

# A Martingale Approach to Optimal Stopping

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# Examples of Optimal Stopping

## a) Detection of a Trend Change

Shiryaev (1961)

$B_t, t \geq 0$  standard Brownian motion

$\tau$  a random time

$$\text{Let } W_t = \begin{cases} B_t & \text{for } t < \tau \\ B_t + \theta(t - \tau) & \text{for } t \geq \tau \end{cases}$$

Issue:

Find a stopping time  $T^*$  such that the expected delay  $E(T - \tau \mid T \geq \tau)$  is minimal given the false alarm probability  $P(T < \tau)$ .

b) Stopping the Brownian Motion at the Maximum

Graversen, Peskir, Shiryaev (2001)

$B_t$ ,  $0 \leq t \leq 1$  standard Brownian motion

$$R(T) := E \left( B_T - \max_{0 \leq s \leq 1} B_s \right)^2$$

Here  $T$  denotes a stopping time of Brownian motion with  $0 \leq T \leq 1$ .

Find a stopping time  $T^*$  with

$$R(T^*) = \min_T R(T).$$

### c) Perpetual American Put Option

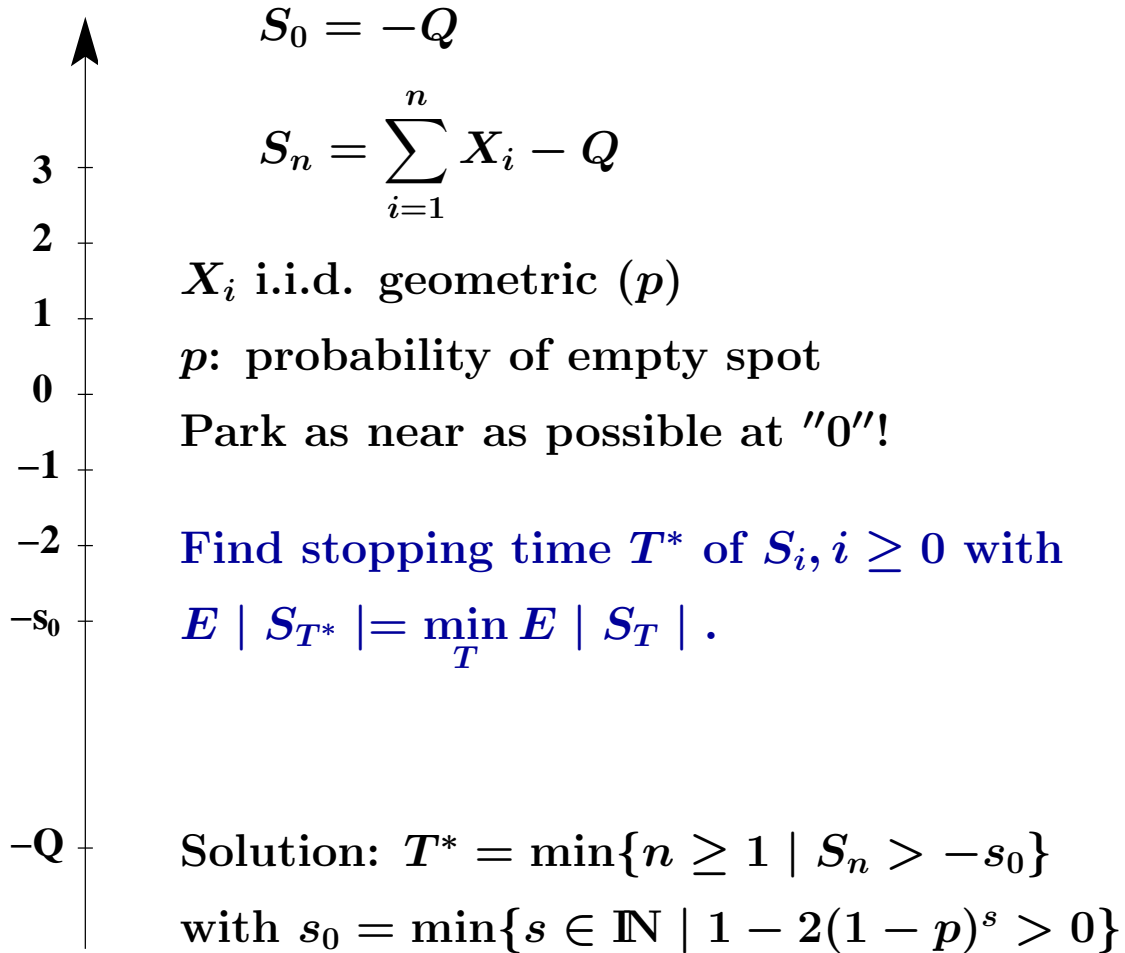
Samuelson (1965), McKean (1965)

$X_t = \sigma B_t + \mu t$  Brownian Motion with drift  $\mu$  and variance  $\sigma^2$ .

Find a stopping time  $T^*$  which maximizes

$$Ee^{-rT}(K - e^{X_T})^+ 1_{\{T < \infty\}}.$$

# The Parking Problem



Chow, Robbins, Siegmund (1968)

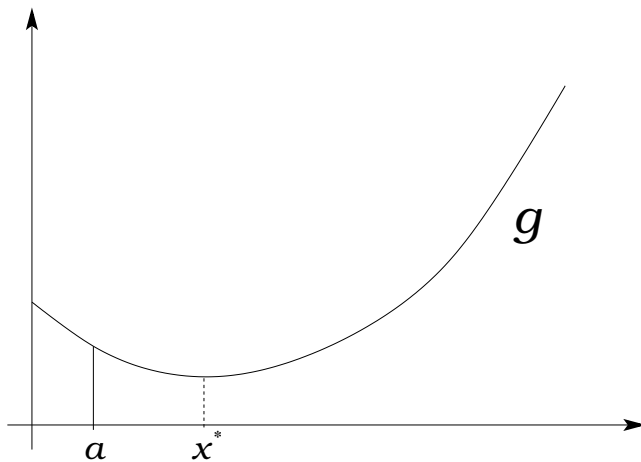
# Generalized Parking Problem (GPP)

Let  $g$  be a convex nonnegative function with a unique minimum at  $x^* > 0$ .

Assume  $X_i$  i.i.d. with  $EX_i > 0$ ,

$$S_n = \sum_{i=1}^n X_i, \quad S_0 = 0.$$

Find a stopping time  $T^*$  with  $Eg(S_{T^*}) = \min_T Eg(S_T)$



Solution (Keener, Lerche, Woodroffe '94):

$$T^* = \min\{n \geq 0 \mid S_n \geq a\} \text{ with } a < x^*$$

with  $a = \sup\{x \mid H^+g(x) < g(x)\}$  where  $H^+$  is the ladder-height distribution of  $S_n$ ;  $n \geq 1$ .

# The Main Idea: OS as GPP

Let  $(Z_t, \mathcal{F}_t; t \geq 0)$  denote a continuous stochastic process on a probability space  $(\Omega, \mathcal{F}, P)$ .

Find a stopping time  $T^*$  with:

$$E_P (Z_{T^*} 1_{\{T^* < \infty\}}) = \max_T E_P (Z_T 1_{\{T < \infty\}})$$

Idea:

Find a process  $(X_t, \mathcal{F}_t; t \geq 0)$ , a probability measure  $Q (Q \ll P)$  and a function  $g$  with unique maximum at  $x^*$  such that

$$Z_t = g(X_t) \frac{dQ}{dP} \Big|_{\mathcal{F}_t}$$

Then

$$\begin{aligned} E Z_T 1_{\{T < \infty\}} &= E \left( g(X_T) \frac{dQ}{dP} \Big|_{\mathcal{F}_T} \right) \\ &= E_Q (g(X_T) 1_{\{T < \infty\}}) \\ &\leq g(x^*) Q (T < \infty) \\ &\leq g(x^*) \end{aligned}$$

With  $T^* = \min\{t \geq 0 \mid X_t = x^*\}$  the inequalities become equalities, if  $Q (T^* < \infty) = 1$ .

# Where does this approach apply?

1. Repeated Significance Test
2. Disruption Problem
3. Cusum Test
4. Test of Power One for Change Point Alternatives
5. Detection with exponentially penalty for delay
6. Optimality of Parabolic Boundaries
7. Stopping near the maximum of a Brownian motion
8. Perpetual Call and Put Options
9. Perpetual Straddle and Strangles
10. Russian Options
11. Perpetual Put with running maximum as strike price
12. Integral Options
13. Random Exponential Discounting of Diffusious

# The Repeated Significance Test is a Bayes-Test

$W(t), t \geq 0$  Brownian motion with drift  $\theta$

Testing sequentially:  $H_0 : \theta < 0$  versus  $H_1 : \theta > 0$

Prior:  $G(d\theta) = \varphi(\tau\theta)\sqrt{\tau}d\theta$

$$R(T, \delta) = \int_{-\infty}^0 \left( P_{\theta}\{\delta \text{ rejects } H_0\} + \frac{c}{2}\theta^2 E_{\theta}T \right) G(d\theta) \\ + \int_0^{\infty} \left( P_{\theta}\{\delta \text{ rejects } H_1\} + \frac{c}{2}\theta^2 E_{\theta}T \right) G(d\theta)$$

Find  $(T^*, \delta^*)$  with  $R(T^*, \delta^*) = \min_{(T, \delta)} R(T, \delta)$ .

$$\delta^* = \delta_T^* = 1_{\{W(T) > 0\}} \quad T^* = ?$$

PNAS, 83 (1986)

$$\begin{aligned}
& \int_{-\infty}^0 P_\theta\{\delta \text{ rejects } H_0\}G(d\theta) + \int_0^\infty P_\theta\{\delta \text{ rejects } H_1\}G(d\theta) \\
&= \int G_{x,T}(-\infty, 0]Q(dx) \\
&\quad + \int G_{x,T}(0, \infty]Q(dx) \\
&\geq \int \min_x(G_{x,T}(-\infty, 0], G_{x,T}(0, \infty])Q(dx) \\
&= \int \Phi\left(-\frac{|W(T)|}{\sqrt{T+r}}\right) dQ
\end{aligned}$$

$$G = N(0, r^{-1}), \quad Q = \int P_\theta G(d\theta)$$

$$\begin{aligned}
& \int \theta^2 E_\theta T G(d\theta) \\
&= \int T \int \theta^2 |G_{W(T),T}(d\theta)| dQ \\
&= \int T \left( \frac{W(T)^2}{(T+r)^2} + \frac{1}{T+r} \right) dQ \\
&= \int (T+r) \left( \frac{W(T)^2}{(T+r)^2} + \frac{1}{T+r} \right) dQ - 1 \\
&= \int \frac{W(T)^2}{T+r} dQ
\end{aligned}$$

$$G_{W(T),T} = N\left(\frac{W(T)}{T+r}, \frac{1}{T+r}\right)$$

Representation of the risk:

$$R(T, \delta_T^*) = \int g\left(\frac{W(T)^2}{T+r}\right) dQ$$

with  $g(x) = \Phi(-\sqrt{x}) + cx/2$

$g$  is convex with unique minimum  $x^*$

$$R(T, \delta_T^*) = \int g\left(\frac{W(T)^2}{T+r}\right) dQ \geq g(x^*)$$

Let  $T^* = \min\{t > 0 \mid W(t)^2/(t+r) = x^*\}$

Since  $Q\{T^* < \infty\} = 1$  it follows

$$R(T^*, \delta_{T^*}^*) = g(x^*).$$

The same type of argument holds for the SPRT.

The representation is  $R(T, \delta_T^*) = \int g(|\theta| |W(T)|) d\bar{Q}$

with  $\bar{Q} = \frac{1}{2}(P_\theta + P_{-\theta})$ ,  $\theta > 0$ .

# Disruption Problem

Dissertation of Shiryaev (1961)

Observations:  $W_t = B_t + \theta(t - \tau)^+$  with

$B_t, t \geq 0$  standard Brownian motion,

$\theta > 0$  fixed

Filtration:  $\mathcal{F}_t = \sigma(W_s; 0 \leq s \leq t)$

Change-point:  $\tau$  random time,

with distribution  $\pi = p\delta_0 + (1-p)F$ ,

where  $F(t) = 1 - e^{-\lambda t}$

Risk:  $R(T) = P_\pi(T < \tau) + cE_\pi(T - \tau)^+$

Find  $T^*$  with  $R(T^*) = \min_T R(T)$

## Theorem

$T^* = \min\{t > 0 \mid \pi_t \geq p^*\}$  with  $\pi_t = P(\tau \leq t \mid \mathcal{F}_t)$

Here  $p^*$  is the unique solution in  $(0, 1)$  of  $G'(p) = 1$ ,

where  $G$  is the (and finite at 0) solution of

$$\frac{\theta}{2}x^2(1-x^2)G''(x) + \lambda(1-x)G'(x) = cx$$

$$\pi_t = \frac{\varphi_t}{e^{-\lambda t} + \varphi_t} \quad \text{posterior}$$

where

$$\varphi_t = \frac{p}{1-p} L_t + \int_0^t \frac{L_t}{L_s} \lambda e^{-\lambda s} ds$$

and

$$L_t = \exp(\theta W_t - \theta^2 t/2)$$

Case  $p = 0$ :

$$P(\tau \leq t) = 1 - e^{-\lambda t} \quad \text{prior}$$

$$R\pi_t = \frac{\pi_t}{1 - e^{-\lambda t}} - 1 \quad \text{relative posterior}$$

## Disruption Problem: The Representation

Let  $\pi_t = P(\tau < t \mid \mathcal{F}_t)$ .

$\pi_t$  is a diffusion with

$$d\pi_t = \lambda(1 - \pi_t)dt + \theta\pi_t(1 - \pi_t)d\bar{W}_t$$

where  $\bar{W}_t$  is a standard Brownian motion.

Ito's formula yields:

$$\begin{aligned} dG(\pi_t) &= G'(\pi_t)d\pi_t + \frac{1}{2}G''(\pi_t)(d\pi_t)^2 \\ &= G'(\pi_t) [\lambda(1 - \pi_t)dt + \theta\pi_t(1 - \pi_t)d\bar{W}_t] \\ &\quad + \frac{1}{2}G''(\pi_t)\theta^2\pi_t^2(1 - \pi_t)^2dt \end{aligned}$$

If  $G$  satisfies the equation

$$\frac{\theta^2}{2}x^2(1 - x)^2G''(x) + \lambda(1 - x)G'(x) = cx$$

then

$$\begin{aligned} G(\pi_t) - G(\pi_0) &= c \int_0^t \pi_s ds + c \int_0^t \theta\pi_s(1 - \pi_s)d\bar{W}_s \\ \Rightarrow E [G(\pi_T) - G(\pi_0)] &= c E \int_0^T \pi_s ds \end{aligned}$$

$$\begin{aligned}
R(T) &= P(T < \tau) + cE(T - \tau)^+ \\
&= E \left[ (1 - \pi_T) + c \int_0^T \pi_s ds \right]
\end{aligned}$$

Then with  $g(x) = (1 - x) + G(x)$  yields

$$R(T) = \int g(\pi_T) dP - g(p)$$

$g$  is convex with a unique minimum at  $p^*$ .

This insight opens a new direction to Bayes tests of power one for change point problems. Cusum and Mixture stopping rules can be derived as Bayes tests. (Beibel 1996, 1997), (Beibel – Lerche 2003).

# Perpetual American Put Option

Samuelson(1965), McKean(1965)

$X_t = \sigma B_t + \mu t$  Brownian Motion with drift  $\mu$  and variance  $\sigma^2$ .

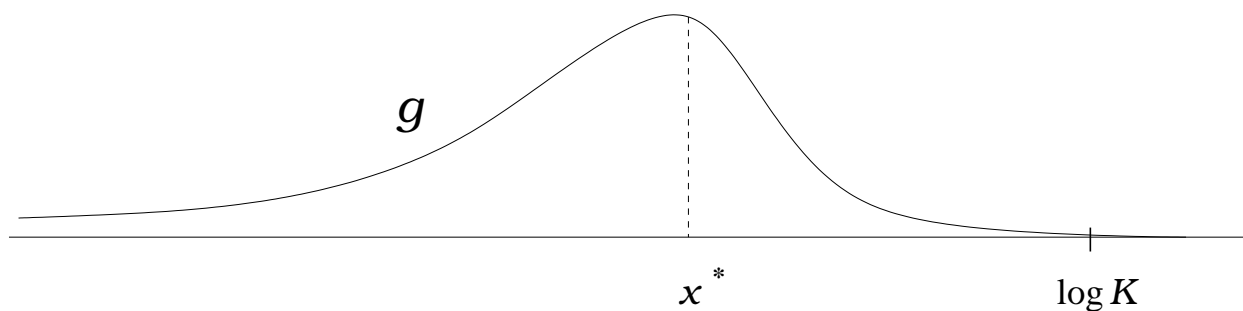
Find a stopping time  $T^*$  which maximizes

$$E_P e^{-rT} (K - e^{X_T})^+ 1_{\{T < \infty\}}.$$

Idea:

Find  $Q$  and  $g$  with  $E_P e^{-rT} (K - e^{X_T})^+ = E_Q g(X_T)$ , where  $Q \ll P$  and  $g$  has a unique maximum at  $x^*$ .

Then  $T^* = \min\{t \geq 0 \mid X(t) = x^*\}$ , if  $Q(T^* < \infty) = 1$ .



Let  $f(x) = (K - e^x)^+$ .

How to find  $M_t = \frac{dQ_t}{dP_t}$  ?

It holds for all  $\alpha \in \mathbb{R}$

$$E_P e^{-rT} f(X_T) = E_P f(X_T) (e^{X_T})^{-\alpha} (e^{X_T})^{\alpha} e^{-rT}.$$

Choose  $g(x) = f(x)e^{-\alpha x}$  and  $\alpha$  such that

$M_t = e^{\alpha X_t} e^{-rt}$  is a martingale.

This holds when

$$\begin{aligned} M_t &= \exp [\alpha(\sigma B_t) + \alpha\mu t - rt] \\ &= \exp [(\alpha\sigma)B_t - t(\alpha\sigma)^2/2]. \end{aligned}$$

$M_t$  is a positive martingale with  $M_0 = 1$

iff  $(\alpha\sigma)^2/2 + \alpha\mu - r = 0$

$\alpha^{\pm} = -\frac{\mu}{\sigma^2} \pm \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}}$  are the two possible values.

Then we have

$$E_P e^{-rT} (K - e^{X_T})^+ 1_{\{T < \infty\}} = E_Q g(X_T)$$

with  $g(x) = f(x)e^{-\alpha^- x}$  and

$$\frac{dQ_t}{dP_t} = M_t = \exp \left[ (\alpha^- \sigma) B_t - \frac{t(\alpha^- \sigma)^2}{2} \right]$$

Let  $K < 1 + (-\alpha^-)^{-1}$ . Then  $g$  has a unique maximum at  $x^* = \log \frac{\alpha^- K}{\alpha^- - 1} < 0$ . Under  $Q$   $X_t$  is Brownian motion with drift

$$\alpha^- \sigma^2 + \mu = -\sigma^2 \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}} < 0.$$

This yields  $Q(T^* < \infty) = 1$ , if  $x^* < 0$ .

If  $\mu < 0$  then  $P(T^* < \infty) = 1$  and

$$\sup_T E_P e^{-rT} (K - e^{X_T})^+ = E_P e^{-rT^*} (K - e^{X_{T^*}})^+.$$

# Optimality of Parabolic Boundaries

Let  $X_t = B_t + x_0$ ,  $t \geq 0$   $B_t$  standard Brownian motion. For a measurable function  $h$  find a stopping time  $T$  that maximizes

$$E \left( (T + 1)^{-\beta} h \left( \frac{X_T}{\sqrt{T + 1}} \right) \right) !$$

(Moerbeke(1974))

Let  $H(x) = \int_0^\infty e^{ux - u^2/2} u^{2\beta-1} du$  with  $\beta > 0$  and assume that there exists a unique point  $x^*$  with

$$\sup_{x \in \mathbb{R}} \frac{h(x)}{H(x)} = \frac{h(x^*)}{H(x^*)} = C^* \quad \text{and } 0 < C^* < \infty$$

## Theorem:

Let  $x_0 < x^*$ . Then

$$\begin{aligned} \sup_T E \left\{ (T + 1)^{-\beta} h \left( \frac{X_T}{\sqrt{T + 1}} \right) \right\} \\ = E \left\{ (T^* + 1)^{-\beta} h \left( \frac{X_{T^*}}{\sqrt{T^* + 1}} \right) \right\} \\ = H(x_0) C^* \end{aligned}$$

where  $T^* = \inf \left\{ t > 0 \mid \frac{X_t}{\sqrt{t+1}} = x^* \right\}$ .

$$M_t = (t + 1)^{-\beta} H \left( \frac{X_t}{\sqrt{t + 1}} \right) = \int_0^\infty e^{uX_t - \frac{u^2}{2}t} e^{-\frac{u^2}{2}} u^{2\beta-1} du$$

is a positive martingale with starting value  $H(x_0)$ .

Then  $M_t = (t + 1)^{-\beta} H(X_t/\sqrt{t + 1})/H(x_0)$  is a positive martingale with  $EM_0 = 1$ .

Then

$$(t + 1)^{-\beta} h \left( \frac{X_t}{\sqrt{t + 1}} \right) = H(x_0) \frac{h \left( \frac{X_t}{\sqrt{t+1}} \right)}{H \left( \frac{X_t}{\sqrt{t+1}} \right)} M_t.$$

Then with  $\bar{P} = M \cdot P$

$$\begin{aligned} E \left( (T + 1)^{-\beta} h \left( \frac{X_T}{\sqrt{T + 1}} \right) \right) &= H(x_0) \bar{E} \frac{h \left( \frac{X_T}{\sqrt{T+1}} \right)}{H \left( \frac{X_T}{\sqrt{T+1}} \right)} \\ &\leq H(x_0) C^*. \end{aligned}$$

But  $EM_{T^*} = 1$  and  $P(T^* < \infty) = 1$  for

$$T^* = \inf \left\{ t > 0 \mid \frac{X_t}{\sqrt{t + 1}} = x^* \right\}.$$

Special case: **Dvoretzky(1965), Chow-Robbins(1965)**

$$h(x) = x, \quad x_0 = 0, \quad \beta = \frac{1}{2}$$

$$E(X_T/(T + 1)) = \max \text{ with}$$

$$T^* = \min \left\{ t > 0 \mid \frac{X_t}{\sqrt{t + 1}} = x^* \right\}$$

$x^*$  is solution of  $x = (1-x^2) \int_0^\infty e^{ux-u^2/2} du$  (**Shepp 1969**)

# Stopping near the Maximum of Brownian motion

Graversen, Peskir, Shiryaev (2001)

$$T^* = \inf\{t \leq 1 \mid S_t - B_t \geq z_* \sqrt{1-t}\}$$

where  $S_t = \max_{0 \leq s \leq t} B_s$ .

$z_*$  is the unique solution of the equation

$$4\Phi(z_*) - 2z_*\varphi(z_*) - 3 = 0, \quad z_* \cong 1,12.$$

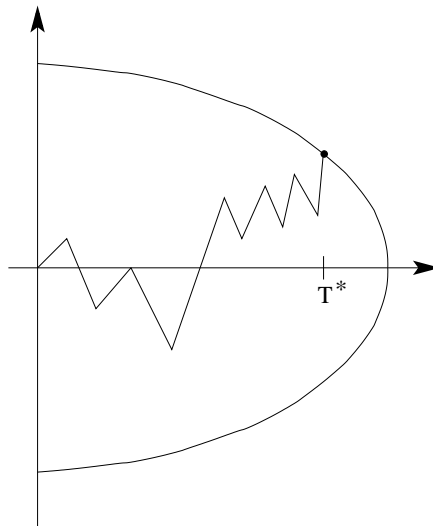
Note:  $\mathcal{L}(S - B) = \mathcal{L}(|B|)$

It holds:

$$ET^* = \frac{z_*^2}{1 + z_*^2} \cong 0,55, \quad \text{Var}T^* \cong 0,05$$

$$ET^* = EB_{T^*}^2 = z_*^2 E(1 - T^*)$$

Idea:



Let  $V_* = R(T^*)$

$$\text{Then } V_* = 2 \inf_{\sigma} E \left( \int_0^{\sigma} e^{-2s} F(|Z_s|) ds \right) + 1$$

where  $F(x) = 4\Phi(x) - 3$  and  $Z_s$  is a diffusion process with  $dZ_t = Z_t dt + \sqrt{2} d\beta_t$  and  $\beta_t, t \geq 0$  is a SBM.

$$\text{Let } W_*(z) = \inf_{\sigma} E_z \left( \int_0^{\sigma} e^{-2s} F(|Z_s|) ds \right).$$

$$\text{Then } V_* = 2W_*(0) + 1$$

To determine  $W_*$  solve

$$\begin{aligned} \text{FBP: } (\mathcal{D} - 2)W(z) &= -F(|z|) \quad \text{for } -z_* \leq z \leq z_* \\ W(\pm z_*) &= 0 \\ W'(\pm z_*) &= 0 \end{aligned}$$

$$\text{with } \mathcal{D} = z \frac{d}{dz} + \frac{d^2}{dz^2}.$$

With Ito's formula

$$E_z (e^{-2\tau} W(|Z_{\tau}|) - W(z)) = -E_z \int_0^{\tau} e^{-2t} F(|Z_t|) dt$$

Find a stopping time  $\sigma^*$  with

$$E_z \left( e^{-2\sigma^*} W(|Z_{\sigma^*}|) \right) = \max_{\tau} E (e^{-2\tau} W(|Z_{\tau}|))$$

$$\text{Solution: } \sigma^* = \inf\{t \geq 0 \mid |Z_t| = z_*\}$$

# Perpetual Russian Options

Shepp, Shiryaev 1993

$$X_t = \exp\left(\sigma W_t + (\mu - \sigma^2/2)t\right), \quad t \geq 0$$

geometric Brownian motion with  $\sigma > 0$ ,  $\mu \in \mathbb{R}$

$$S_t = \max_{0 \leq s \leq t} X_s$$

Find a stopping time  $T$  that maximizes

$$E(e^{-rT} S_t), \quad \text{when } r > \mu.$$

Let  $\gamma_{1,2} = \frac{\sigma^2/2 - \mu \mp \sqrt{(\sigma^2/2 - \mu)^2 + 2\sigma^2 r}}{\sigma^2}$  and  $\alpha = \left(\frac{1 - 1/\gamma_1}{1 - 1/\gamma_2}\right)^{1/(\gamma_2 - \gamma_1)}$

Then  $T^* = \inf\{t > 0 \mid S_t/X_t = \alpha\}$

**Proof:** The continuous semimartingale  $\frac{X_t}{S_t}$  satisfies the stochastic differential equation

$$d\frac{X_t}{S_t} = \frac{1}{S_t}dX_t - \frac{X_t}{S_t^2}dS_t$$

since  $S$  has increasing paths. For sufficiently smooth functions  $h$  we therefore obtain

$$\begin{aligned} & e^{-rt}S_t h\left(\frac{X_t}{S_t}\right) \\ &= \int_0^t e^{-ru}S_u \left\{ -rh\left(\frac{X_u}{S_u}\right) + \mu\left(\frac{X_u}{S_u}\right)h'\left(\frac{X_u}{S_u}\right) \right. \\ & \quad \left. + \frac{\sigma^2}{2}\left(\frac{X_u}{S_u}\right)^2 h''\left(\frac{X_u}{S_u}\right) \right\} du \\ & \quad + \int_0^t e^{-ru} \left\{ -\left(\frac{X_u}{S_u}\right)h'\left(\frac{X_u}{S_u}\right) + h\left(\frac{X_u}{S_u}\right) \right\} dS_u \\ & \quad + \sigma \int_0^t e^{-ru} X_u h'\left(\frac{X_u}{S_u}\right) dW_u. \end{aligned}$$

The process  $S$  is flat off the set  $\{t \mid S_t = X_t\}$ .

Therefore

$$\begin{aligned} & \int_0^t e^{-ru} \left\{ -\left(\frac{X_u}{S_u}\right)h'\left(\frac{X_u}{S_u}\right) + h\left(\frac{X_u}{S_u}\right) \right\} dS_u \\ &= \int_0^t e^{-ru} \{-h'(1) + h(1)\} dS_u. \end{aligned}$$

Hence for sufficiently smooth functions  $h$  the process  $e^{-rt}S_t h\left(\frac{X_t}{S_t}\right)$  is a local martingale if  $h$  satisfies

$$\begin{aligned} 0 &= -rh(x) + \mu x h'(x) + \frac{\sigma^2}{2} x^2 h''(x) \text{ for all } x \in (-\infty, 1] \\ 0 &= h'(1) - h(1). \end{aligned}$$

Let  $h(x) = \frac{1}{\gamma_2 \alpha^{\gamma_1} - \gamma_1 \alpha^{\gamma_2}} (\gamma_2 (\alpha x)^{\gamma_1} - \gamma_1 (\alpha x)^{\gamma_2})$  with  $\gamma_{1,2}$  and  $\alpha$  as above.

Then  $\inf_{0 < x \leq 1} h(x) = h(\alpha^{-1}) > 0$  and

$M_t = e^{-rt} S_t h\left(\frac{X_t}{S_t}\right)$  is a positive local martingale.

Then

$$\begin{aligned} E(e^{-rT} S_T) &= E h\left(\frac{X_T}{S_T}\right)^{-1} M_T \\ &\leq \frac{1}{h(\alpha^{-1})} E M_T \\ &\leq \frac{1}{h(\alpha^{-1})} \end{aligned}$$

Since  $P(T^* < \infty) = 1$  and  $E M_{T^*} = 1$ , the inequalities become equalities for  $T^*$ .

# Put Options with strike $\max_{0 \leq s \leq t} X_s$

$X_t$  geometric Brownian motion

$$S_t = \max_{0 \leq u \leq t} X_u$$

Find a stopping time  $T^*$  which maximizes

$$E \left( e^{-rT} (S_t - X_T) \mathbf{1}_{(T < \infty)} \right).$$

Let  $\gamma_{1,2} = - \left( \frac{\mu}{\sigma^2} - \frac{1}{2} \right) \mp \sqrt{\frac{2r}{\sigma^2} + \left( \frac{\mu}{\sigma^2} - \frac{1}{2} \right)^2}$  and

$$\alpha = \left( \frac{1-1/\gamma_1}{1-1/\gamma_2} \right)^{1/(\gamma_2-\gamma_1)}.$$

Let  $h(x) = \frac{1}{\gamma_2 \alpha^{\gamma_1} - \gamma_1 \alpha^{\gamma_2}} (\gamma_2 (\alpha x)^{\gamma_1} - \gamma_1 (\alpha x)^{\gamma_2})$ .

Then

$$E e^{-rT} (S_T - X_T) = \bar{E} \frac{1 - X_T/S_T}{h(X_T/S_T)},$$

where  $\bar{P} = M \cdot P$ .

Let  $\mu \leq r/2$  and

let  $x^* = \operatorname{argmax}\{(1-x)/h(x) \mid x \in (0, 1)\}$ .

Then  $T^* = \inf\{t > 0 \mid X_t = x^* S_t\}$ .

# One-Sided Boundaries

Let  $h$  be measurable,  $X_t = \sigma B_t + \mu t$  Brownian motion with drift  $\mu$  and variance  $\sigma^2$ . Find a stopping time  $T^*$  which maximizes

$$Ee^{-rT}h(X_T)1_{\{T<\infty\}}.$$

Let  $\alpha_{1,2} = -\frac{\mu}{\sigma^2} \pm \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}}$  ( $\alpha_2 < 0 < \alpha_1$ ).

Then  $M_t^{(i)} = e^{-rt}e^{\alpha_i X_t}$ ,  $i = 1, 2$  are positive martingales.

## Theorem 1:

If  $0 < C_1 = \sup_{x \in \mathbb{R}}(e^{-\alpha_1 x} h(x)) < \infty$  and  $C_1 = e^{-\alpha_1 x_1} h(x_1)$  for some  $x_1 > 0$ , then

$$\sup_T Ee^{-rT}h(X_T)1_{\{T<\infty\}} = C_1,$$

$$T^* = \inf\{t > 0 \mid X_t = x_1\}.$$

## Theorem 2:

If  $0 < C_2 = \sup_{x \in \mathbb{R}}(e^{-\alpha_2 x} h(x)) < \infty$  and  $C_2 = e^{-\alpha_2 x_2} h(x_2)$  for some  $x_2 > 0$ , then

$$\sup_T Ee^{-rT}h(X_T)1_{\{T<\infty\}} = C_2,$$

$$T^* = \inf\{t > 0 \mid X_t = x_2\}.$$

# Two-Sided Boundaries

Let  $h(x)$  be nonnegative and measurable with

$$\text{a) } \sup_{x \leq 0} (e^{-\alpha_1 x} h(x)) > \sup_{x \geq 0} (e^{-\alpha_1 x} h(x)) > 0 \text{ and}$$

$$\text{b) } \sup_{x \geq 0} (e^{-\alpha_2 x} h(x)) > \sup_{x \leq 0} (e^{-\alpha_2 x} h(x)) > 0.$$

Examples:

$$1.) \quad h(x) = x^2$$

$$2.) \quad h(x) = \max\{(L - e^x)^+, (e^{-x} - K)^+\}$$

Let  $p \in [0, 1]$ . Let  $M_t = pM_t^{(1)} + (1-p)M_t^{(2)}$ . Then

$$Ee^{-rT} h(X_t) = EM_T \frac{h(X_T)}{pe^{\alpha_1 X_T} + (1-p)e^{\alpha_2 X_T}}.$$

Lemma: If a) and b) holds, there exists a  $p^* \in (0, 1)$

with  $\sup_{x \geq 0} G_{p^*}(x) = \sup_{x \leq 0} G_{p^*}(x)$ , where

$$G_p(x) = \frac{h(x)}{pe^{\alpha_1 x} + (1-p)e^{\alpha_2 x}}.$$

Theorem 3:

Let  $C^* = \sup_{x \in \mathbb{R}} G_{p^*}(x)$ . If there exists points  $x_1 > 0$  and  $x_2 < 0$  with  $G_{p^*}(x_1) = C^* = G_{p^*}(x_2)$ , then

$$\sup_T Ee^{-rT} h(X_T) 1_{\{T < \infty\}} = C^*$$

and

$$T^* = \inf\{t > 0 \mid X_t = x_1 \text{ or } X_t = x_2\}.$$

# Stopping of Diffusions with Random Exponential Discounting

$B_t, t \geq 0$  SBM

$X_t$  diffusion with  $X_0 = x$  and

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$$

$h : \mathbb{R} \rightarrow \mathbb{R}_+$  a continuous function

Find a stopping time  $T^*$  of  $X$  with

$$E\left(e^{-A(T)}h(X_T)1_{\{T<\infty\}}\right) = \max$$

$A(s)$ : additive continuous stochastic process adapted to  $\mathcal{F}^X$

$$A(s+t) = A(s) + A(t) \circ \theta_s$$

Example:  $r(x) \geq 0, \alpha > 0$

$$E\left(\exp\left\{-\int_0^T r(B_t)dt\right\}(B_T)^\alpha 1_{\{T<\infty\}}\right) = \max$$

$$X(0) = x_0, \quad x_0 \in I$$

Assume  $\psi_+(x_0) = \psi_-(x_0) = 1$

Then

$$e^{-At}h(X_t) = M_t \frac{h(X_t)}{p\psi_+(X_t) + (1-p)\psi_-(X_t)}$$

with  $M_t = e^{-At} (p\psi_+(X_t) + (1-p)\psi_-(X_t))$

for any  $p \in [0, 1]$  and  $0 \leq t < \infty$ .

$M_t$  is a positive local martingale and hence

$$E(M_T 1_{\{T < \infty\}}) \leq 1.$$

Problem:

Maximize  $\frac{h(x)}{p\psi_+(x) + (1-p)\psi_-(x)}$  over all  $x \in I$  with a proper  $p$ .

## How to choose the martingales?

$$\psi_+(x) = \begin{cases} E_x \left( e^{-A(T_{x_0})} \mathbf{1}_{\{T_{x_0} < \infty\}} \right) & \text{for } x \leq x_0 \\ [E_{x_0} \left( e^{-A(T_x)} \mathbf{1}_{\{T_x < \infty\}} \right)]^{-1} & \text{for } x \geq x_0 \end{cases}$$

$$\psi_-(x) = \begin{cases} [E_{x_0} \left( e^{-A(T_x)} \mathbf{1}_{\{T_x < \infty\}} \right)]^{-1} & \text{for } x \leq x_0 \\ E_x \left( e^{-A(T_{x_0})} \mathbf{1}_{\{T_{x_0} < \infty\}} \right) & \text{for } x \geq x_0 \end{cases}$$

$$\begin{aligned} M_t^{(+)} &= e^{-A(t)} \psi_+(X_t) \\ M_t^{(-)} &= e^{-A(t)} \psi_-(X_t) \end{aligned} \quad \text{are u.i. martingales with}$$

$$E_x(M_{T_b}^{(+)} \mathbf{1}_{\{T_b < \infty\}}) = \psi_+(x) \quad \text{for } b \geq x \text{ on } 0 \leq t \leq T_b$$

$$E_x(M_{T_a}^{(-)} \mathbf{1}_{\{T_a < \infty\}}) = \psi_-(x) \quad \text{for } x \geq a \text{ on } 0 \leq t \leq T_a.$$

### Note:

If  $A(t) = \int_0^t r(X_s) ds$  with  $r(x) \geq 0$ , then  $\psi_{\pm}(x)$  are the solutions of  $\mathcal{D}\psi = r \cdot \psi$  with appropriate boundary conditions.

$$\mathcal{D} = \mu(x) \frac{\partial}{\partial x} + \frac{1}{2} \sigma(x) \frac{\partial^2}{\partial x^2}.$$

## Distinguish the following Cases

$$1) \quad \sup_{x \geq x_0, x \in I} (h(x)/\psi_+(x)) = \infty$$

$$2) \quad \sup_{x \leq x_0, x \in I} (h(x)/\psi_-(x)) = \infty$$

$$3) \quad 0 < C^* = \sup_{x \in I} \frac{h(x)}{\psi_+(x)} = \sup_{x \geq x_0} \frac{h(x)}{\psi_+(x)}$$

$$4) \quad 0 < C^* = \sup_{x \in I} \frac{h(x)}{\psi_-(x)} = \sup_{x \leq x_0, x \in I} \frac{h(x)}{\psi_-(x)}$$

$$5) \quad 0 < \sup_{x \geq x_0, x \in I} (h(x)/\psi_+(x)) < \infty$$

$$0 < \sup_{x \leq x_0, x \in I} (h(x)/\psi_-(x)) < \infty$$

and

$$\sup_{x \leq x_0, x \in I} \frac{h(x)}{\psi_+(x)} > \sup_{x \geq x_0, x \in I} \frac{h(x)}{\psi_+(x)} \text{ and}$$

$$\sup_{x \geq x_0, x \in I} \frac{h(x)}{\psi_-(x)} > \sup_{x \leq x_0, x \in I} \frac{h(x)}{\psi_-(x)}$$

In case 5) there exists a  $p^* \in (0, 1)$  such that

$$\sup_{x \geq x_0, x \in I} \frac{h(x)}{p^*\psi_+(x) + (1-p^*)\psi_-(x)} = \sup_{x \leq x_0, x \in I} \frac{h(x)}{p^*\psi_+(x) + (1-p^*)\psi_-(x)}.$$

## Case 3

Theorem “3”:

If  $0 < C^* = \sup_{x \in I} \frac{h(x)}{\psi_+(x)} = \sup_{x \geq x_0, x \in I} \frac{h(x)}{\psi_+(x)} < \infty$

Then

$$\sup_T E_{x_0} \{ e^{-A_T} h(X_T) 1_{\{T < \infty\}} \} = C^* \quad (+)$$

If there exists a point  $x^* \geq x_0$  with  $C^* = h(x^*)/\psi_+(x^*)$ , then the supremum in (+) is attained by

$$T^* = \inf \{ t \geq 0 \mid X_t = x^* \}.$$

This holds since

$$E_{x_0} e^{-A_{T^*}} h(X_{T^*}) = E_{x_0} \frac{h(X_{T^*})}{\psi_+(x)} M_{T^*} = C^*.$$

## Case 5

Theorem “5”:

Let  $p^*$  be such that

$$\begin{aligned} 0 &< \sup_{x \geq x_0, x \in I} \frac{h(x)}{p^* \psi_+(x) + (1 - p^*) \psi_-(x)} \\ &= \sup_{x \leq x_0, x \in I} \frac{h(x)}{p^* \psi_+(x) + (1 - p^*) \psi_+(x)} \end{aligned}$$

then  $\sup_T E_{x_0} (e^{-A_T} h(X_T) \mathbf{1}_{\{T < \alpha\}}) = C^*$ .

If there exist points  $x_1 > x_0$  and  $x_2 < x_0$  such that

$$\begin{aligned} &\frac{h(x_1)}{p^* \psi_+(x) + (1 - p^*) \psi_-(x)} \\ &= \frac{h(x_2)}{p^* \psi_+(x) + (1 - p^*) \psi_-(x)} = C^*, \end{aligned}$$

then the supremum is attained for

$T^* = \inf\{t > 0 \mid X_t = x_1, X_2 = x_2\}$ .

# Generalized Parking Problem: Discrete Case, Details

$X_1, X_2, \dots$  i.i.d. with  $EX_i > 0$ ,

$$S_n = \sum_{i=1}^n X_i, \quad S_0 = 0.$$

Find stopping time  $T^*$  with  $Eg(S_{T^*}) = \min_T Eg(S_T)$

Solution:  $T^* = \min\{n \geq 0 \mid S_n \geq a\}$

with

$$a = \sup\{x \mid H^+g(x) < g(x)\}$$

$$H^+g(x) := \int g(x+y)H^+(dy)$$

$$H^+(y) := P(S_\eta \leq y)$$

$$\eta := \min\{n > 0 \mid S_n > 0\}$$

$H^+$ : the distribution of the first ladder height  $S_\eta$ .

Let  $K(x) = \int_0^x \frac{1 - H^+(y)}{\gamma_1} dy$   
with  $\gamma_i = \int y^i dH^+(y)$ ,  $i \in \mathbb{N}$ .

### Theorem

If  $Kg(x) < \infty$  for all  $0 \leq x < \infty$ , then  $Kg(x)$  is minimized at  $x = a$ .

### Example 1:

If  $g(x) = |x - b| \forall x \in \mathbb{R} \Rightarrow b - a = \text{med}(K)$

### Example 2:

If  $g(x) = (x - b)^2 \forall x \in \mathbb{R}$   
 $\Rightarrow b - a = \text{mean}(K) = \gamma_2/\gamma_1$

### Example 3:

If  $g(x) = e^{-x} + cx \forall x \in \mathbb{R}$ ,  $0 < c < 1$   
 $\Rightarrow b = \log(1/c)$ .

If  $\int x^2 H^+(dx) < \infty$  and if  $\kappa := \int_0^\infty e^{-x} K(dx)$   
 $\Rightarrow Kg(x) = \kappa e^{-x} + c(x + \frac{\gamma_2}{2\gamma_1})$  and is minimized  
when  $x = \log(\kappa/c)$ .

# Related Literature

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