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On a Class of Extremal Problems in Statistics

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Summary: Let m denote the infimum of the integral of a function φ w. r. t. all probability measures with given marginals. The determination of m is of interest for a series of stochastic problems. In the present paper we prove a duality theorem for the determination of m and give some examples for its application. We consider especially the problem of extremal variance of sums of random variables and prove a theorem for the existence of random variables with given marginal distributions, such that their sum has variance zero.

1. Introduction

A basic problem of dealing with dependent random variables is the following one. Let φ be a function of n variables and let $P_1, \ldots, P_n$ be n one-dimensional probability measures; then determine the minimum and the maximum of the integral of φ w.r.t. all probability measures with marginals $P_1, \ldots, P_n$. By means of results of this type one can describe the influence of dependence on a stochastic problem which is defined by the function φ.

Solutions for this stochastic optimization problem are known only in very few special cases. A very nice solution in the case $\varphi(x_1, \ldots, x_n) = \max \{x_i \mid 1 \leqq i \leqq n\}$ has been given in recent papers by LAI and ROBBINS [10], [11]. Their result is that $\max \{x_i \mid 1 \leqq i \leqq n\}$ is for arbitrary dependent random variables not much larger than for independent random variables. So they are able to prove limit theorems for the maximum of a sequence of arbitrary dependent random variables.

In section two of this paper we prove a duality theorem for the general optimization problem and give some examples where solutions are found by an application of this theorem. A useful aspect of this duality theorem is that on one hand it allows to give bounds for the minimum and on the other hand it describes the support of an 'optimal measure'.

In section three we consider the problem of extremal variance of sums of random variables which is of the type described above. For some cases we are able to give a solution of this problem. We isolate the more special question for

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the existence of n random variables with constant sum and with given marginals, prove a duality theorem for this question, and get solutions in some special cases.

2. A Duality Theorem

Let $\varphi: [0, 1]^n \to R^1$ be a continuous function and let $P_1, \ldots, P_n$ be n probability measures on $[0, 1]\, \mathfrak{B}^1$, where $\mathfrak{B}^1$ is the Borel σ-Algebra on R^1. Define

$$\mathfrak{H}(P_1, \ldots, P_n) := \{P \mid P \text{ is a probability measure on } [0, 1]^n\, \mathfrak{B}^n \text{ with marginals } P_1, \ldots, P_n\}.$$

$\mathfrak{H}(P_1, \ldots, P_n)$ is a convex set of probability measures which is compact w.r.t. the vague topology.

The '*primal*' problem we want to consider is to determine

$$m := \inf \{\int \varphi \mathrm{d}P \mid P \in \mathfrak{H}(P_1, \ldots, P_n)\}. \tag{1}$$

It is clear from the note above that there exists *a* $P^* \in \mathfrak{H}(P_1, \ldots, P_n)$ with $m = \int \varphi \mathrm{d}P^*$. The problem of determining $\sup \{\int \varphi \mathrm{d}P \mid P \in \mathfrak{H}(P_1, \ldots, P_n)\}$ is included in this formulation of the optimization problem by taking $-\varphi$ instead of φ.

For a topological space X we denote by $C(X)$ the set of all continuous, real functions on X. In the following duality theorem we give an optimization problem which is 'dual' to the primal optimization problem.

Theorem 1:

$$m = \sup \left\{ \sum_{i=1}^{n} \int f_i \mathrm{d}P_i \mid f_i \in C[0, 1], \quad 1 \leqq i \leqq n, \right. \tag{2}$$

$$\left. \sum_{i=1}^{n} f_i(x_i) \leqq \varphi(x_1, \ldots, x_n), \quad \forall x = (x_1, \ldots, x_n) \in [0, 1]^n \right\}.$$

Proof: The proof of Theorem 1 is based on the following duality theorem of ISII [8], [9], Th. 2.3, in topological vectorspaces (cf. also DIETER [2], GOLSTEIN [5], Kapitel 2, 3). Let X be a convex cone with vertex 0 in a real vectorspace, let Z be a topological vectorspace, $z_0 \in Z$ and let a pseudoorder be induced on Z by a convex cone $\mathfrak{E}$ with vertex 0. Let $F: X \to R^1$, $\psi: X \to Z$ be linear functions such that $\mathring{\mathfrak{E}} \neq \emptyset$ and $0 \in \overline{\psi(X) - \mathring{\mathfrak{E}} + z_0}$, where $\mathring{A}$ denotes the interior of A. Then,

$$\sup \{F(x) \mid x \in X, \psi(x) + z_0 \geqq 0\} = \inf \{z^*(z_0); \tag{3}$$

$$z^* \in Z^*, z^* \geqq 0, z^*(\psi(x)) + F(x) \leqq 0, \forall x \in X\}$$

where Z^* is the topological dual of Z.

We define: $X := C^n[0, 1]$, $Z := C[0, 1]^n$,

$$F(f_1, \ldots, f_n) := \sum_{i=1}^{n} \int f_i \mathrm{d}P_i, \quad \psi(f_1, \ldots, f_n)\,(x) := -\sum_{i=1}^{n} f_i(x_i)$$

for $x=(x_1, \ldots, x_n)$, $z_0:=\varphi$ and the cone $\mathfrak{C}:=\{f\in C[0,1]^n \mid f\geqq 0\}$. The topology on $Z=C[0,1]^n$ is given by the norm $\|f\|:=\sup\{|f(x)| \mid x\in[0,1]^n\}$. The left side of (3) is identical to our dual problem

$$M_1:=\sup\left\{\sum_{i=1}^n \int f_i \mathrm{d}P_i \mid f_i\in C[0,1],\ 1\leqq i\leqq n,\quad \sum_{i=1}^n f_i\circ\pi_i\leqq\varphi\right\},$$

where π_i denotes the projection on the i-th component, $1\leqq i\leqq n$.

By Riesz' representation theorem (cf. Dunford, Schwartz [3], Th. 3, pg. 265) the dual space of $C[0,1]^n$ is the space of signed measures on $[0,1]^n$ $\mathfrak{B}^n$ and hence $\{z^*\in Z^* \mid z^*\geqq 0\}$ equals the set of measures on $[0,1]^n$. Therefore, the right hand side of (3) is identical to

$$M_2:=\inf\left\{\int\varphi\mathrm{d}\mu \mid \mu \text{ is a measure on } [0,1]^n\ \mathfrak{B}^n \text{ and}\right.$$
$$\left.-\int\sum_{i=1}^n f_i\circ\pi_i\mathrm{d}\mu\leqq-\sum_{i=1}^n\int f_i\mathrm{d}P_i,\ \forall f_i\in C[0,1],\ 1\leqq i\leqq n\right\}.$$

Since $f_i\in C[0,1]$ implies $-f_i\in C[0,1]$, we get

$$M_2:=\inf\left\{\int\varphi\mathrm{d}\mu \mid \mu \text{ is a measure with } \int\sum_{i=1}^n f_i\circ\pi_i\mathrm{d}\mu=\sum_{i=1}^n\int f_i\mathrm{d}P_i,\right.$$
$$\left.\forall f_i\in C[0,1],\ 1\leqq i\leqq n\right\}.$$

Taking $f_i\equiv 1$, $1\leqq i\leqq n$ we get that admissible μ are normalized on 1. With $f_i\equiv 0$ for $i\neq j$ we obtain further that admissible μ have as j-th marginal P_j, since by Theorem 1.3 of Billingsley [1] continuous functions are measure determining. Therefore, the right hand side of (3) is equal to our primal problem

$$M_2=\inf\{\int\varphi\mathrm{d}\mu \mid \mu\in\mathfrak{H}(P_1,\ldots,P_n)\}.$$

We have to check the regularity conditions. Obviously it holds $\mathring{\mathfrak{C}}\neq\emptyset$. It remains to show that 0 is an interior point of $\psi(X)-\mathfrak{C}+z_0$, i.e. for $f\in C[0,1]^n$ with $\|f\|\leqq\varepsilon$ there exist $g\in C^n[0,1]$ and $h\in C[0,1]^n$, $h\geqq 0$ such that $f=\psi(g)-h+\varphi$. Since φ is continuous, it is bounded, $|\varphi|\leqq K$. With $g_i:\equiv-\frac{1}{n}(K+\varepsilon)$, $1\leqq i\leqq n$ and $g:=(g_1,\ldots,g_n)$ we have that

$$h:=-\sum_{i=1}^n g_i\circ\pi_i-f+\varphi\geqq 0.$$

Therefore, by Isii's Theorem $M_1=M_2$ which is identical to (2). ∎

The following proposition shows that there exists a solution of the *dual problem* in (2) if one enlarges the space $C[0,1]$.

Let $B[0,1]$ be the set of all Borel-measurable, real, bounded functions on $[0,1]$.

Proposition 2:

$$m=\sup\left\{\sum_{i=1}^n\int f_i\mathrm{d}P_i \mid f_i\in B[0,1],\ 1\leqq i\leqq n,\ \sum_{i=1}^n f_i\circ\pi_i\leqq\varphi\right\},$$

and the supremum is attained.

Proof. For $f_i \in B[0, 1]$, $1 \leqq i \leqq n$, with $\sum_{i=1}^{n} f_i \circ \pi_i \leqq \varphi$ and for $P \in \mathfrak{H}(P_1, \ldots, P_n)$ we obtain

$$\sum_{i=1}^{n} \int f_i \mathrm{d}P_i = \sum_{i=1}^{n} \int f_i \circ \pi_i \mathrm{d}P = \int \left(\sum_{i=1}^{n} f_i \circ \pi_i \right) \mathrm{d}P \leqq \int \varphi \mathrm{d}P ,$$

and hence

$$m \geqq \sup \left\{ \sum_{i=1}^{n} \int f_i \mathrm{d}P_i \mid f_i \in B[0, 1],\ 1 \leqq i \leqq n,\ \sum_{i=1}^{n} f_i \circ \pi_i \leqq \varphi \right\}.$$

Since $C[0, 1] \subset B[0, 1]$ Theorem 1 implies equality.

For the existence of functions $f_1^*, \ldots, f_n^*$, for which the supremum is attained, we first note that there exists a $K \in R_+^1$ such that

$$m = \sup \left\{ \sum_{i=1}^{n} \int f_i \mathrm{d}P_i \mid f_i \in B[0, 1],\ |f_i| \leqq K,\ 1 \leqq i \leqq n,\ \sum_{i=1}^{n} f_i \circ \pi_i \leqq \varphi \right\}. \tag{4}$$

Let $f_1, \ldots, f_n \in B[0, 1]$ with $\sum_{i=1}^{n} f_i \circ \pi_i \leqq \varphi$ be given and let $b_i := \sup f_i$, $1 \leqq i \leqq n$, and let $a := \inf \varphi$, $A := \sup \varphi$. Defining

$$g_i := f_i + \frac{1}{n} \sum_{j=1}^{n} b_j - b_i, \quad 1 \leqq i \leqq n ,$$

we obtain

$$\sum_{i=1}^{n} g_i \circ \pi_i \leqq \varphi, \quad \sum_{i=1}^{n} \int g_i \mathrm{d}P_i = \sum_{i=1}^{n} \int f_i \mathrm{d}P_i$$

and

$$\sup g_i = \frac{1}{n} \sum_{i=1}^{n} b_j = \bar{b}, \quad 1 \leqq i \leqq n .$$

It follows that $\bar{b} \leqq \frac{A}{n}$. In the next step define $h_i(x) := \max \{g_i(x), c\}$ with

$$c := \min_{1 \leqq r \leqq n} \left(\frac{a}{r} - \frac{n-r}{nr} A \right) = a - \frac{n-1}{n} A .$$

Then

$$\sum_{i=1}^{n} h_i(x_i) = \sum_{i=1}^{n} g_i(x_i) \leqq \varphi(x_1, \ldots, x_n) ,$$

if $g_i(x_i) \geqq c$, $1 \leqq i \leqq n$. Let now $g_i(x_i) < c$ for exactly r indices i, $1 \leqq r \leqq n$, then

$$\sum_{i=1}^{n} h_i(x_i) \leqq rc + (n-r)\, \bar{b} \leqq rc + \frac{n-r}{n} A \leqq a \leqq \varphi(x_1, \ldots, x_n) ,$$

and $g_i \leqq h_i$, $1 \leqq i \leqq n$, implies

$$\sum_{i=1}^{n} \int g_i \mathrm{d}P_i \leqq \sum_{i=1}^{n} \int h_i \mathrm{d}P_i .$$

With $K:=\max\left\{|c|, \frac{|A|}{n}\right\}$ we obtain (4). Now let $(f_1^{(k)}, \ldots, f_n^{(k)})$, $k\in \boldsymbol{N}$, be a sequence such that

$$f_i^{(k)}\in B_K[0,1]:=\{f\in B[0,1] \mid |f|\leqq K\}, \quad 1\leqq i\leqq n, \quad \sum_{i=1}^n f_i^{(k)}\circ\pi_i\leqq\varphi$$

and

$$\lim_{k\to\infty}\sum_{i=1}^n \int f_i^{(k)}\mathrm{d}P_i = m\,.$$

Then since $\bigotimes_{i=1}^n B_K[0,1]$ is a sequentially compact subset of $\bigotimes_{i=1}^n L^1(P_i)$ supplied with the weak topology $\sigma\left(\bigotimes_{i=1}^n L^1(P_i), \bigotimes_{i=1}^n L^\infty(P_i)\right)$ there exist $(\bar f_1, \ldots, \bar f_n)\in \bigotimes_{i=1}^n B_K$ [0,1] and a subsequence of $(f_1^{(k)}, \ldots, f_n^{(k)})_{k\in N}$ converging to $(\bar f_1, \ldots, \bar f_n)$ w.r.t. the weak topology. Now one can proceed as in Landers, Rogge [12]. ∎

Corollary 3: *Let $P^*\in\mathfrak{H}(P_1, \ldots, P_n)$. Then P^* is a solution of* (1) *if and only if there exist $f_1^*, \ldots, f_n^*\in B[0,1]$ with $\sum_{i=1}^n f_i^*\circ\pi_i\leqq\varphi$ and $\sum_{i=1}^n f_i^*\circ\pi_i=\varphi[P^*]$.*

Proof: Let $f_1^*, \ldots, f_n^*\in B[0,1]$ with $\sum_{i=1}^n f_i^*\circ\pi_i\leqq\varphi$ and $\sum_{i=1}^n\int f_i^*\mathrm{d}P_i=m$. Then $P^*\in\mathfrak{H}(P_1, \ldots, P_n)$ is a solution of (1) if and only if $\int\left(\varphi-\sum_{i=1}^n f_i^*\circ\pi_i\right)\mathrm{d}P^*=0$ which is equivalent to $\sum_{i=1}^n f_i^*\circ\pi_i=\varphi[P^*]$. ∎

Remark 1: Clearly Theorem 1, Proposition 2, and Corollary 3 remain true if the interval [0, 1] is replaced by an arbitrary compact metric space E. Then φ is a continuous real function on E^n, $P_1, \ldots, P_n$ are probability measures on $(E, \mathfrak{B}(E))$, $\mathfrak{B}(E)=$ Borel σ-Algebra on E, and $\mathfrak{H}(P_1, \ldots, P_n)$ is the set of all probability measures on $(E^n, \mathfrak{B}(E^n))$ with marginals $P_1, \ldots, P_n$. As a special case let $P_1, \ldots, P_n$ be probability measures on $(R^1, \mathfrak{B}^1)$. By $P_i\{+\infty\}=P_i\{-\infty\}==0$, $1\leqq i\leqq n$, the P_n are extended on $(\bar R^1, \bar{\mathfrak{B}}^1)$. Taking $E=\bar R^1$ and observing that $C(\bar R^n)$ can be identified with the set of functions

$$C'(R^n):=\{f\in C(R^n) \mid \lim_{\substack{x\in R^n\\ x\to x_0}} f(x) \text{ exists and is finite for each } x_0\in \bar R^n\backslash R^n\}$$

we have by Theorem 1 and Proposition 2 for $\varphi\in C'(R^n)$

$$\begin{aligned}&\inf\{\textstyle\int\varphi\mathrm{d}P \mid P\in\mathfrak{H}(P_1, \ldots, P_n)\}\\ &=\sup\left\{\sum_{i=1}^n\int f_i\mathrm{d}P_i \mid f_i\in C'(R^1),\ 1\leqq i\leqq n,\quad \sum_{i=1}^n f_i\circ\pi_i\leqq\varphi\right\}\\ &=\sup\left\{\sum_{i=1}^n\int f_i\mathrm{d}P_i \mid f_i\in B(R^1),\ 1\leqq i\leqq n,\quad \sum_{i=1}^n f_i\circ\pi_i\leqq\varphi\right\},\end{aligned}$$

where $\mathfrak{H}(P_1, \ldots, P_n)=$ set of all probability measures P on $(R^n, \mathfrak{B}^n)$ with marginals $P_1, \ldots, P_n$ and $B(R^1)$ is the set of real, bounded measurable functions on R^1.

Example 1: a) Let $P_1 = P_2 = Q$ be the uniform distribution on $[0, 1]\ \mathfrak{B}^1$ and $\varphi(x_1, x_2) = \varphi_1(x_1) \cdot \varphi_2(x_2)$, where φ_1 and φ_2 are continuous increasing functions on $[0, 1]$. We consider the primal problem of minimizing

$$\int \varphi_1(x_1)\, \varphi_2(x_2)\, \mathrm{d}P(x_1, x_2), \quad P \in \mathfrak{H}(Q, Q)\,.$$

This includes the well-known problem of finding random variables X_1 and X_2 with prescribed distribution functions F_1 and F_2, such that EX_1X_2 gets a minimum, where the F_i are continuous, strictly increasing on a compact interval $[a_i, b_i]$, and $F_i(t) = 0$ for $t \leqq a_i$, $F_i(t) = 1$ for $t \geqq b_i$, $i = 1, 2$. Then $EX_1X_2 = \int F_1^{-1}(x_1)\, F_2^{-1}(x_2)\, \mathrm{d}P$, where P is the distribution of $(F_1(X_1), F_2(X_2))$, and hence $P \in \mathfrak{H}(Q, Q)$.

Since the φ_i are increasing functions we get by an heuristic argument that $\varphi_1(x_1)$ and $\varphi_2(x_2)$ must be ordered in opposite senses on the support of an optimal measure $P^* \in \mathfrak{H}(Q, Q)$. Hence P^* must be the distribution of $(U, 1-U)$, where U is a $R(0, 1)$-distributed random variable. To prove this by Corollary 3 we have to look for functions $f_1, f_2 \in B[0, 1]$ such that

$$f_1(x_1) + f_2(x_2) \leqq \varphi_1(x_1)\, \varphi_2(x_2), \quad x_1, x_2 \in [0, 1]\,, \tag{5}$$

with equality for $x_2 = 1 - x_1$. Assuming the existence of the derivatives f_i', φ_i' for the moment and putting $H(x_1, x_2) := f_1(x_1) + f_2(x_2) - \varphi_1(x_1)\, \varphi_2(x_2)$ we get from (5):

$$\frac{\partial}{\partial x_1} H(x_1, x_2)\Big|_{x_2 = 1 - x_1} = 0, \quad \frac{\partial}{\partial x_2} H(x_1, x_2)\Big|_{x_2 = 1 - x_1} = 0\,,$$

and hence

$$f_1'(x_1) = \varphi_1'(x_1)\, \varphi_2\,(1 - x_1), \quad f_2'\,(1 - x_1) = \varphi_1(x_1)\, \varphi_2'\,(1 - x_1)$$

or

$$f_1(x_1) = c_1 + \int_0^{x_1} \varphi_2\,(1-t)\, \mathrm{d}\varphi_1(t)\,, \tag{6}$$

$$f_2(x_2) = c_2 + \int_0^{x_2} \varphi_1\,(1-t)\, \mathrm{d}\varphi_2(t)\,.$$

Now let the functions $f_1, f_2 \in C[0, 1]$ be given by (6) in terms of STIELTJES integrals (which do not require the differentiability conditions on the φ_i), where the constants c_i will be chosen according to (5). By partial integration we get

$$\int_0^{x_1} \varphi_2\,(1-t)\, \mathrm{d}\varphi_1(t) = \varphi_1(x_1)\, \varphi_2\,(1 - x_1) - \varphi_1(0)\, \varphi_2(1) + \int_{1-x_1}^{1} \varphi_1\,(1-t)\, \mathrm{d}\varphi_2(t)$$

and hence

$$f_1(x_1) + f_2(x_2) = c_1 + c_2 + \varphi_1(x_1)\, \varphi_2\,(1 - x_1) - \varphi_1(0)\, \varphi_2(1) + \int_{1-x_1}^{1} \varphi_1\,(1-t)\, \mathrm{d}\varphi_2(t) + \int_0^{x_2} \varphi_1\,(1-t)\, \mathrm{d}\varphi_2(t)\,.$$