Optimal Stopping beyond the Free Boundary Approach

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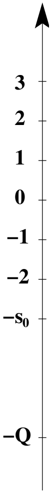
A Parking Problem



$$S_0 = -Q$$

$$S_n = \sum_{i=1}^{n} X_i - Q$$

X_i i.i.d. geometric (p)

p: probability of empty spot

Park as near as possible at $''0''$!

Find a stopping time T^* of $S_i, i \geq 0$ with

$$E \mid S_{T^*} \mid = \min_T E \mid S_T \mid .$$

Solution: $T^* = \min\{n \geq 1 \mid S_n > -s_0\}$

with $s_0 = \min\{s \in \mathbb{N} \mid 1 - 2(1-p)^s > 0\}$

Chow, Robbins, Siegmund (1971): Great Expectations, p. 45

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Generalized Parking Problem (GPP)

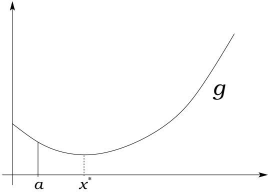
Let g be a convex nonnegative function with a unique minimum at $x^* > 0$.

Assume X_i i.i.d. with $EX_i > 0$,

$$S_n = \sum_{i=1}^{n} X_i, \quad S_0 = 0.$$

Find a stopping time T^* with

$$Eg(S_{T^*}) = \min_T Eg(S_T).$$



Solution (Woodroofe, Lerche, Keener '94):

$$T^* = \min\{n \geq 0 \mid S_n \geq a\}$$

with $a = \sup\{x \mid H^+ g(x) < g(x)\}$ where H^+ is the ladder-height distribution of S_n; $n \geq 1$ and $H^+ g(x) = \int g(x+y) H(dy)$.

Lorden's Result on the SPRT

Let
$$R(T) = P_0(T < \infty) + cIE_1 T,$$
where $I = E_1 \log \frac{dP_1}{dP_0}(X)$.

Let
$$S_n = \log \frac{dP_1^n}{dP_0^n}.$$

Then by Wald' identity
$$R(T) = \int g(S_T)\, dP_1$$
with $g(x) = e^{-x} + cx$.

g is a nonnegative convex function with a unique minimum at $\log \frac{1}{c}$.

Then $T^* = \min\{n \geq 1 | S_n \geq \log(\frac{\kappa}{c})\}$

where $\kappa = \lim\limits_{a \to \infty} E_1 \exp(-(S_{\tau_a} - a))$

and $\tau_a = \min\{n \geq 1 | S_n \geq a\}$. Lorden(AS 1977)

The Repeated Significance Test as Bayes Test (RST)

W_t, $t \geq 0$ Brownian motion with drift θ

Testing sequentially: $H_0 : \theta < 0$ versus $H_1 : \theta > 0$

Prior: $G = N(\mu, r^{-1})$

$$R(T,\delta) = \int_{-\infty}^{0} \left(P_\theta\{\delta \text{ rejects } H_0\} + \frac{c}{2}\theta^2 E_\theta T \right) G(d\theta) + \int_{0}^{\infty} \left(P_\theta\{\delta \text{ rejects } H_1\} + \frac{c}{2}\theta^2 E_\theta T \right) G(d\theta)$$

Find (T^*, δ^*) with $R(T^*, \delta^*) = \min_{(T,\delta)} R(T, \delta)$.

$\delta^* = \delta^*_T = 1_{\{W_T + r\mu > 0\}}$ $\qquad T^* = \ ?$

PNAS, 83 (1986)

Representation of the risk:

$$R(T,\delta_T^*) = \int g\left(\frac{(W_T + r\mu)^2}{T+r}\right) dQ$$

with $g(x) = \Phi(-\sqrt{x}) + cx/2, \quad Q = \int P_\theta \, G(d\theta), \quad G = N\left(0, r^{-1}\right).$

g is convex with unique minimum x^* and

$$R(T,\delta_T^*) = \int g\left(\frac{(W_T + r\mu)^2}{T+r}\right) dQ \geq g(x^*)$$

Let $T^* = \min\{t > 0 \mid W_t^2/(t+r) = x^*\}$.

Since $Q\{T^* < \infty\} = 1$ it follows $R(T^*, \delta_{T^*}^*) = g(x^*)$ if $r\mu^2 \leq x^*$.

Let $r\mu^2 > x^*$. We show that for any stopping time T such that

$Q(T < \infty) = 1$, it holds $R(T, \delta^*) \geq g(r\mu^2)$.

Consider the Q-Martingale

$$N_t = \frac{dP_0}{dQ}\bigg|_{\mathcal{F}_t} = \sqrt{\frac{t+r}{r}} \exp\left(-\frac{(W_t + r\mu)^2}{2(t+r)} + \frac{r\mu^2}{2}\right),$$

where P_0 is the measure of B.M. without drift.

Since $h(x) = 2\log(1/x)$ is convex

$$h(N_t) \qquad \text{and} \qquad Z_t = \frac{(W_t + r\mu)^2}{t+r}$$

is a Q-submartingale.

Since g is convex on $[0, \infty)$ and increasing on $[x^*, \infty)$ for any bounded

stopping time T it holds,

$$R(T, \sigma^*) = E_Q\, g(Z_T) \geq g(E_Q Z_t) \geq g(Z_0) = g(r\mu^2).$$

Disruption Problem

Shiryaev (1961) studied the following problem.

Observations:	$W_t = B_t + \theta(t-\tau)^+$ with
	$B_t,\ t \geq 0$ standard Brownian motion,
	$\theta > 0$ fixed
Filtration:	$\mathcal{F}_t = \sigma(W_t;\ 0 \leq s \leq t)$
Change-point:	τ random time, independent of B
	with distribution $\pi = p\delta_0 + (1-p)F$,
	where $F(t) = 1 - e^{-\lambda t}$
Risk:	$R(T) = P_\pi(T < \tau) + cE_\pi(T-\tau)^+$
	Find T^* with $R(T^*) = \min_T R(T)$.

Let $\pi_t = P(\tau \leq t \mid \mathcal{F}_t)$.

Then

$$\pi_t = \frac{\varphi_t}{e^{-\lambda t} + \varphi_t}$$

where

$$\varphi_t = \frac{p}{1-p} L_t + \int_0^t \frac{L_t}{L_s} \lambda e^{-\lambda s} \, ds$$

with

$$L_t = \exp(\theta W_t - \theta^2 t/2).$$

π_t is a diffusion with $d\pi_t = \lambda(1-\pi_t)dt + \theta\pi_t(1-\pi_t)d\overline{W_t}$ with $\overline{W_t}$ is a standard Brownian motion. Itô‘s formula yields:

$$\begin{aligned} dG(\pi_t) &= G'(\pi_t)d\pi_t + \frac{1}{2}G''(\pi_t)(d\pi_t)^2 \\ &= G'(\pi_t)\left[\lambda(1-\pi_t)dt + \theta\pi_t(1-\pi_t)d\overline{W_t}\right] \\ &\quad + \frac{1}{2}G''(\pi_t)\theta^2\pi_t^2(1-\pi_t)^2 dt \end{aligned}$$

If G satisfies the equation

$$\frac{\theta^2}{2}x^2(1-x)^2G''(x) + \lambda(1-x)G'(x) = cx$$

and behaves well at 0, then

$$G(\pi_t) - G(\pi_0) = c\int_0^t \pi_s ds + c\int_0^t \theta\pi_s(1-\pi_s)d\overline{W_s}$$

$$\Rightarrow E\left[G(\pi_T) - G(\pi_0)\right] = c\,E\int_0^T \pi_s ds$$

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Then one obtains

$$\begin{aligned} R(T) &= P(T < \tau) + cE(T-\tau)^+ \\ &= E\left[(1-\pi_T) + c\int_0^T \pi_s ds\right] \end{aligned}$$

with $g(x) = (1-x) + G(x)$

$$R(T) = \int g(\pi_T) dP - g(p)$$

g is convex with a unique minimum at p^*.

Theorem

$T^* = \min\{t > 0 \mid \pi_t \geq p^*\}$ with $\pi_t = P(\tau \leq t \mid \mathcal{F}_t)$

Here p^* is the unique solution in $(0,1)$ of $G'(p) = 1$, where G is the (finite at 0) solution of

$$\frac{\theta}{2} x^2 (1-x^2) G''(x) + \lambda(1-x) G'(x) = cx.$$

The Basic Idea: OS as GPP

Let $(Z_t, \mathcal{F}_t; t \geq 0)$ denote a continuous stochastic process on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$.

Find a stopping time T^* with

$$E_P\left(Z_{T^*} 1_{\{T^* < \infty\}}\right) = \max_T E_P\left(Z_T 1_{\{T < \infty\}}\right).$$

Idea:

Find a process $(X_t, \mathcal{F}_t; t \geq 0)$, a nonnegative martingale $(M_t, \mathcal{F}_t; t \geq 0)$ with $EM_0 = 1$ and a function g with unique maximum at x^* such that $Z_t = g(X_t) M_t$.

Then

$$\begin{aligned} EZ_T 1_{\{T<\infty\}} &= E\left(g(X_T) M_T 1_{\{T<\infty\}}\right) \\ &\leq g(x^*) EM_T 1_{\{T<\infty\}} \\ &\leq g(x^*) \end{aligned}$$

With $T^* = \min\{t \geq 0 \mid X_t = x^*\}$ the inequalities become equalities, if $EM_{T^*} 1_{\{T^*<\infty\}} = 1$.

Perpetual American Put Option

Samuelson (1965), McKean (1965)

$X_t = \sigma B_t + \mu t$, $t \geq 0$ Brownian Motion with drift μ and variance σ^2.

Find a stopping time T^* which maximizes

$$E_P\, e^{-rT}(K - e^{X_T})^+\, 1_{\{T<\infty\}}.$$

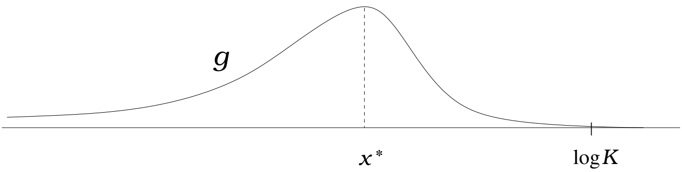
Idea:

Find M and g with $E_P\, e^{-rT}(K - e^{X_T})^+\, 1_{\{T<\infty\}} = E_P\, g(X_T)\, M_T\, 1_{\{T<\infty\}}$,

where g has a unique maximum at x^*.

Then

$$T^* = \min\{t \geq 0 \mid X_t = x^*\} \text{ if } E_P\, M_{T^*} = 1.$$



Let $f(x) = (K - e^x)^+$. How to find M_t ?

It holds for all $\alpha \in \mathbb{R}$

$$f(X_T)\, e^{-rT} = f(X_T)(e^{X_T})^{-\alpha}(e^{X_T})^{\alpha} e^{-rT}.$$

Choose $g(x) = f(x)e^{-\alpha x}$ and α such that $M_t = e^{\alpha X_t} e^{-rt}$ is a martingale.

This holds when

$$\begin{aligned} M_t &= \exp\left[\alpha(\sigma B_t) + t(\alpha\mu - r)\right] \\ &= \exp\left[(\alpha\sigma)B_t - t(\alpha\sigma)^2/2\right]. \end{aligned}$$

M_t is a positive marginale with $M_0 = 1$ iff $(\alpha\sigma)^2/2 + \alpha\mu - r = 0$

$\alpha^{\pm} = -\frac{\mu}{\sigma^2} \pm \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}}$ are the two possible values.

Then we have

$$E_P\, e^{-rT}\left(K-e^{X_T}\right)^{+} 1_{\{T<\infty\}} = E_Q\, g(X_T)\, 1_{\{T<\infty\}}$$

with $g(x) = \dfrac{f(x)}{e^{\alpha^- x}}$ and $\quad \dfrac{dQ_t}{dP_t} = M_t$.

Let $K < 1 + (-\alpha^-)^{-1}$. Then g has a unique maximum at

$x^* = \log \frac{\alpha^- K}{\alpha^- - 1} < 0$. Under $Q \quad X$ is Brownian motion with drift

$\alpha^- \sigma^2 + \mu = -\sigma^2 \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}} < 0$.

This yields $Q(T^* < \infty) = 1$ for $T^* = \inf\{t > 0 \mid X_t = x^*\}$.

Then

$$\sup_T E_P\left(e^{-rT}(K - e^{X_T})^{+} 1_{\{T<\infty\}}\right) = E_Q\, g(X_{T^*}) = C^*$$

with $C^* = \dfrac{(K - e^{x^*})}{e^{\alpha^- x^*}}$.

Put Options with strike $\max\limits_{0\le s\le t} X_s$

$X_t = \exp\left(\sigma B_t + \left(\mu - \sigma^2/2\right) t\right)$, $t > 0$ geometric Brownian motion with $\sigma > 0$, $\mu \in \mathbb{R}$, $S_t = \max\limits_{0\le u\le t} X_u$.

Find a stopping time T^* which maximizes

$$E\left(e^{-rT}(S_T - X_T)\right), \quad \text{when } \mu < r \text{ and } r > 0.$$

Let $\gamma_{1,2} = -\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right) \mp \sqrt{\frac{2r}{\sigma^2} + \left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2}$ and $\alpha = \left(\frac{1-1/\gamma_1}{1-1/\gamma_2}\right)^{1/(\gamma_2-\gamma_1)}$.

Let $h(x) = \frac{1}{\gamma_2\alpha^{\gamma_1} - \gamma_1\alpha^{\gamma_2}}\left(\gamma_2(\alpha x)^{\gamma_1} - \gamma_1(\alpha x)^{\gamma_2}\right)$.

Then $M_t = e^{-rt} S_t\, h\left(\frac{X_t}{S_t}\right)$ is a nonnegative local martingale and

$$Ee^{-rT}\left(S_T - X_T\right) = E\,\frac{1 - X_T/S_T}{h(X_T/S_T)} M_T \le \frac{1 - x^*}{h(x^*)},$$

where $x^* = \operatorname{argmax}\{(1-x)/h(x) \mid x \in (0,1)\}$.

Then $T^* = \inf\{t > 0 \mid X_t/S_t \le x^*\}$ is optimal.

Optimality of Parabolic Boundaries

Let $X_t = B_t + x_0$, $t \geq 0$ with B_t standard Brownian motion. For a measurable function g find a stopping time T that maximizes

$$E\left((T+1)^{-\beta} g\left(\frac{X_T}{\sqrt{T+1}}\right)\right). \qquad \text{(Moerbeke (1974))}$$

Let $H(x) = \int_0^\infty e^{ux-u^2/2} u^{2\beta-1}\, du$ with $\beta > 0$

and assume that there exists a unique point x^* with

$$\sup_{x\in\mathbb{R}} \frac{g(x)}{H(x)} = \frac{g(x*)}{H(x*)} = C^* \quad \text{and } 0 < C^* < \infty$$

Let $x_0 < x^*$. Then

$$\sup_T E\left\{(T+1)^{-\beta} g\left(\frac{X_T}{\sqrt{T+1}}\right)\right\} = E\left\{(T^*+1)^{-\beta} g\left(\frac{X_{T^*}}{\sqrt{T^*+1}}\right)\right\}$$

$$= H(x_0)C^*$$

where $T^* = \inf\left\{t > 0 \mid \frac{X_t}{\sqrt{t+1}} = x^*\right\}$.

$$(t+1)^{-\beta} H\left(\frac{X_t}{\sqrt{t+1}}\right) = \int_0^\infty e^{uX_t - \frac{u^2}{2}t}\left(e^{-\frac{u^2}{2}} u^{2\beta-1}\right) du$$

is a positive martingale with starting value $H(x_0)$.

Thus $M_t = (t+1)^{-\beta} H\left(\frac{X_t}{\sqrt{t+1}}\right) \Big/ H(x_0)$ is a positive martingale with

$EM_0 = 1$.

Then

$$\begin{aligned} E_P\left((T+1)^{-\beta} g\left(\frac{X_T}{\sqrt{T+1}}\right)\right) &= H(x_0) E_P \frac{g\left(\frac{X_T}{\sqrt{T+1}}\right)}{H\left(\frac{X_T}{\sqrt{T+1}}\right)} M_T \\ &\leq H(x_0) C^*. \end{aligned}$$

But for $E_P M_{T^*} = 1$

$$T^* = \inf\left\{t > 0 \mid \frac{X_t}{\sqrt{t+1}} = x^*\right\}.$$

Special case:

$$g(x) = x,\ x_0 = 0,\ \beta = \frac{1}{2}$$

$$E(X_T/(T+1)) = \max \ \text{with}$$

$$T^* = \min\left\{ t > 0 \mid \frac{X_t}{\sqrt{t+1}} = x^* \right\}$$

x^* is solution of $\quad x = (1-x^2)\int_0^\infty e^{ux-u^2/2}\,du.$ (Shepp 1969)

Two-Sided Boundaries

Let g be measurable, $X_t = \sigma B_t + \mu t$ Brownian motion with drift μ and variance σ^2. $X_0 = 0$ find a stopping time T^* which maximizes

$$Ee^{-rT} g(X_T) 1_{\{T<\infty\}}.$$

Let $\alpha_{1,2} = -\frac{\mu}{\sigma^2} \pm \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}} \quad (\alpha_2 < 0 < \alpha_1)$.

Then $M_t^{(i)} = e^{-rt} e^{\alpha_i X_t}$, $i = 1, 2$ are positive martingales.

We consider boundaries of the type

1.) $g(x) = x^2$

2.) $g(x) = \max\{(L - e^x)^+, (e^x - K)^+\}$

Let $p \in [0, 1]$. Let $M_t = pM_t^{(1)} + (1 - p)M_t^{(2)}$. Then

$$Ee^{-rT} g(X_t) = EM_T \frac{g(X_T)}{pe^{\alpha_1 X_T} + (1 - p)e^{\alpha_2 X_T}}.$$

Let $g(x)$ be nonnegative and measurable with

a) $\sup_{x \leq 0}(e^{-\alpha_1 x} g(x)) > \sup_{x \geq 0}(e^{-\alpha_1 x} g(x)) > 0$

b) $\sup_{x \geq 0}(e^{-\alpha_2 x} g(x)) > \sup_{x \leq 0}(e^{-\alpha_2 x} g(x)) > 0.$

Lemma

If a) and b) holds, there exists a $p^* \in (0,1)$ *with* $\sup_{x \geq 0} G_{p^*}(x) = \sup_{x \leq 0} G_{p^*}(x)$, *where*

$$G_p(x) = \frac{g(x)}{pe^{\alpha_1 x} + (1-p)e^{\alpha_2 x}}.$$

Theorem

Let $C^* = \sup_{x \in \mathbb{R}} G_{p^*}(x)$. *If there exists points* $x_1 > 0$ *and* $x_2 < 0$ *with* $G_{p^*}(x_1) = C^* = G_{p^*}(x_2)$.

Then $\sup_T Ee^{-rT} g(X_T) = C^*$

and $T^* = \inf\{t > 0 \mid X_t = x_1 \ \text{or} \ X_t = x_2\}.$

Stopping of Diffusions with Random Exponential Discounting

X diffusion with $X_0 = x$ and $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$

and B standard Brownian motion,

$g : \mathbb{R} \to \mathbb{R}_+$ a continuous function.

Find a stopping time T^* of X with

$$E_x\left(e^{-A(T)} g(X_T) 1_{\{T<\infty\}}\right) = \max.$$

$A(s)$: additive continuous stochastic process adapted to $\mathcal{F}^X$

$$A(s+t) = A(s) + A(t) \circ \theta_s$$

Example:

$$E_x\left(\exp\left\{-\int_0^T B_t^2 \, dt\right\} (B_T^+)^\alpha \, 1_{\{T<\infty\}}\right) = \max$$

How to choose the martingales?

$$\psi_+(x) = \begin{cases} E_x\left(e^{-A(T_{x_0})}1_{\{T_{x_0}<\infty\}}\right) & \text{for } x \le x_0 \\ \left[E_{x_0}\left(e^{-A(T_x)}1_{\{T_x<\infty\}}\right)\right]^{-1} & \text{for } x \ge x_0 \end{cases}$$

$$\psi_-(x) = \begin{cases} \left[E_{x_0}\left(e^{-A(T_x)}1_{\{T_x<\infty\}}\right)\right]^{-1} & \text{for } x \le x_0 \\ E_x\left(e^{-A(T_{x_0})}1_{\{T_{x_0}<\infty\}}\right) & \text{for } x \ge x_0. \end{cases}$$

$$\begin{aligned} M_t^{(+)} &= e^{-A(t)}\psi_+(X_t) \\ M_t^{(-)} &= e^{-A(t)}\psi_-(X_t) \end{aligned} \quad \text{are u.i. martingales with}$$

$$E_x\left(M_{T_b}^{(+)}1_{\{T_b<\infty\}}\right) = \psi_+(x) \quad \text{for } b \ge x \text{ on } 0 \le t \le T_b$$

$$E_x\left(M_{T_a}^{(-)}1_{\{T_a<\infty\}}\right) = \psi_-(x) \quad \text{for } x \ge a \text{ on } 0 \le t \le T_a.$$

Note:

If $A(t) = \int_0^t r(X_s)ds$ with $r(x) \ge 0$, then $\psi_\pm(x)$ are the solutions of $\mathcal{D}\psi = r \cdot \psi$ with

$$\mathcal{D} = \mu(x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma(x)\frac{\partial^2}{\partial x^2}.$$

Example 1:

$r(x) = rx^2$ with $r > 0$, $x = 0$.

Let $\psi(t) = e^{-x^2/2} \frac{2^{5/4}}{\Gamma(1/2)} \int_0^\infty e^{xt - t^2/2} \frac{1}{\sqrt{t}} dt$

Then $\psi_+(x) = \psi\left(\sqrt{8/r}\, x\right)$ is a solution of

$$\frac{1}{2}\psi''(x) = rx^2\psi(x) \qquad \text{with } \psi(0) = 1.$$

Then $\exp\left(-r\int_0^t X_s^2 ds\right)\psi_+(X_t)$ is a local martingale and

$$\sup_{x\in\mathbb{R}}\left[(x^+)^\alpha/\psi_+(x)\right] = \sup_{x\geq 0}\left[(x^+)^\alpha/\psi_+(x)\right] < \infty.$$

$T^* = \inf\{t > 0 \mid X_t = x^*\}$ with $x^* = \arg\max_x \left[(x^+)^\alpha/\psi_+(x)\right] > 0$

Distinguish the following cases for state space *I* :

1) $\sup_{x\geq x_0, x\in I} (g(x)/\psi_+(x)) = \infty$

2) $\sup_{x\leq x_0, x\in I} (g(x)/\psi_-(x)) = \infty$

3) $0 < C^* = \sup_{x\in I} \frac{g(x)}{\psi_+(x)} = \sup_{x\geq x_0, x\in I} \frac{g(x)}{\psi_+(x)}$

4) $0 < C^* = \sup_{x\in I} \frac{g(x)}{\psi_-(x)} = \sup_{x\leq x_0, x\in I} \frac{g(x)}{\psi_-(x)}$

5) $0 < \sup_{x\geq x_0, x\in I} (g(x)/\psi_+(x)) < \infty$

$0 < \sup_{x\leq x_0, x\in I} (g(x)/\psi_-(x)) < \infty$

and

$\sup_{x\leq x_0, x\in I} \frac{g(x)}{\psi_+(x)} > \sup_{x\geq x_0, x\in I} \frac{g(x)}{\psi_+(x)}$ and $\sup_{x\geq x_0, x\in I} \frac{g(x)}{\psi_-(x)} > \sup_{x\leq x_0, x\in I} \frac{g(x)}{\psi_-(x)}$

In case 5) there exists a $p^* \in (0,1)$ such that

$$\sup_{x\geq x_0, x\in I} \frac{g(x)}{p^*\psi_+(x) + (1-p^*)\psi_-(x)} = \sup_{x\leq x_0, x\in I} \frac{g(x)}{p^*\psi_+(x) + (1-p^*)\psi_-(x)}.$$

Case 5)

Theorem “5”:

Let x_0 be such, that $\psi_+(x_0) = 1 = \psi_-(x_0)$

1) Let p^* be such that

$$0 < \sup_{x \geq x_0, x \in I} \frac{g(x)}{p^*\psi_+(x)+(1-p^*)\psi_-(x)} = \sup_{x \leq x_0, x \in I} \frac{g(x)}{p^*\psi_+(x)+(1-p^*)\psi_-(x)}.$$

Then $\sup_T E_{x_0}\left(e^{-A_T} g(X_T) 1_{\{T<\infty\}}\right) = C^*$.

2) If there exist points $x_1 > x_0$ and $x_2 < x_0$ such that

$$\frac{g(x_1)}{p^*\psi_+(x_1) + (1-p^*)\psi_-(x_1)} = \frac{g(x_2)}{p^*\psi_+(x_2) + (1-p^*)\psi_-(x_2)} = C^*,$$

then the supremum is attained for $T^* = \inf\{t > 0 \mid X_t = x_1 \text{ or } X_t = x_2\}$.

3) Let $x_1 \leq x \leq x_2$. Then

$$\sup_T E_x\, e^{-A_T} g(X_T) = E_x\, e^{A_{T^*}} g(X_{T^*}) = C^*(p^*\psi_+(x) + (1-p)\psi_-(x))$$

Case $r(x) = r$

Remarks:

1) When g is twice continuously differentiable up to a finite number of points and when $T^* = \inf\{t > 0 \mid X_t \notin (x_1^*, x_2^*)\}$ is optimal. Then the FB-approach and the BL-approach yield the same value-function in the continuation set.
2) A complete characterization for all points of the stopping set has been given by Christensen in his dissertation (2010). He showed by using a Choquet-representation result for r-harmonic funtions that the optimal stopping set S^* can be characterized as

$$S^* = \left\{ x \mid \exists\, f \; r\text{-harmonic with } x = \arg\max_y \frac{g(y)}{f(y)} \right\}.$$

Multiplicative Minimax Characterization

Let $(Z_t, \mathcal{F}_t; t \geq 0)$ denote a right-continuous process of class D.

Theorem (Jamshidian (2007))

1) Let $m > 0$ and $Z_m \geq 0$ a.s. Then

$$\sup_{T \leq m} EZ_T = \inf_{M \in \mathcal{M}_+} E\left(M_m \sup_{0 \leq t \leq m} \frac{Z_t}{M_t}\right)$$

$\mathcal{M}_+$ denotes the class of positive $\mathcal{F}$-adapted martingales with $M_0 = 1$. If $Z_m > 0$ a.s. then the infimum is attained.

2) Let $Z_\infty := \overline{\lim_{t \to \infty}} Z_t \geq 0$. Then

$$\sup_{0 \leq T < \infty} EZ_T = \inf_{M \in C_+} E\left(M_\infty \sup_{0 \leq t < \infty} \frac{Z_t}{M_t}\right)$$

C_+ denotes the subclass of $\mathcal{M}_+$ consisting of the uniformly integrable martingales M with strictly positive limit.

Relation to the BL-approach

Let $Z_t = g(X_t) \cdot M_t$, with g a continuous function and X a continuous stochastic process and M a continuous positive martingale. Assume g has a unique maximum at x^*.

Then

$$\sup_{0 \leq t < \infty} \frac{Z_t}{M_t} \leq g(x^*).$$

Let $T^* = \inf\{t > 0 \,|\, X_t = x^*\} < \infty$ and $EM_{T^*} = 1$. Then

$$EZ_{T^*} = EM_{T^*}\, g(X_{T^*}) = g(x^*). \tag{E1}$$

But $M_{T^* \wedge t}$, $t \geq 0$ is <u>not</u> necessary a minimizing martingale in the sense of Jamshidian, namely:

$$\sup_T EZ_T = g(x^*) < EM_{T^*} \sup_{0 \leq t < \infty} \frac{Z_t}{M_{T^* \wedge t}}.$$

Note that for many examples of option pricing M is not uniformly integrable and

$$P\left(\sup_{t>T^*}\left(g(X_t)M_t\right) > g(x^*)M_{T^*}\right) > 0$$

holds.

Nevertheless $M_{m\wedge t}$ will approach the infimum in many cases when $m \to \infty$.

A Modified Minimax Duality

Theorem (Lerche-Urusov(2010))

Let the equation (E1) hold and let the process $(Z^{-}_{T^*\wedge t})$ *belong to class D. Then it holds*

$$\text{(E2)} \qquad \sup_{T} EZ_T = g(x^*) = \inf_{N\in C^+} E\left(N_{T^*} \sup_{t\geq 0} \frac{Z_{T^*\wedge t}}{N_{T^*\wedge t}}\right)$$

Furthermore the sequence $N^{(n)} = M_{T^*} + \frac{1}{n} \in C^+$ *is a minimizing sequence of (E2). If* $M_{T^*} > 0$ *a.s., then* $(M_{T^*\wedge t}, t \geq 0) \in C^+$ *and it is a minimizing martingale. Finally* $(g(x^*)M_{T^*\wedge t}, t \geq 0)$ *is the Snell-envelope of* Z *on* $\{T^* > 0\}$.

Sequential Test for Change Point

Let B be standard Brownian motion.

Observations:

a) $W_t = B_t \qquad \forall\, t > 0$

or

b) $W_t = B_t + D\,1_{\{t>\tau\}} \qquad \text{with } D = \begin{cases} +1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2} \end{cases}$

τ has distribution ϱ.

B, D, τ are independent.

P_∞: measure of $W = B$,

Q: measure of W for case b)

Consider

$$R_c(T) = P_\infty(T < \infty) + cE_Q(T - \tau)^+.$$

Find a stopping time T_c^*, which minimizes $R_c(T)$ for $c > 0$.

Then

$R_c(T) = E_Q\left(g_c(L_T) + c/I(\theta)\, V_T\right)$ with $g_c(x) = e^{-x} + c/I(\theta)\, x$,

$I(\theta) = \theta^2/2$ and $L_t = \log \left.\frac{dQ}{dP_\infty}\right|_{\mathcal{F}_t}$, with $\mathcal{F}_t = \sigma(W_s; 0 \le s \le t)$ and

$(V_t; t \ge 0)$ is a nonnegative increasing process.

This is a non-Markovian stopping problem since

$$L_t = \int_0^\infty \exp\left(-\theta^2/2\,(t-s)^+\right) \cosh(\theta(W_t - W_{t\wedge s}))\, \varrho(ds)$$

g_c has a minimum at $x^* = I(\theta)/c$.

Remark:

For $\tau = 0$: $\quad V_t = \frac{1}{2}\int_0^t \frac{\theta^2}{\cosh^2(\theta W_s)}\, ds$

Theorem

Let $S_b = \inf\{t > 0 \mid L_t \ge b\}$.

Then

$$\inf_T R_c(T) = R_c\left(S_{I(\theta)/c}\right) + o(c) \quad \textit{if } c \to 0.$$

Appendix

Generalized Parking Problem: Discrete Case, Details

$X_1, X_2, \ldots$ i.i.d. with $EX_1 > 0$, $\quad S_n = \sum_{i=1}^{n} X_i, \quad S_0 = 0.$

Find a stopping time T^* with $Eg(S_{T^*}) = \min_T Eg(S_T)$, where g is nonnegative convex with a unique minimum at b.

Solution:

$T^* = \min\{n \geq 0 \mid S_n \geq a\}$ with $a = \sup\{x \mid H^+ g(x) < g(x)\}$.

$$H^+ g(x) := \int g(x+y) H^+(dy)$$

$$H^+(y) := P(S_\eta \leq y)$$

$$\eta := \min\{n > 0 \mid S_n > 0\}$$

H^+: the distribution of the first ladder height S_η.

Let $K(z) = \int_0^z \frac{1 - H^+(y)}{\nu_1} \, dy$

with $\nu_i = \int y^i \, dH^+(y)$, $i \in \mathbb{N}$ and $\nu_1 = E(X_1) \exp\left(\sum_{i=1}^{\infty} \frac{1}{n} P[S_n < 0] \right)$.

Theorem

If $Kg(x) < \infty$ for all $0 \leq x < \infty$, then $Kg(x)$ is minimized at $x = a$.

Example 1:

If $g(x) = |x - b|$ for $x \in \mathbb{R} \quad \Rightarrow \quad a = b - \operatorname{med}(K)$

Example 2:

If $g(x) = (x - b)^2$ for $x \in \mathbb{R} \quad \Rightarrow \quad a = b - \frac{\nu_2}{\nu_1}$

Example 3:

If $g(x) = e^{-x} + cx$ for $x \in \mathbb{R}$, with $0 < c < 1 \quad \Rightarrow \quad b = \log\left(\frac{1}{c}\right)$.

If $\int x^2 H^+(dx) < \infty$ and if $\kappa := \int_0^\infty e^{-x} K(dx)$

$\Rightarrow Kg(x) = \kappa e^{-x} + c\left(x + \frac{\nu_2}{2\nu_1}\right)$

and is minimized when $a = \log\left(\frac{\kappa}{c}\right) = b - \log\left(\frac{1}{\kappa}\right)$.

$$Kg_c(b) = c\left[\left(1 + \log\frac{1}{c}\right) + \log\kappa + \frac{\nu_2}{2\nu_1}\right]$$

Nonlinear Parking Problem: Discrete Case

$Z_1, Z_2, \ldots$ a perturbed random walk, say

$$Z_n = S_n + \xi_n \quad \text{for } n = 0, 1, 2, \ldots ,$$

where

$$S_n = \sum_{i=1}^{n} X_i , \quad n \geq 1$$

with

$$X_1, X_2, \ldots \text{ i.i.d. with } EX_1 > 0,$$

having a non-arithmetic distribution.

ξ_n are slowly changing in the sense of "Woodroofe, SIAM, 1982".

Let g_c, $0 < c \leq 1$ denote convex functions. Find T_c^* with

$$Eg_c\left(Z_{T_c^*}\right) = \min_T Eg_c(Z_T).$$

For each $0 < c \leq 1$ let g_c be a convex function with a unique minimum at $b = b_c \geq 0$. Assume $\lim\limits_{c \downarrow 0} b_c = \infty$ and there exists a convex function $h_0 : \mathbb{R} \to \mathbb{R}$ with minimum at zero and with

$$h_c(x) := \frac{g_c(b+x) - g_c(b)}{c} \to h_0(x) < \infty.$$

Let $\quad K(y) = \int_0^y \frac{1 - H(x)}{\gamma_1} dx \quad$ for S_n; $n \geq 1$ and H

as in the *Generalized Parking Problem*.

Theorem

Let $\quad \gamma = \operatorname*{argmin}_x Kh(-x) \quad$ and $\quad T_{b-\gamma} = \min\{n \geq 1 \mid Z_n \geq b - \gamma\}$.

Then as $c \to 0$

$$\begin{aligned} \inf_T Eg_c(Z_T) &= Eg_c(Z_{T_{b-\gamma}}) + o(c) \\ &= g_c(b) + cKh_0(-\gamma) + o(c). \end{aligned}$$

Repeated Significance Test Again

$$R_c(T) = \int g_c\left(\frac{S_T^2}{T+r}\right) dQ\,,$$

with $$\begin{aligned} G_c(z) &= \Phi(-z) + cz/2, \\ Q &= \int P_\theta G(d\theta) \quad \text{with } G = N(0, r^{-1}), \\ b_c &= \arg\min_z g_c(z). \end{aligned}$$

Then

$$\inf_T R_c(T) \geq g_c(b_c) + c\left(\int_{-\infty}^{\infty} \inf_T E_\theta h(Z_T - b_c)\, G(d\theta) + o(1)\right)\,,$$

where $Z_n = \dfrac{S_n^2}{2(n+r)}$ and $h(z) = z + e^{-z} - 1$.

Then by the *Theorem* above

$$\begin{aligned}\inf_T E_\theta h(Z_T - b_c) &\geq K^\theta h(-\gamma(\theta)) + o(1) \\ &= \log(\kappa(\theta)) + \frac{\nu_2(\theta)}{2\nu_1(\theta)} + o(1).\end{aligned}$$

Note that $Z_n = \left(\theta S_n - \frac{n\theta^2}{2}\right) + \xi_n$

Finally:

$$\inf_T R_c(T) \geq g_c(b_c) + c\int_{-\infty}^{\infty}\left(\log(\kappa(\theta)) + \frac{\nu_2(\theta)}{2\nu_1(\theta)}\right) G(d\theta) + o(1).$$

Related Literature

The Main Idea

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Appendix
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