for the sum of $d = 4$ Pareto(2)-distributed risks, coupled by a Frank copula with parameter $\delta = 2$. Numerical ranges for the bounds $M_{\mathfrak{H}}(s)$ are also provided.

we define the weighted random variables $Y_r^{\vec{\alpha}}$ as

$$Y_r^{\vec{\alpha}} = \alpha_r X_1 + X_r, r = 2, \ldots, d.$$

The distribution $H_r^{\vec{\alpha}}$ of Y_r is given by

$$H_r^{\vec{\alpha}} = \int_0^{x/\alpha_r} P\left(X_r \le x - \alpha_r x_1 | X_1 = x_1\right) dF_1(x_1) = \int_0^{x/\alpha_r} F_{r|x_1}(x - \alpha_r x_1)\, dF_1(x_1).$$

For the fixed bivariate marginal distributions $F_{1r}, r = 2, \ldots, d$, we keep the Frank-Pareto model described in Section 5.1, but we let the tail parameters of the Pareto marginals vary.

In order to obtain the reduced upper dual bound (4.6), one should minimize the dual bound $\mathrm{D}_{\mathfrak{H}}^{\vec{\alpha}}$ over all vectors $\vec{\alpha}$ satisfying (5.4). If possible at all, this is a rather onerous task from a computational viewpoint. Instead, we perform the minimization over the lattice $\mathfrak{A}$, defined as

$$\mathfrak{A} = \left\{ (\alpha_2, \ldots, \alpha_d) \in [0,1]^n : \alpha_r \in \{0, 1/N, \ldots, (N-1)/N, 1\}, \sum_{r=2}^{d} \alpha_r = 1 \right\},$$

for some fixed integer N. Increasing N increases the accuracy of the bound (4.6), but also, and exponentially, the computational time used to calculate it. In the application to follow, we choose $N = 9$.

In Figure 2, we plot the reduced upper standard bounds $\mathrm{S}_{\mathfrak{H}}$ (see (3.11a)), the reduced dual bounds $\mathrm{D}_{\mathfrak{H}}$ (see (3.12a)) and parameterized dual bounds $\mathrm{D}^*_{\mathfrak{H}}$ (see (4.6)) on $M_{\mathscr{E}^\star}$ for a random vector of $d = 4$ Pareto-distributed risks under the star-like marginal system described above. The parameter of the Frank copula is set to $\delta = 2$, while the tail parameters of the Pareto are $\theta_1 = 1.5, \theta_2 = 2, \theta_3 = 2.5$. Recall that the standard (3.11a) and dual (3.12a) bounds can be seen as obtained within a parameterized Fréchet class in which $\alpha_2 = \alpha_3 = \alpha_4 = 1/3$. It is interesting to see that the parameterized dual bound (4.6) improves the corresponding reduced dual bound (3.12a) at all thresholds, coherently with the fact that

$$M_{\mathscr{E}^\star}(s) \leq \mathrm{D}^*_{\mathfrak{H}}(s) \leq \mathrm{D}_{\mathfrak{H}}(s) \leq \mathrm{S}_{\mathfrak{H}}(s), s \in \mathbb{R}.$$

It is possible to use the rearrangement algorithm to provide numerical upper bounds on $\overline{D}(s)$. These upper bounds, which we report in Table 1, show that the parameterized dual bounds $\mathrm{D}^*_{\mathfrak{H}}$ are of good quality. We remark that the computation of analytical dual bounds in the inhomogeneous setting is handable only for low dimensions $d \leq 5$. If one needs less accuracy, the rearrangement algorithm can produce reasonable bounds up to dimensions $d \leq 30$.

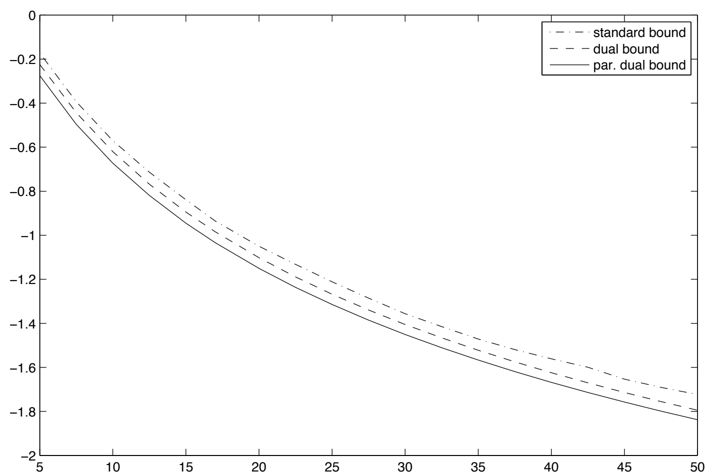

Figure 2: Reduced standard bounds (in log scale) $\mathrm{S}_{\mathfrak{H}}$ (see (3.11a)), reduced dual bounds $\mathrm{D}_{\mathfrak{H}}$ (see (3.12a)) and parameterized dual bounds $D^*_{\mathfrak{H}}$ (see (4.6)) for the sum of $d = 4$ Pareto-distributed risks, coupled by a Frank copula. The parameter of the Frank copula is set to $\delta = 2$, while the tail parameters of the Pareto are $\theta_1 = 1.5, \theta_2 = 2, \theta_3 = 2.5$.

	Standard bound	Dual bound	Par. dual bound	Num. upper bound
$s = 10$	0.269193	0.239977	0.212608	0.218750
$s = 15$	0.155761	0.127512	0.113569	0.117188
$s = 20$	0.089151	0.079053	0.070731	0.078125
$s = 25$	0.061485	0.053968	0.048500	0.054688
$s = 30$	0.044010	0.039332	0.035422	0.039062
$s = 35$	0.033779	0.030035	0.027107	0.031250

Table 1: Reduced standard bounds $\overline{S}_{\mathscr{E}_d^*}(s)$ (see (3.11a)), reduced dual bounds $\overline{D}_{\mathscr{E}_d^*}(s)$ (see (3.12a)) and parameterized dual bounds $\overline{D}(s)$ (see (4.6)) for the same random vector as in Figure 2. Upper bounds for $\overline{D}(s)$, calculated numerically via the algorithm introduced in [PR] (2011), are also provided.

6 Concluding remarks

In this paper, we discuss and suggest various methods and results for the problem of obtaining good upper and lower bounds for the risk distribution of the joint portfolio of dependent risks. We discuss also numerical aspects of the problem.

In the simple marginal case, we describe the role of rearrangements and optimal couplings. We give a general extension of the conditional moment method which is easy to calculate. The dual bounds improve generally the standard bounds. They are easy to calculate in the homogeneous case. In the inhomogeneous case, for dimension $d \geq 5$, the rearrangement algorithm for the approximation of Fréchet bounds is preferable while, in bounded domains, the conditional bounds give good alternatives.

The bounds can be essentially improved if further information on the dependence structure is available, for instance in the case of positive dependence in the portfolio. For the case of information on (overlapping) higher order marginals, we present several classical and new reduction results. Of particular importance seems to be the reduction of general marginal structures by a weighting method. This leads to some new classes of reduced bounds for general marginal systems.

We discuss and demonstrate the quality of various bounds in some examples.

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