# The Distribution of Returns at Longer Horizons

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**Abstract**

Longer horizon returns are constructed from data on daily returns. Observed drawbacks to running a Lévy process are a sharp decrease in higher moments. Drawbacks to scaling are a constancy of higher moments. A strategy that combines some exposure to independent increments and some exposure to scaling is developed in the context of self decomposable daily return distributions. Estimations are conducted on 400 stocks and we report that a good strategy for constructing longer horizon returns can even be that of running as a Lévy process half the daily return while scaling the remainder at rate one half.

A statistical analysis of data on daily returns can provide some assessment of the centered daily return distribution but many questions of financial interest require a knowledge of distributional properties at a longer horizon like a week, a month, a quarter or even a year. This is particularly true when considering investment decisions in stocks or options thereof as the typical investment horizon or rebalancing period is more than just a day. One could in principle estimate on daily data a fully dynamic return generating process reflecting time varying conditional moments that can then be simulated to the horizon of interest. However, such an approach is time consuming and considerably hampered by the many questions that need to be answered in building dynamic models. A more direct alternative is the construction of the long horizon return directly from the short horizon return. Such an approach is investigated in Eberlein and Özkan (2003) where a Lévy process distribution is estimated at the shorter horizon and the process is run to the longer horizon to get the latter distribution. An implication of the independent increment property of Lévy processes is that volatilites should scale with the square root of the return horizon. These principles are widely adopted and noted in Diebold, Hickman, Inoue and Schuermann (1998) as advocated by Basel II documents, though they warn against the wide spread use of square root scaling for volatilities. The objections are consistent with the wide spread rejection of the hypothesis of independent and identically distributed increments in empirical work (Lo and MacKinlay (1988), and Lo (2002)). We note however, that this hypothesis may be accepted after a devolatilization procedure has been implemented as described in Eberlein, Kallsen, and Kristen (2003).

Apart from running a Lévy process to the option maturity, numerous authors have advocated the hypothesis of self similarity and scaling and we cite Peters (1991, 1999), Mantegna and Stanley (1995), Cont, Bouchaud and Potters (1991), Mandelbrot (1997), Heyde (1997), Shiryaev (1999) and Cont (2001). These scaling principles were employed in Bakshi and Madan (2006) and the scaling coefficient was estimated to lie just under one half. On the hypothesis of self similarity one gets the distribution of returns at the longer horizon as equal in law to $h^{\gamma}$ times the lower horizon return where $h$ is the ratio of the two horizons. A convenient choice for the coefficient $\gamma$, in line with the square root scaling of variances, is the value 0.5.

The two approaches are widely differentiated with respect to their effects on skewness and kurtosis. For example, it is known that for a Lévy process skewness falls like the reciprocal of the square root of the horizon while excess kurtosis falls like the reciprocal of the horizon itself. On the other hand, with scaling both skewness and kurtosis, are constant for all horizons. In this paper we present evidence in support of both hypotheses, demonstrating that scaling tends to dominate the alternative of running the Lévy process, when we consider Kolmogorov Smirnov (KS) statistics at longer horizons. However, we also observe that skewness and kurtosis fall at longer horizons and these observations are not consistent with the constancy implied by scaling. They just do not fall as fast as predicated by running a Lévy process. This set of observations leads us to formulate a mixed approach that allows for both aspects to be present. We proceed to decompose the return into two components, one of which we let run like a Lévy process while the other we scale. The mixed approach of partially running a Lévy process and scaling the remaining component, is shown to yield superior KS statistics and also provides for slower decline in skewness and kurtosis. There are two parameters to be estimated in the mixed approach, and this is the proportion that is run like a Lévy process and the scaling coefficient of the portion that is scaled. We find that an adequate and superior performance to either running or scaling is delivered by just running half the return as a Lévy process and scaling the remainder with a scaling coefficient of one half. Hence, in the absence of estimation opportunities for the longer horizon we advocate running half the return as a Lévy process and scaling the remainder at a scaling coefficient of one half.

The outline of the paper is as follows. Section 1 presents preliminary evidence in support to some degree for both hypotheses that also addresses why a strict adherence to either hypothesis is questionable. In Section 2 we develop our mixed approach that accomodates both ideas at some level. Estimation strategies and results for the combined model are described in Section 3. Section 4 describes the improvement attained by the mixed model, along with what is possible without estimation, by running half the return as a Lévy process and scaling by one half the remainder. Section 5 concludes.

# 1 Preliminary Evidence on Scaling and Running a Lévy process

We obtained from CRSP (Chicago Center for Research in Security Prices) data on stock prices for 477 stocks with 3000 observations starting on Jan. 31 1996 and finishing on December 31 2007. We shall use this data to construct from daily returns 10 and 20 day returns by simulating both operations of running a process of independent increments and scaling the daily return. We then ask how well the simulated long horizon returns describe the properties of the actual 10 and 20 day nonoverlapping returns in the data.

The first step is then to build the empirical distribution function for daily returns for each of the 477 stocks. This was done by binning the data in the interquantile range of a tenth of a percent to the 99.9 percentile into 100 regularly sized bins. For purposes of comparison to actual long horizon returns we also extracted data on 10 and 20 day nonoverlapping returns.

For simulating the running of a Lévy process for 10 or 20 days, we draw from the empirical daily return distribution 10000 paths of length 10 and 20 respectively. The long horizon return is then built by merely summing these draws over the length of the paths.

For simulating the scaling operation we draw from the daily return distribution just once and scale the outcome by the square root of 10 or 20 to build the sample for the scaled long horizon returns.

Our first comparison between the simulated long horizon returns and the actual ones is to perform a $KS$ test for whether the actual non-overlapping 10 and 20 day return outcomes come from the two candidate distributions that we call Lévy for running the process of independent increments, and Scaling for just scaling the daily return. We record the 477 p-values for each of the two methods, Lévy and Scaling, at each of the two horizons 10, and 20 days. We graph in figure (1) the proportion of stocks with a p-value exceeding a particular level.

This graph supports the superiority of Scaling over running a Lévy process. At both horizons the proportion of stocks with high p-values is greater under Scaling though we note that almost 70% of the stocks fail to make a 10% p-value in all the cases. The support for scaling is therefore not that strong and there appears to be a substantial room for improvement in performance.

Our second comparison focuses attention on skewness and kurtosis for daily returns and the same for the set of nonoverlapping 10 and 20 day returns. Under a Lévy process construction we know that long horizon skewness decreases at a rate proportional to the square of the number of periods while excess kurtosis decreases like the reciprocal of the number of periods (Konikov and Madan (2000)). On the other hand under scaling these moments are constant. To evaluate these properties we regressed the 10 and 20 days skewness and excess

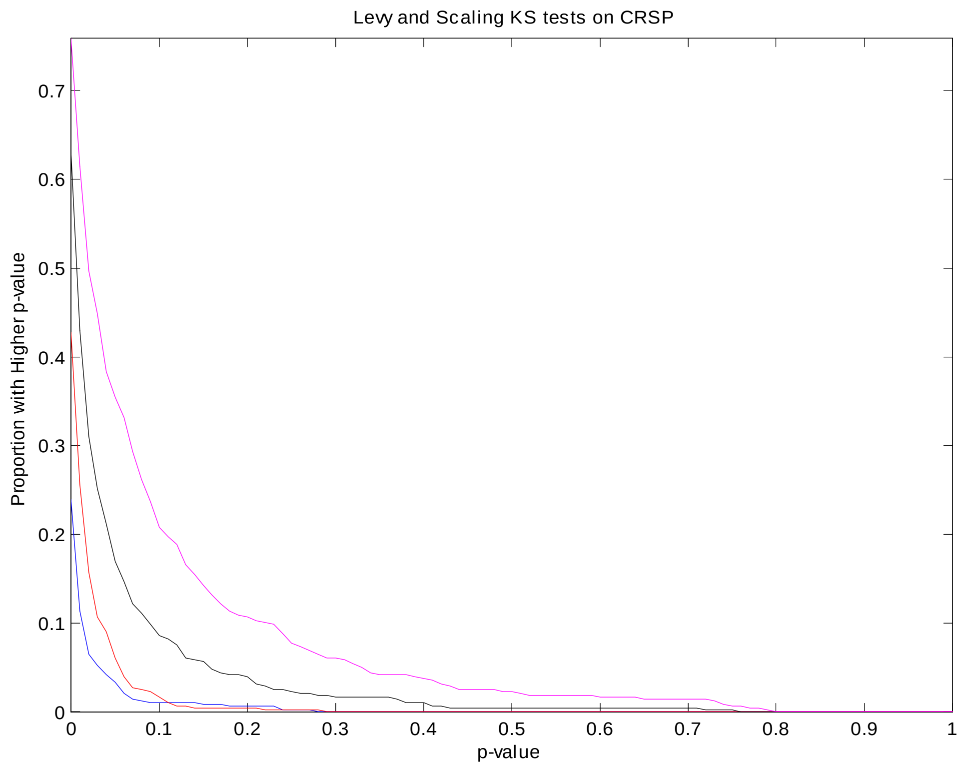


Figure 1: Independent Draws for 10 and 20 days in blue and red respectively. Scaling for 10 and 20 days in black and magenta.

kurtosis across the 477 stocks on the daily skewness and excess kurtosis.

For the validity of Lévy process approach we expect, in this regression, a zero intercept with slope coefficients equal to $\frac{1}{\sqrt{10}} = .3162$ and $\frac{1}{\sqrt{20}} = .2236$ for the skewness regression. For the excess kurtosis regression the expected slope coefficients are .1 and .05 while the intercepts are zero. On the other hand for the validity of scaling we should also observe zero intercepts and unit slope coefficients. The estimated coefficients are given in Table 1.

TABLE 1
Higher Moment Regressions

| | skw10 | skw 20 | krt10 | krt20 |
|---|---|---|---|---|
| intercept | .0621 | .1670 | 1.2293 | 1.7014 |
| slope | .4098 | .3205 | .1657 | .0429 |
| $R^2$ | 39.14% | 19.40% | 55.77% | 9.69% |

We report in addition $t-statistics$ for the null hypothsis of Lévy and Scaling in Tables 2. We test here for the departure of estimated coefficients from the values expected under the null hypothesis.

TABLE 2
Lévy and Scaling t-statistics

| | skw10 | skw20 | krt10 | krt20 |
|---|---|---|---|---|
| intercept | 3.57 | 7.51 | 9.07 | 14.13 |
| slope Lévy | 3.98 | 3.23 | 9.67 | -1.17 |
| slope Scaling | -25.10 | -22.60 | -122.93 | -158.79 |

It is clear that both skewness and kurtosis are falling with slope coefficients below unity, though they are not falling at a rate commensurate with a Lévy process. The fact that they are falling is also evidenced by the large and highly significant t-statistics associated with scaling. That the rate is lower than that of a Lévy process is evidenced by the positive and significant Lévy t-statistics. We take this evidence of declining skewness and kurtosis to be in favor of the hypothesis of independent increments being present in some form.

A preliminary investigation of the relationship between daily return distributions and the distribution of longer horizon returns provides partial support for both operations of scaling and running a Lévy process. The latter accumulates the effects of independent shocks while the former reflects the attributes of self similarity that preserve distributional structure but recognize that the longer horizon is a scaled version of the shorter horizon. The next section considers combining these properties.

# 2 Combining Running and Scaling

The arguments in favor of running a Lévy process relate to the arrival of sequences of independent news shocks that get incorporated into stock price revisions in markets. The arguments in favor of scaling relate to issues of self similarity whereby we recognize the occurence of a change of scale but no change in the structure of the underlying probability law. A natural approach to synthesize these two ideas is to attempt to split the random variable at unit time into two components, one of which we run as a Lévy process as it incorporates the news shocks while the second we scale as it represents the self similar structural component. We are then interested in splitting our unit time random variable. This brings us to the property of self-decomposability and its relationship to limit laws.

The study of limit laws goes back to Lévy (1937) and Khintchine (1938) who asked what were the class of all random variables that may be obtained as the limits of averages scaled at rates other than the square root of the number of terms. It is well known that square root scaling yields the Gaussian distribution of the central limit theorem. The class of other limit laws is the class of self-decomposable random variables that is a subclass of the infinitely divisible laws. These laws have been effectively used to synthesize the surface of option prices in Carr, Geman, Madan and Yor (2007). Here we postulate that the daily return distribution is such a limit law. It is important to note that jump diffusion processes popularly used in many financial models are not limit laws as the latter are processes of infinite activity as defined in Carr, Geman, Madan and Yor (2002), with an infinite number of jumps in any interval. Jump diffusions are finite actvity processes and hence cannot be limit laws.

It follows from the definition of self-decomposability of a random variable $X$ (Sato (1999)), representing here a daily return, that for every $c, 0 < c < 1$ there exists a random variable denoted $X^{(c)}$, independent of $X$ such that

$$X \stackrel{(d)}{=} cX + X^{(c)}.$$

We therefore have for such self-decomposable laws a natural splitting into two well understood components. Let $Y_h$ denote the long horizon return with the number of short horizon or unit time periods in the long horizon being $h$. We propose to construct this long horizon return by running the Levy process $cX$ to $h$ units of time and scaling the independent component $X^{(c)}$ by the coefficient $\gamma$. Hence our candidate model is

$$Y_h = cX(h) + h^\gamma X^{(c)}.$$

where $cX(h)$ represents the Lévy process with unit time distribution $cX$ now taken at time $h$. The characteristic function for $Y_h$ may then be easily constructed from that of $X$. Specifically let

$$\phi(u) = E\left[\exp\left(iuX\right)\right] = \exp\left(\psi(u)\right)$$

We then have that

$$E\left[e^{iuY_h}\right] = \exp\left(h\psi(cu) + \psi(uh^\gamma) - \psi(cuh^\gamma)\right) \tag{1}$$

and we have two extra parameters to be estimated for the long horizon return after we have estimated and identified the daily return characteristic function.

We now establish that the mixed Lévy scaling model with characteristic function given by (1) is capable of a slower reduction in skewness and kurtosis than would be seen in a full Lévy model.

**Proposition 1** *The variance of $Y_h$ at horizon $h$, $\nu_h$, is given by*

$$\nu_h = \nu \left(c^2 h + (1-c^2)h^{2\gamma}\right) \tag{2}$$

*where $\nu$ is the variance for $h = 1$. The corresponding equations for skewness and excess kurtosis are*

$$\begin{aligned} Skewness(Y_h) &= \frac{Skewness(X)}{\sqrt{h}} \left[\frac{c^3 + (1-c^3)h^{3\gamma-1}}{(c^2 + (1-c^2)h^{2\gamma-1})^{\frac{3}{2}}}\right] \\ Excess\ Kurtosis(Y_h) &= \frac{Excess\ Kurtosis(X)}{h} \left[\frac{c^4 + \left(1-c^4\right)h^{4\gamma-1}}{(c^2 + (1-c^2)h^{2\gamma-1})^2}\right] \end{aligned}$$

***Proof.*** *Consider the case of centered variates. In this case we know that*

$$\psi''(0) = -E[X^2]$$

*We get on twice differentiation with respect to $u$ that*

$$-E\left[X^2 e^{iuX}\right] = \exp\left(\psi(u)\right)\left(\psi''(u) + \psi'(u)^2\right)$$

*If we differentiate one and two more times then on the left we have*

$$-iE\left[X^3 e^{iuX}\right],\ E\left[X^4 e^{iuX}\right]$$

*and on the right, ignoring odd terms for the fourth power, we have*

$$\begin{aligned} &\exp(\psi(u))(\psi'''(u) + 3\psi''(u)\psi'(u) + \psi'(u)^3) \\ &\exp\left(\psi(u)\right)\left(\psi^{(iv)}(u) + 3\psi''(u)^2\right) \end{aligned}$$

*Evaluating at $u = 0$ we get for a centered variate*

$$E[X^3] = i\psi'''(0)$$

$$E[X^4] = \psi^{(iv)}(0) + 3\left(E\left[X^2\right]\right)^2$$

*Equivalently we may write for the fourth power that excess kurtosis is*

$$\frac{E\left[X^4\right]}{\left(E\left[X^2\right]\right)^2} - 3 = \frac{\psi^{(iv)}(0)}{\left(\psi''(0)\right)^2}$$

*We now apply these results to the derivatives of the characteristic exponent of $Y_h$ to learn that*

$$\psi_Y^{''}(0) = \psi^{''}(0)\left(c^2 h + \left(1-c^2\right)h^{2\gamma}\right)$$

*The result (2) for the variance of $Y_h$ follows. The result for skewness and excess kurtosis follow on noting that*

$$\frac{\psi_Y'''(0)}{(-\psi_Y''(0))^{\frac{3}{2}}} = \frac{\psi'''(0)\left[c^3h + (1-c^3)h^{3\gamma}\right]}{(-\psi''(0))^{\frac{3}{2}}\left[c^2h + (1-c^2)h^{2\gamma}\right]^{\frac{3}{2}}}$$
$$\frac{\psi_Y^{(iv)}(0)}{\left(\psi_Y''(0)\right)^2} = \frac{\psi^{(iv)}(0)\left(c^4h + \left(1-c^4\right)h^{4\gamma}\right)}{\left(\psi''(0)\right)^2\left(c^2h + \left(1-c^2\right)h^{2\gamma}\right)^2}$$

■

We remark that variances per unit time implied by variances for the horizon $h$, i.e. $\frac{v_h}{h}$ are therefore rising for $\gamma > \frac{1}{2}$ and falling otherwise. When $c = 1$ we have skewness decreasing at rate $\frac{1}{\sqrt{h}}$ while excess kurtosis decreases at rate $\frac{1}{h}$.

With $\gamma = \frac{1}{2}$ we have

$$\begin{aligned}
\frac{\psi_Y'''(0)}{\left(-\psi_Y''(0)\right)^{\frac{3}{2}}} &= \frac{\psi'''(0)}{(-\psi''(0))^{\frac{3}{2}}}\frac{c^3h + (1-c^3)h^{\frac{3}{2}}}{h^{\frac{3}{2}}} \\
&= \frac{\psi'''(0)}{(-\psi''(0))^{\frac{3}{2}}}\left(1 - c^3\left(1 - \frac{1}{\sqrt{h}}\right)\right) \\
\frac{\psi_Y^{(iv)}(0)}{\left(\psi_Y''(0)\right)^2} &= \frac{\psi^{(iv)}(0)}{\left(\psi''(0)\right)^2}\frac{c^4h + (1-c^4)h^2}{h^2} \\
&= \frac{\psi^{(iv)}(0)}{\left(\psi''(0)\right)^2}\left((1-c^4) + \frac{c^4}{h}\right) \\
&= \frac{\psi^{(iv)}(0)}{\left(\psi''(0)\right)^2}\left(1 - c^4\left(1 - \frac{1}{h}\right)\right)
\end{aligned}$$

The partial with respect to $c$ is negative (as $h > 1$) and so for $c < 1$ the skewness and excess kurtosis do not fall as fast as they do for $c = 1$ and we may obtain the required slower reduction in skewness and excess kurtosis with our strategy of running as a Lévy process the fraction $c$ of the daily return and scaling the rest.

## 3 Estimating the Scaling Coefficients $c, \gamma$

We estimate on daily return data for all 477 stocks, both the variance gamma $VG$ model, (Madan and Seneta (1990), Madan, Carr and Chang (1998)) and the $CGMY$ model (Carr, Geman, Madan and Yor (2002)). We then estimate for each stock $c$ and $\gamma$ by maximum likelihood constructed on inverting the characteristic function (1) by Fourier inversion, using as data for the long horizon return, the nonoverlapping 10 and 20 day returns.

The parameters for the $VG$ model were estimated on all 477 stocks but we had convergence for the $CGMY$ model on 400 stocks only. We report scaling coefficient constructions on these 400 stocks. We present in Tables 1 and 2 summary statistics on the parameter estimates along with a sample of estimated parameter values for a few selected stocks. We also graph in Figures (2 to 7), a subsample of the empirical and estimated density functions for both the $VG$ and $CGMY$ models. The former has at the daily return level a more peaked return density.

TABLE 1
VG Parameter Estimates
on 400 Daily CRSP Stock Returns

| | $\sigma$ | $\nu$ | $\theta$ in basis points |
|---|---|---|---|
| mean | 0.0228 | 0.7016 | 25.7817 |
| standard deviation | 0.00687 | 0.1681 | 19.6708 |
| quantile .25 | 0.0179 | 0.5916 | 12.5691 |
| quantile .75 | 0.0267 | 0.8108 | 34.1186 |
| minimum | 0.0112 | 0.1909 | -19.1058 |
| maximum | 0.0478 | 1.1786 | 107.8367 |
| BA | 0.0197 | 0.5585 | 8.6394 |
| C | 0.0208 | 0.7921 | 20.7277 |
| CSCO | 0.0293 | 0.7997 | 31.0627 |
| MMM | 0.0154 | 0.6909 | 14.1126 |
| WMT | 0.0190 | 0.6234 | 23.7023 |

TABLE 2
CGMY Parameter Estimates
on 400 Daily CRSP Stock Returns

| | C | G | M | Y |
|---|---|---|---|---|
| mean | 0.0169 | 37.1987 | 31.1039 | 0.9795 |
| standard deviation | 0.0249 | 14.0814 | 13.3636 | 0.2121 |
| quantile .25 | 0.0036 | 27.7377 | 21.8479 | 0.8394 |
| quantile .75 | 0.0176 | 45.2781 | 38.2068 | 1.1417 |
| minimum | 0.0010 | 6.9253 | 5.8521 | 0.4229 |
| maximum | 0.1586 | 100.0635 | 103.0807 | 1.4288 |
| BA | 0.0012 | 25.6027 | 22.3962 | 1.3404 |
| C | 0.0305 | 44.1240 | 38.3503 | 0.7077 |
| CSCO | 0.1586 | 42.7787 | 35.5464 | 0.4229 |
| MMM | 0.0068 | 51.8982 | 42.6672 | 0.9623 |
| WMT | 0.0069 | 43.9618 | 33.4598 | 1.0104 |

We next take these marginals and estimate $c$, the proportion of the random variable that is run as a Lévy process and $\gamma$, the scaling coefficient for the component that is scaled as a self similar process. The estimation is done via maximum likelihood using data on nonoverlapping 10 and 20 day returns. Tables 3 and 4 provide summary statistics on these values for both the $VG$ and $CGMY$ models along with sample values for the stocks reported in Tables 1

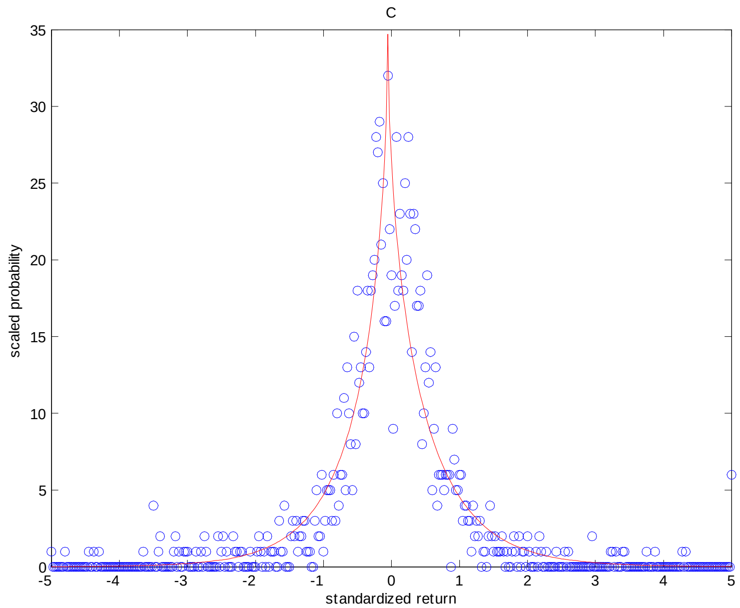


Figure 2: VG fit to C

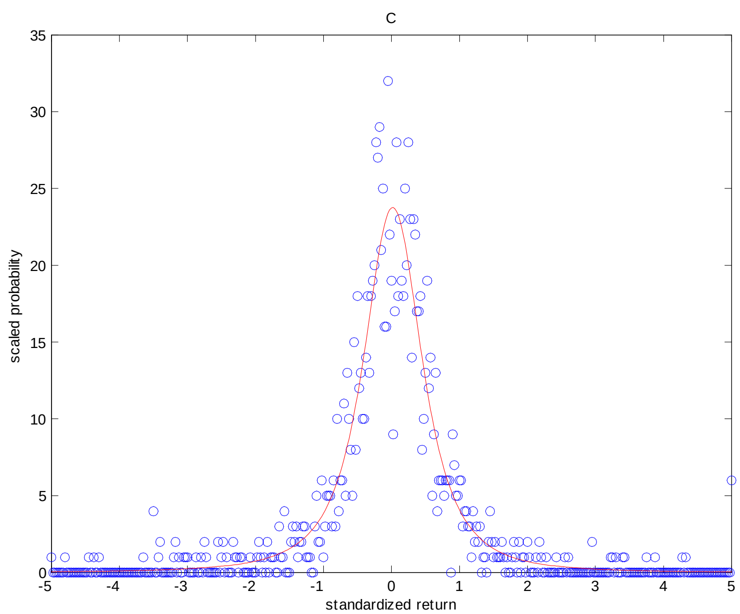


Figure 3: CGMY fit to C

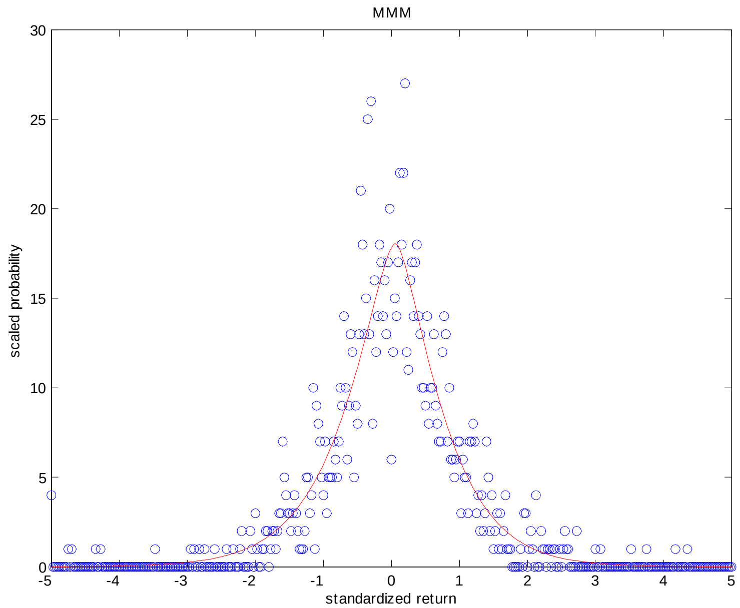


Figure 4: VG fit to MMM

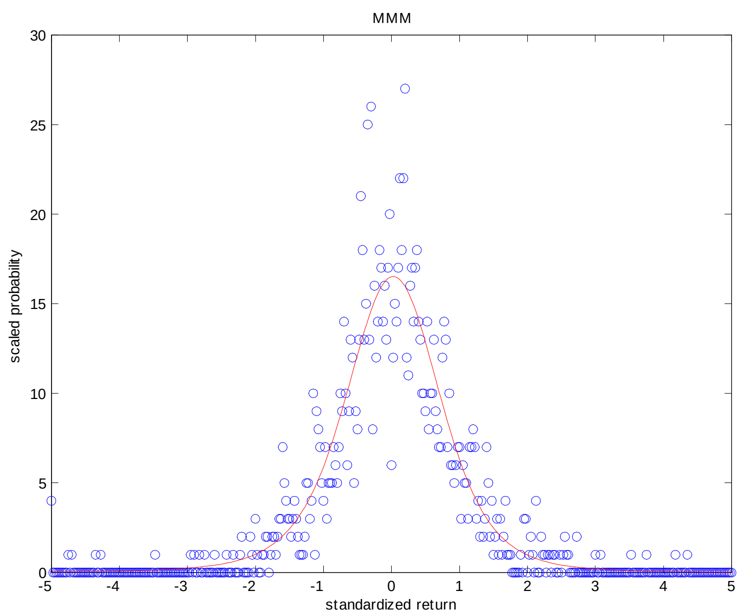


Figure 5: CGMY fit to MMM

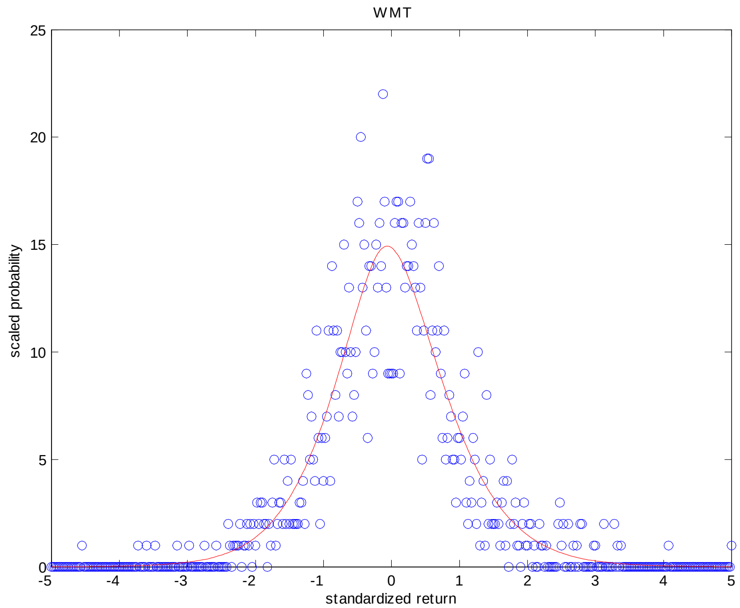


Figure 6: VG fit to WMT

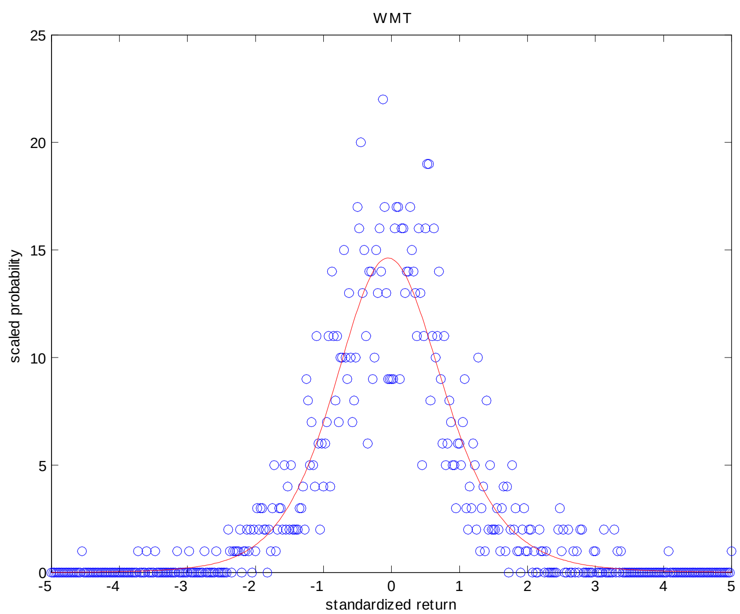


Figure 7: CGMY fit to WMT

and 2.

Scaling VG

| | 10 day | | 20 day | |
|---|---|---|---|---|
| | $c$ | $\gamma$ | $c$ | $\gamma$ |
| mean | 0.4687 | 0.4737 | 0.5387 | 0.4545 |
| standard deviation | 0.2458 | 0.1055 | 0.2623 | 0.1362 |
| quantile .25 | 0.3125 | 0.4560 | 0.3812 | 0.4522 |
| quantile .75 | 0.6334 | 0.5096 | 0.7220 | 0.5121 |
| BA | 0.6496 | 0.4831 | 0.6344 | 0.4989 |
| C | 0.3860 | 0.4832 | 0.6633 | 0.4924 |
| CSCO | 0.2265 | 0.4975 | 0.0786 | 0.5261 |
| MMM | 0.6943 | 0.4311 | 0.4431 | 0.5025 |
| WMT | 0.5313 | 0.3786 | 0.5225 | 0.4281 |

Scaling CGMY

| | 10 day | | 20 day | |
|---|---|---|---|---|
| | $c$ | $\gamma$ | $c$ | $\gamma$ |
| mean | 0.4175 | 0.4595 | 0.4823 | 0.4338 |
| standard deviation | 0.2932 | 0.0957 | 0.3044 | 0.1443 |
| quantile .25 | 0.1004 | 0.4458 | 0.2452 | 0.4322 |
| quantile .75 | 0.6369 | 0.5074 | 0.7179 | 0.5065 |
| BA | 0.5996 | 0.4681 | 0.5116 | 0.4902 |
| C | 0.2802 | 0.4866 | 0.6404 | 0.9889 |
| CSCO | 0.0027 | 0.5004 | 0.0525 | 0.5256 |
| MMM | 0.7136 | 0.4127 | 0.1700 | 0.5095 |
| WMT | 0.6806 | 0.2816 | 0.7787 | 0.2492 |

# 4 Performance of the Joint model

We performed two analyses of the performance of the Mixed Lévy Scaling Model for constructing the centered distribution of 10 and 20 returns from daily returns. The first analysis uses the estimated scaling parameters for each horizon and each model to construct the distribution function of the scaled return from the analytical characteristic function (1). This distribution function is then employed to generate 10000 readings using the inverse uniform method and then we compute the p-value for the KS statistic designed to test if the associated observed nonoverlapping returns come from a distribution with this simulated data. The second analysis just uses the average parameters reported in Table 1 and 2 for getting to the horizon of interest.

We present two graphs (Figures (8 and 9)) for the proportion of p-values exceeding a particular level among the 400 stocks for each of these two constructions. There are four curves in each graph, for VG 10 day in blue, VG 20 day in red, CGMY 10 day in black and CGMY 20 day in magenta.

We see that the optimized parameters deliver a superior performance as is to be expected, but the average parameters perform reasonably well and better

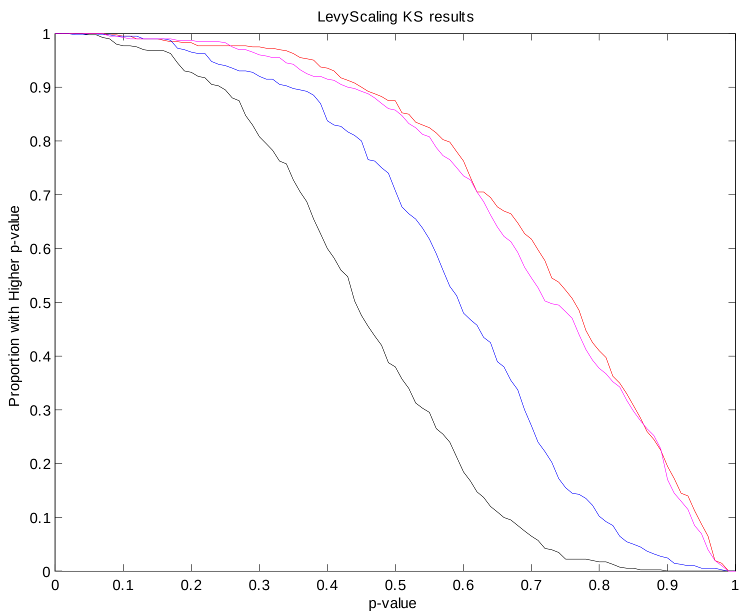


Figure 8: Proportion of stocks with a higher p-value for 10 and 20 day return distributions modeled by the Mixed Lévy Scaling Model using VG and CGMY distributions

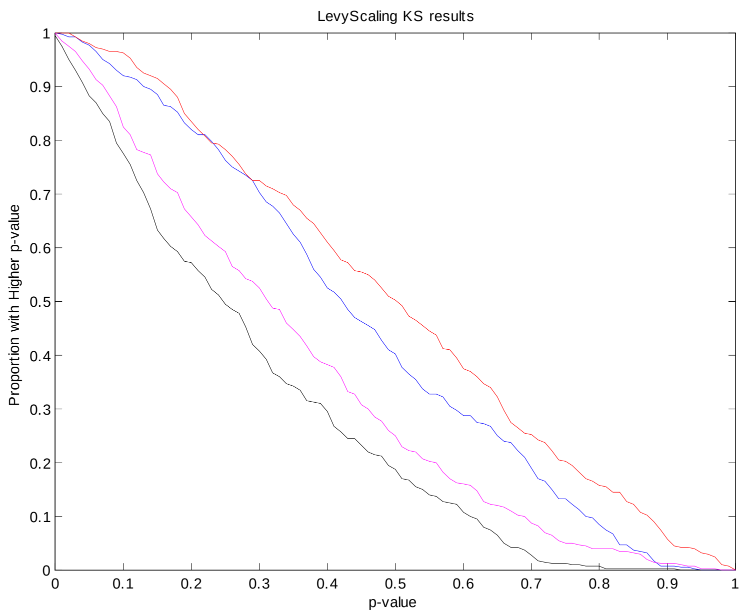


Figure 9: Proportion of stocks with a higher p-value for 10 and 20 day return distributions modeled by the Mixed Lévy Scaling Model at average parameter values using VG and CGMY distributions.

than using either full Lévy or full Scaling. One could let half the daily random variable run as an independent Lévy process while the remainder is scaled with a scaling coefficient of 0.5. This appears to be a better strategy than either employing full Lévy or full scaling.

### 4.1 Skewness and Kurtosis in the Joint model

We see that the model that combines scaling with running a fraction of the unit time random variable as a Lévy process reflects possibly some decline in skewness and kurtosis, but not at the speed implied by a full Lévy process with no scaling component. We generated 10000 paths for all three approaches for 10 and 20 day returns using both VG and the CGMY parameters on all the 400 stocks.

| | Skewness | | |
|---|---|---|---|
| | Lévy | Mixed | Scaling |
| VG 10 | .3385 | .6352 | .6383 |
| VG 20 | .2517 | .5813 | .5478 |
| CGMY 10 | .3757 | .6759 | .6592 |
| CGMY 20 | .5239 | .6068 | .5537 |
| | Kurtosis | | |
| VG 10 | 2.87 | 4.04 | 4.12 |
| VG 20 | 2.79 | 3.92 | 3.86 |
| CGMY 10 | 3.06 | 4.43 | 4.47 |
| CGMY 20 | 3.71 | 4.09 | 4.07 |

## 5 Conclusion

The problem of constructing return distributions at longer horizons from data on daily returns is considered. It is shown that the hypothesis of independent increments leads to a decline in skewness and kurtosis that is much faster than what we observe in data. Furthermore the hypothesis of scaling daily return distributions to the longer horizon appears to have more support, based on Kolmogorov-Smirnov tests conducted on data for nonoverlapping 10 and 20 day returns. However, the scaling hypothesis implies that skewness and kurtosis are constant with respect to the time horizon and it is observed that they fall in the data.

A strategy for combining some exposure to independent increments and some exposure to scaling or self similarity over the longer horizon is developed in the context of daily return distributions that are self-decomposable. This strategy splits the daily return into a fraction of itself plus an independent component. We then run the Lévy process associated with the fraction and scale the residual independent component using a scaling coefficient. The longer horizon return then requires the estimation of the fraction and the scaling coefficient. Maximum likelihood procedures are developed for the estimation of these two parameters.

Estimations are conducted on 400 stocks and we report that a good strategy for constructing longer horizon distributions can even be to run half the daily return as a Lévy process and scale the remainder at rate one half.

## References


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