

# MARKET MODELS FOR CDOs DRIVEN BY TIME-INHOMOGENEOUS LÉVY PROCESSES

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ABSTRACT. This paper considers a top-down approach for CDO valuation and proposes a market model. We extend previous research on this topic in two directions: on the one side, we use as driving process for the interest rate dynamics a time-inhomogeneous Lévy process, and on the other side, we do not assume that all maturities are available in the market. Only a discrete tenor structure is considered, which is in the spirit of the classical Libor market model. We create a general framework for market models based on multidimensional semimartingales. This framework is able to capture dependence between the default-free and the defaultable dynamics, as well as contagion effects. Conditions for absence of arbitrage and valuation formulas for tranches of CDOs are given.

## 1. INTRODUCTION

Contrary to the single-obligor credit risk models, portfolio credit risk models consider a pool of credits with different obligors and the adequate risk quantification for the whole portfolio becomes a challenge. The most difficult and important task is to determine the dependence structure in the portfolio (also termed default correlation).

The main question which we consider here is the valuation of tranches of *collateralized debt obligations* (CDOs) and related derivatives. CDOs are structured asset-backed securities, whose value and payments depend on a pool of underlying assets (such as bonds or loans) called the *collateral*. They consist of different *tranches* representing different risk classes, ranging from *senior* tranches with the lowest risk, over *mezzanine* tranches, to *equity* tranches which carry the highest risk. If there are defaults in the collateral, the occurring losses are transferred to investors in order of seniority, starting by equity tranches.

Among various portfolio credit risk models, there are two main approaches to be distinguished: the *bottom-up* approach where the default event of each individual obligor is modeled, and the *top-down* approach where the aggregate loss process of a given portfolio is modeled and the individual obligors in the portfolio are not identified. The latter approach was investigated in a series of recent papers, among which we mention Sidenius, Piterbarg, and Andersen (2008), Ehlers and Schönbucher (2006), Ehlers and Schönbucher (2009) and Filipović, Overbeck, and Schmidt (2009). All these papers are

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in the spirit of the Heath–Jarrow–Morton framework for modeling of the term structure of continuously compounded forward rates, extended to the defaultable setting in an appropriate way.

In this paper we present a *market model* which studies portfolio credit risk in a top-down setting. It generalizes the defaultable Libor market model (see Eberlein, Kluge, and Schönbucher (2006)) to portfolio credit risk. As in Filipović, Overbeck, and Schmidt (2009) we utilize  $(T, x)$ -bonds. A  $(T, x)$ -bond pays 1 if the aggregate losses do not exceed the level  $x$  at maturity  $T$ , and zero otherwise. In practice not all maturities are available and hence, we work on a discrete tenor structure  $0 = T_0 < T_1 < \dots < T_n$  and extend the classical definition of (risk-free) forward Libor rates to defaultable forward  $(T_k, x)$ -Libor rates defined via  $(T_k, x)$ -bonds. The need for this approach is illustrated in Carpentier (2009), and to our knowledge only Bennani and Dahan (2004) studied market models for CDOs. In particular, Section 6.1 below shows how to extract the  $(T_k, x)$ -Libor rates from traded CDOs.

The model we propose is given in a very general setting. The driving process is a multidimensional special semimartingale consisting of two components: a multidimensional time-inhomogeneous Lévy process representing market fluctuations and a component which is the compensated aggregate loss process. In this way we incorporate the influence of the aggregate loss process on the price dynamics of defaultable assets in the portfolio, which is known as *contagion*. This covers the empirical phenomenon that a default in the market affects the credit spreads of other obligors. Moreover, we include a direct dependence of the risk-free interest rates and the aggregate loss process.

The paper is structured as follows. In Section 2 we introduce the setting and basic notions. In Section 3 we describe the aggregate CDO loss process  $A$  and specify the driving process  $X$ . Section 4 is devoted to the risk-free Libor rates and the corresponding forward measures, whereas Section 5 contains the construction of the forward  $(T_k, x)$ -Libor rates. Furthermore, we derive conditions for the absence of arbitrage. As a special case, we recall the defaultable Lévy Libor model of Eberlein, Kluge, and Schönbucher (2006), which can be embedded in this framework by considering a portfolio consisting of only one defaultable bond. In Section 6 we show how derivative valuation can be facilitated by using appropriate defaultable forward measures and present a valuation formula for a single tranche CDO, which is the standard instrument for investing in a CDO-pool. Finally, we discuss the relation of CDOs and Libor rates under assumptions used in practice.

## 2. BASIC NOTIONS AND DEFINITIONS

Let  $T^* > 0$  be a fixed time horizon and assume that the tenor structure  $0 = T_0 < T_1 < \dots < T_n = T^*$  is given. Set  $\delta_k := T_{k+1} - T_k$ , for  $k = 0, \dots, n-1$ .

We assume that default-free zero coupon bonds with maturities  $T_1, \dots, T_n$  are traded on the market and denote by  $P(t, T_k)$  the time- $t$  price of a default-free zero coupon bond with maturity  $T_k$ . For default-free zero coupon bonds  $P(T_k, T_k) = 1$  for all  $k$ . Furthermore we assume that  $P(t, T_k) > 0$  for any  $0 \leq t \leq T_k$  and all  $k$ .

**Definition 2.1.** The (*forward*)  $T_k$ -Libor rate at time  $t \leq T_k$  is defined by

$$L(t, T_k) := \frac{1}{\delta_k} \left( \frac{P(t, T_k)}{P(t, T_{k+1})} - 1 \right). \quad (1)$$

Throughout we call this and similar rates Libor rates and leave "forward" aside. This rate is default-free as the  $T_k$ -bonds are.

On the defaultable side, we consider a pool of credit risky assets. The overall nominal is normalized to 1. We denote by  $A = (A_t)_{t \geq 0}$  the increasing *aggregate CDO-loss process*. Let  $\mathcal{I} = [0, 1]$  denote the set of attainable loss levels.<sup>1</sup> Then  $A$  takes values in  $\mathcal{I}$  and  $x \in \mathcal{I}$  represents the level where  $100x\%$  of the overall nominal has defaulted.

**Remark 2.2.** The relation to a bottom-up approach is as follows: Denote by  $\tau_1, \dots, \tau_m$  the default times of the credit risky securities in the collateral and their (possibly random) loss given default by  $q_1, \dots, q_m$ . Then

$$A_t = \sum_{i=1}^m q_i \mathbf{1}_{\{\tau_i \leq t\}}.$$

Following Filipović, Overbeck, and Schmidt (2009), we introduce the following concept. Denote  $\mathcal{T} := \{T_0, \dots, T_n\}$  and  $\overline{\mathcal{T}} := \{T_1, \dots, T_{n-1}\}$ .

**Definition 2.3.** If  $(T_k, x) \in \mathcal{T} \times \mathcal{I}$ , a security which pays  $\mathbf{1}_{\{A_{T_k} \leq x\}}$  at  $T_k$  is called  $(T_k, x)$ -bond. Its price at time  $t \leq T_k$  is denoted by  $P(t, T_k, x)$ .

If the market is free of arbitrage,  $P(t, T_k, x)$  is nondecreasing in  $x$  and

$$P(t, T_k, 1) = P(t, T_k).$$

We postulate that the price of a  $(T_k, x)$ -bond can be written as

$$P(t, T_k, x) = p(t, T_k, x) \mathbf{1}_{\{A_t \leq x\}}, \quad (2)$$

where  $(p(t, T_k, x))_{0 \leq t \leq T_k}$  is a strictly positive special semimartingale with  $p(T_k, T_k, x) = 1$ . This process represents the pre-default value of the bond.

Using  $(T_k, x)$ -bonds we extend the definition of the Libor rate and introduce the following defaultable, simply compounded interest rate:

**Definition 2.4.** The  $(T_k, x)$ -Libor rate at time  $t \leq T_k$  is given by

$$L(t, T_k, x) := \mathbf{1}_{\{A_t \leq x\}} \frac{1}{\delta_k} \left( \frac{p(t, T_k, x)}{p(t, T_{k+1}, x)} - 1 \right); \quad (3)$$

the  $(T_k, x)$ -credit spread is defined by

$$H(t, T_k, x) := \mathbf{1}_{\{A_t \leq x\}} \frac{L(t, T_k, x) - L(t, T_k)}{1 + \delta_k L(t, T_k)}. \quad (4)$$

The  $(T_k, x)$ -forward price is given by

$$F(t, T_k, x) := \frac{P(t, T_k, x)}{P(t, T_k)}. \quad (5)$$

<sup>1</sup>It is straightforward to generalize to true subsets  $\mathcal{I} \subset [0, 1]$ .

Here,  $L(t, T_k, x)$  is a defaultable forward interest rate that one can contract at time  $t$ , given that  $A_t \leq x$ , on a defaultable forward investment of one unit of cash from  $T_k$  to  $T_{k+1}$ . As  $A_t \leq 1$ , we obtain  $L(t, T_k) = L(t, T_k, 1)$ . Furthermore, on the set  $\{A_t \leq x\}$ ,

$$1 + \delta_k H(t, T_k, x) = \frac{1 + \delta_k L(t, T_k, x)}{1 + \delta_k L(t, T_k)}. \quad (6)$$

Together with (1) and (3) we obtain, again on  $\{A_t \leq x\}$ ,

$$1 + \delta_k H(t, T_k, x) = \frac{p(t, T_k, x)}{P(t, T_k)} \left( \frac{p(t, T_{k+1}, x)}{P(t, T_{k+1})} \right)^{-1} > 0. \quad (7)$$

By (7), on  $\{A_t \leq x\}$  only,

$$H(t, T_k, x) = \frac{1}{\delta_k} \left( \frac{F(t, T_k, x)}{F(t, T_{k+1}, x)} - 1 \right).$$

The following remark illustrates relation between discrete-tenor and continuous-tenor credit spread.

**Remark 2.5.** The quantities  $H(t, T_k, x)$  represent the discrete-tenor analogs of credit spreads in the defaultable HJM models. More precisely, expressed in terms of  $(T, x)$ -forward-rates (see Filipović, Overbeck, and Schmidt (2009)) we have that

$$P(t, T_k, x) = e^{-\int_t^{T_k} f(t, u, x) du}.$$

The risk-free case is obtained with  $x = 1$ . By the definition of the Libor rate

$$\begin{aligned} L(t, T_k) &= \frac{1}{\delta_k} \left( e^{\int_{T_k}^{T_{k+1}} f(t, u, 1) du} - 1 \right) \\ &\approx \frac{1}{\delta_k} \int_{T_k}^{T_{k+1}} f(t, u, 1) du. \end{aligned}$$

This shows that the Libor rate is approximately the average forward rate over the time interval  $[T_k, T_{k+1}]$ , as expected. Furthermore, for a defaultable  $(T_k, x)$ -bond we have by (7) on  $\{A_t \leq x\}$

$$\begin{aligned} H(t, T_k, x) &= \frac{1}{\delta_k} \left( e^{-\int_{T_k}^{T_{k+1}} (f(t, u, x) - f(t, u, 1)) du} - 1 \right) \\ &\approx \frac{1}{\delta_k} \int_{T_k}^{T_{k+1}} (f(t, u, x) - f(t, u, 1)) du, \end{aligned}$$

which is approximately the average forward credit spread over the time interval  $[T_k, T_{k+1}]$ .

### 3. THE DRIVING PROCESS

Let a complete stochastic basis  $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q}_{T^*})$  be given, where  $\mathcal{G} = \mathcal{G}_{T^*}$  and  $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T^*}$  is some filtration satisfying the usual conditions. For simplicity we write  $\mathbb{Q}^*$  for  $\mathbb{Q}_{T^*}$ . The expectation w.r.t.  $\mathbb{Q}^*$  is denoted by  $\mathbb{E}^*$ .

Let us now describe the driving process for the model. A realistic assumption is that the dynamics of defaultable quantities related to the assets in

the given portfolio is influenced by the aggregate loss process  $A$ . This means that when a default happens in the portfolio, the default intensities of the other assets may be affected as well. Moreover, the risk-free interest rates can also depend on the aggregate loss process. In order to incorporate these features, we design a model where two sources of randomness appear:

- (1) a time-inhomogeneous Lévy process  $\tilde{X}$  representing the market noise, which is driving the risk-free and the pre-default dynamics
- (2) the aggregate loss process  $A$  for the given pool of credits.

These two processes are, in general, *not mutually independent*. Therefore, we will consider as the driving process such a multidimensional semimartingale whose components will be (1) and (2).

The definition and main properties of time-inhomogeneous Lévy processes can be found for example in Eberlein and Kluge (2006). For general semimartingale theory we refer to the book by Jacod and Shiryaev (2003), whose notation we adopt throughout the paper.

Before giving a precise characterization of the driving process, let us describe the aggregate loss process  $A$  in more detail. It takes values in  $\mathcal{I} = [0, 1]$ . We assume that  $A_t = \sum_{s \leq t} \Delta A_s$  is an  $\mathcal{I}$ -valued increasing marked point process with absolutely continuous  $\mathbb{Q}^*$ -compensator

$$\nu^A(dt, dy) = F_t^A(dy)dt, \quad (8)$$

where  $F^A$  is a transition kernel from  $(\Omega \times [0, T^*], \mathcal{P})$  into  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  and  $\mathcal{P}$  denotes the predictable  $\sigma$ -algebra on  $\Omega \times [0, T^*]$ .

Note that  $A$  is a semimartingale with finite variation and with canonical representation

$$A = x * \mu^A = x * (\mu^A - \nu^A) + x * \nu^A,$$

where  $\mu^A$  denotes its random measure of jumps. Recall that  $A$  is a special semimartingale since its jumps are bounded by 1 (cf. Lemma I.4.24 in Jacod and Shiryaev (2003)).

The indicator process  $\mathbf{1}_{\{A_t \leq x\}}$  is a càdlàg, decreasing process with intensity process

$$\lambda(t, x) = \nu^A(t, (x - A_t, 1] \cap \mathcal{I});$$

i.e. the process

$$M_t^x = \mathbf{1}_{\{A_t \leq x\}} + \int_0^t \mathbf{1}_{\{A_s \leq x\}} \lambda(s, x) ds \quad (9)$$

is a  $\mathbb{Q}^*$ -martingale (see Filipović, Overbeck, and Schmidt (2009), Lemma 3.1).

Let

$$X = (X^1, \dots, X^d, X^{d+1})$$

be an  $\mathbb{R}^{d+1}$ -valued special semimartingale on the stochastic basis  $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q}^*)$  with  $X_0 = 0$  a.s. and canonical representation given by

$$X_t = \int_0^t \sqrt{c_s} dW_s + \int_0^t \int_{\mathbb{R}^{d+1}} x(\mu - \nu)(ds, dx), \quad (10)$$

hence the truncation function is the identity on  $\mathbb{R}^{d+1}$ ;  $W$  is a  $(d+1)$ -dimensional standard Brownian motion with respect to  $\mathbb{Q}^*$ ,  $\mu$  is the random measure of jumps of  $X$  and  $\nu$  is its  $\mathbb{Q}^*$ -compensator. Moreover,

- (i)  $c_s$  is a symmetric, non-negative definite real-valued  $(d+1) \times (d+1)$ -matrix such that  $(c_s)^{i,d+1} = (c_s)^{d+1,i} = 0$ ,  $i = 1, \dots, d+1$ ;
- (ii)  $\nu$  is absolutely continuous, i.e.  $\nu(dt, dy) = F_t(dy)dt$ , for some transition kernel  $F$  from  $(\Omega \times [0, T^*], \mathcal{P})$  into  $(\mathbb{R}^{d+1}, \mathcal{B}(\mathbb{R}^{d+1}))$ . We assume that for every  $t \in [0, T^*]$ , the projection on  $\mathbb{R}^d$  is

$$F_t(E \times \mathbb{R}) = \tilde{F}_t(E), \quad E \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}), \quad (11)$$

for some Lévy measure  $\tilde{F}_t$  on  $\mathbb{R}^d$  (i.e.  $\tilde{F}_t(\{0\}) = 0$  and  $\int_{\mathbb{R}^d} (|x|^2 \wedge 1) \tilde{F}_t(dx) < \infty$ ). The projection on  $\mathbb{R}$  is

$$F_t(\mathbb{R}^d \times E) = F_t^A(E), \quad E \in \mathcal{B}(\mathbb{R} \setminus \{0\}), \quad (12)$$

where  $F^A$  is defined in (8).

The triplet of predictable  $\mathbb{Q}^*$ -characteristics of  $X$  is given by

$$B_t = 0, \quad C_t = \int_0^t c_s ds, \quad \nu(dt, dy) = F_t(dy)dt, \quad (13)$$

with the associated triplet of local characteristics given by  $(0, c, F)$ .

Due to assumptions (i) and (ii), the first  $d$  components  $(X^1, \dots, X^d)$  of  $X$  form a  $d$ -dimensional time-inhomogeneous Lévy process with the triplet of local characteristics given by  $(0, c_t, \tilde{F}_t)$  and the last component  $X^{d+1}$  is the compensated loss process  $x * (\mu^A - \nu^A)$  with triplet  $(0, 0, F_t^A)$ . In the following proposition we prove these statements.

**Proposition 3.1.** *Let  $X$  be an  $\mathbb{R}^{d+1}$ -valued special semimartingale with predictable characteristics  $(B, C, \nu)$  given by (13) and satisfying (i) and (ii). Denote by  $\tilde{X} := (X^1, \dots, X^d)$  the semimartingale consisting of the first  $d$  components of  $X$ . Then:*

- (a)  $\tilde{X}$  and  $X^{d+1}$  are special semimartingales,
- (b)  $\tilde{X}$  is a time-inhomogeneous Lévy process with the  $\mathbb{Q}^*$ -local characteristics  $(b^{\tilde{X}}, c^{\tilde{X}}, F^{\tilde{X}})$  given by

$$\begin{aligned} b_s^{\tilde{X}} &= 0, \\ c_s^{\tilde{X}} &= [(c_s)^{ij}]_{i,j=1,\dots,d}, \\ F_s^{\tilde{X}}(E) &= \tilde{F}_s(E), \quad E \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}), \end{aligned} \quad (14)$$

- (c)  $X^{d+1}$  is a purely discontinuous local martingale with the  $\mathbb{Q}^*$ -local characteristics  $(b^{X^{d+1}}, c^{X^{d+1}}, F^{X^{d+1}})$  given by

$$\begin{aligned} b_s^{X^{d+1}} &= 0, \\ c_s^{X^{d+1}} &= 0, \\ F_s^{X^{d+1}}(E) &= F_s^A(E), \quad E \in \mathcal{B}(\mathbb{R} \setminus \{0\}). \end{aligned} \quad (15)$$

*Proof:* The proof of the proposition relies on Proposition 2.4 and Lemma 2.5 in Eberlein, Papapantoleon, and Shiryaev (2009), where linear transformations of multidimensional semimartingales are studied. More precisely, for a given  $(d+1)$ -dimensional semimartingale  $X$ , semimartingales of the form  $UX$ , where  $U$  is an  $n \times (d+1)$ -dimensional real-valued matrix, are considered. We are interested in two particular semimartingales of this form:

$$\tilde{X} = U'X,$$

where  $U'$  is a  $d \times (d+1)$ -matrix with  $(u')^{ii} = 1$ , for  $i = 1, \dots, d$ , and  $(u')^{ij} = 0$  otherwise, and

$$X^{d+1} = U''X,$$

where  $U''$  is a  $1 \times (d+1)$ -matrix with  $(u'')^{1,d+1} = 1$  and  $(u'')^{1,i} = 0$  otherwise.

By virtue of Lemma 2.5 in the aforementioned paper, if  $X$  is a special semimartingale, then  $U'X$  and  $U''X$  are also special semimartingales and hence, we deduce (a).

Let us now establish (b). Applying Proposition 2.4 in Eberlein, Papapantoleon, and Shiryaev (2009) to  $\tilde{X} = U'X$  we obtain

$$b_s^{\tilde{X}} = U'b_s + \int_{\mathbb{R}^{d+1}} (h^{\tilde{X}}(U'y) - U'y)F_s(dy) = 0,$$

since  $b_s = 0$  by assumption and the truncation function  $h^{\tilde{X}}$  can be chosen to be the identity since  $\tilde{X}$  is a special semimartingale by (a). Furthermore,

$$c_s^{\tilde{X}} = U'c_s = [(c_s)^{ij}]_{i,j=1,\dots,d}$$

and for  $E \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$  we have

$$\begin{aligned} F_s^{\tilde{X}}(E) &= \int_{\mathbb{R}^{d+1}} \mathbf{1}_E(U'y)F_s(dy) \\ &= \int_{\mathbb{R}^{d+1}} \mathbf{1}_E(y^1, \dots, y^d)F_s(dy) \\ &= F_s(E \times \mathbb{R}) = \tilde{F}_s(E), \end{aligned}$$

by (11). Thus, we have shown (14).

It remains to prove that  $\tilde{X}$  is a time-inhomogeneous Lévy process. To do so, we have to check if it satisfies the properties stated in the definition given in Section 2.1 in Eberlein and Kluge (2006). Obviously,  $\tilde{X}$  is an adapted, càdlàg process such that  $\tilde{X}_0 = 0$  a.s. Moreover, its triplet of local characteristics  $(0, c^{\tilde{X}}, F^{\tilde{X}})$  is deterministic, which by Theorem II.4.15 in Jacod and Shiryaev (2003) implies that its increments are independent. Finally, the characteristic function of the random variable  $\tilde{X}_t$  is given by

$$\begin{aligned} \mathbb{E}^*[e^{i\langle u, \tilde{X}_t \rangle}] &= \exp \int_0^t \left( i\langle u, b_s^{\tilde{X}} \rangle - \frac{1}{2}\langle u, c_s^{\tilde{X}} u \rangle \right. \\ &\quad \left. + \int_{\mathbb{R}^d} \left( e^{i\langle u, x \rangle} - 1 - i\langle u, h^{\tilde{X}}(x) \rangle \right) F_s^{\tilde{X}}(dx) \right) ds, \end{aligned}$$

which again follows directly from Theorem II.4.15, equation (4.16), in Jacod and Shiryaev (2003). Therefore,  $\tilde{X}$  is by definition a time-inhomogeneous Lévy process.

Let us establish (c). Another application of Proposition 2.4 in Eberlein, Papapantoleon, and Shiryaev (2009), this time to  $X^{d+1} = U''X$ , yields

$$b_s^{X^{d+1}} = U''b_s + \int_{\mathbb{R}^{d+1}} (h^{X^{d+1}}(U''y) - U''h(y))F_s(dy) = 0$$

by similar arguments as in (b). Furthermore,

$$c_s^{X^{d+1}} = U''c_s = (c_s)^{d+1,d+1} = 0,$$

by (i). Finally, for  $E \in \mathcal{B}(\mathbb{R} \setminus \{0\})$  we have

$$\begin{aligned} F_s^{X^{d+1}}(E) &= \int_{\mathbb{R}^{d+1}} \mathbf{1}_E(U''y)F_s(dy) \\ &= \int_{\mathbb{R}^{d+1}} \mathbf{1}_E(y^{d+1})F_s(dy) \\ &= F_s(\mathbb{R}^d \times E) = F_s^A(E), \end{aligned}$$

by (12) and therefore, we have obtained (15). It follows that the canonical representation of  $X^{d+1}$  is given by

$$X^{d+1} = x * (\mu^A - \nu^A)$$

and it is a purely discontinuous  $\mathbb{Q}^*$ -local martingale by definition.  $\square$

#### 4. RISK-FREE FORWARD LIBOR RATES

The essential tools in market models are *forward measures*. Under the  $T_k$ -forward measures the default-free zero coupon bonds with maturity  $T_k$  serve as numeraires. In this section we utilize the usual backward induction procedure to obtain the forward measures and the risk-free Libor rates. We basically generalize Eberlein and Özkan (2005) to our setting.

The driving process  $X$  is the  $(d+1)$ -dimensional special semimartingale described in the previous section. We assume that

**(A1)** The local characteristics  $(0, c, F)$  satisfy

$$\sup_{0 \leq t \leq T^*, \omega \in \Omega} \left( \|c_t\| + \int_{\mathbb{R}^{d+1}} (|y|^2 \wedge 1)F_t(dy) \right) < \infty$$

and there exist constants  $C, \varepsilon > 0$  such that

$$\sup_{0 \leq t \leq T^*, \omega \in \Omega} \left( \int_{|y| > 1} \exp\langle u, y \rangle F_t(dy) \right) < \infty,$$

for every  $u \in [-(1+\varepsilon)C, (1+\varepsilon)C]^{d+1}$ .

This assumption is needed to ensure the martingale property of the Libor rates in the backward induction. In particular, it entails the *existence of exponential moments* of  $X$ , i.e.  $\mathbb{E}^*[\exp\langle u, X_t \rangle] < \infty$ , for all  $t \in [0, T^*]$  and  $u$  as above (to see this note that the random variable  $u^{d+1}(X^{d+1})_t = u^{d+1}x * (\mu^A - \nu^A)_t$  is bounded for every  $t$  and then apply Lemma 6 in Eberlein and Kluge (2006) to the time-inhomogeneous Lévy process  $\tilde{X} = (X^1, \dots, X^d)$  and the Lévy measures  $\tilde{F}_t$ ).

Furthermore, we impose the following assumptions:

- (A2) There are deterministic,  $\mathcal{B}(\mathbb{R}_+)$ -measurable functions  $\sigma(\cdot, T_k) : [0, T^*] \rightarrow \mathbb{R}_+^{d+1}$ ,  $k = 1, \dots, n$ , such that

$$\sum_{k=1}^{n-1} \sigma^j(s, T_k) \leq C,$$

for all  $s \in [0, T^*]$  and every coordinate  $j \in \{1, \dots, d+1\}$ , where  $C$  is the constant from (A1). Moreover,  $\sigma(s, T_k) = 0$  for  $s > T_k$ .

- (A3) The initial bond prices satisfy

$$P(0, T_1) > \dots > P(0, T_n) > 0.$$

The function  $\sigma(\cdot, T_k)$  represents the *volatility* of the risk-free  $T_k$ -Libor rate.

**Remark 4.1.**

- (1) Note that the specification made in (A2) allows *direct dependence* between the risk-free interest rates and the aggregate loss process through the last component  $X^{d+1}$  of the driving process  $X$  and the volatility parameter  $\sigma^{d+1}(\cdot, T_k)$ . This means that the loss process  $A$  of the considered portfolio affects fluctuations of the risk-free Libor rates, thus reflecting the empirical fact that multiple defaults in the economy will likely have an impact on the government (risk-free) interest rates. This feature may be removed letting  $\sigma^{d+1}(\cdot, T_k) \equiv 0$  for all  $1 \leq k \leq n$ .
- (2) The volatilities in Assumption (A2) can be generalized, i.e.  $\sigma(\cdot, T_k)$  can be any stochastic process in  $L(X)$  such that  $\sigma(\cdot, T_k) \cdot X$  is exponentially special and that a certain integrability condition is satisfied.

The forward measures and the risk-free forward Libor rates are constructed by backward induction. The measure  $\mathbb{Q}^* = \mathbb{Q}_{T^*} = \mathbb{Q}_{T_n}$  plays the role of the forward measure associated with the settlement date  $T_n$  and is called the *terminal forward measure*. We also write  $W^{T_n}$  for  $W$  and  $\nu^{T_n}$  for  $\nu$  to emphasize the dependence on the measure  $\mathbb{Q}_{T_n}$ .

We start by specifying the dynamics of the  $T_{n-1}$ -Libor rate under the measure  $\mathbb{Q}_{T_n}$ . Then we proceed recursively. In each step a new forward measure  $\mathbb{Q}_{T_{k+1}}$  is constructed and the next Libor rate is specified under this measure. More precisely, the dynamics of the forward Libor rate for the time period  $[T_k, T_{k+1}]$  is given by

$$L(t, T_k) = L(0, T_k) \exp \left( \int_0^t b^L(s, T_k) ds + \int_0^t \sigma(s, T_k) dX_s^{T_{k+1}} \right) \quad (16)$$

with the initial condition

$$L(0, T_k) = \frac{1}{\delta_k} \left( \frac{P(0, T_k)}{P(0, T_{k+1})} - 1 \right).$$

The process  $X^{T_{k+1}}$  is a special semimartingale obtained from  $X$  by a change from the forward measure  $\mathbb{Q}_{T_n}$  to the forward measure  $\mathbb{Q}_{T_{k+1}}$  defined below. Its canonical representation is given by

$$X_t^{T_{k+1}} = \int_0^t \sqrt{c_s} dW_s^{T_{k+1}} + \int_0^t \int_{\mathbb{R}^{d+1}} x(\mu - \nu^{T_{k+1}})(ds, dx), \quad (17)$$

where  $W^{T_{k+1}}$  is a standard  $(d+1)$ -dimensional Brownian motion with respect to  $\mathbb{Q}_{T_{k+1}}$  and  $\nu^{T_{k+1}}$  is the  $\mathbb{Q}_{T_{k+1}}$ -compensator of  $\mu$ . The drift term  $b^L(\cdot, T_k)$  is specified in such a way that  $L(\cdot, T_k)$  becomes a  $\mathbb{Q}_{T_{k+1}}$ -local martingale, i.e.

$$\begin{aligned} b^L(s, T_k) &= -\frac{1}{2} \langle \sigma(s, T_k), c_s \sigma(s, T_k) \rangle \\ &\quad - \int_{\mathbb{R}^{d+1}} \left( e^{\langle \sigma(s, T_k), x \rangle} - 1 - \langle \sigma(s, T_k), x \rangle \right) F_s^{T_{k+1}}(dx). \end{aligned}$$

With this drift specification,  $L(\cdot, T_k)$  actually becomes a true martingale with respect to  $\mathbb{Q}_{T_{k+1}}$  (this can be proved using Assumptions **(A1)** and **(A2)** similarly to the proof of Proposition 1.24 in Grbac (2010)).

The forward measure  $\mathbb{Q}_{T_{k+1}}$  is defined on  $(\Omega, \mathcal{G}_{T_{k+1}})$  by its Radon–Nikodym derivative with respect to  $\mathbb{Q}_{T_{k+2}}$ . Iterating backwards, it can be expressed via

$$\frac{d\mathbb{Q}_{T_{k+1}}}{d\mathbb{Q}_{T_n}} = \prod_{j=k+1}^{n-1} \frac{1 + \delta_j L(T_{k+1}, T_j)}{1 + \delta_j L(0, T_j)} = \frac{P(0, T_n)}{P(0, T_{k+1})} \prod_{j=k+1}^{n-1} (1 + \delta_j L(T_{k+1}, T_j)). \quad (18)$$

Note that the process  $\prod_{j=k+1}^{n-1} (1 + \delta_j L(\cdot, T_j))$  is a  $\mathbb{Q}_{T_n}$ -martingale (this follows by virtue of Proposition III.3.8(a) in Jacod and Shiryaev (2003) and from the backward induction). Hence, for  $t \leq T_{k+1}$ ,

$$\begin{aligned} \left. \frac{d\mathbb{Q}_{T_{k+1}}}{d\mathbb{Q}_{T_n}} \right|_{\mathcal{G}_t} &= \frac{P(0, T_n)}{P(0, T_{k+1})} \prod_{j=k+1}^{n-1} (1 + \delta_j L(t, T_j)) \\ &= \frac{P(0, T_n)}{P(0, T_{k+1})} \frac{P(t, T_{k+1})}{P(t, T_n)}, \end{aligned} \quad (19)$$

where the second equality follows from (1). Define

$$\ell(s, T_j) := \frac{\delta_j L(s, T_j)}{1 + \delta_j L(s, T_j)},$$

for  $1 \leq j \leq n$  and  $s \leq T_j$ . Using Girsanov's theorem for semimartingales we deduce that

$$W_t^{T_{k+1}} := W_t^{T_n} - \int_0^t \sqrt{c_s} \left( \sum_{j=k+1}^{n-1} \alpha(s, T_j) \right) ds \quad (20)$$

with

$$\alpha(s, T_j) := \ell(s-, T_j)\sigma(s, T_j), \quad (21)$$

is a  $(d + 1)$ -dimensional standard Brownian motion with respect to  $\mathbb{Q}_{T_{k+1}}$ , and

$$\nu^{T_{k+1}}(ds, dy) := \left( \prod_{j=k+1}^{n-1} \beta(s, T_j, y) \right) \nu^{T_n}(ds, dy) =: F_s^{T_{k+1}}(dy)ds \quad (22)$$

with

$$\beta(s, T_j, y) := \ell(s-, T_j) \left( e^{\langle \sigma(s, T_j), y \rangle} - 1 \right) + 1, \quad (23)$$

is the  $\mathbb{Q}_{T_{k+1}}$ -compensator of  $\mu$  (cf. Eberlein and Özkan (2005), Section 4, pp. 340–341).

**Remark 4.2.** As we have pointed out, if the  $(d + 1)$ st-component of the volatility  $\sigma(\cdot, T_k)$  in Assumption **(A2)** is chosen to be zero, the loss process  $A$  does not influence the dynamics of risk-free Libor rates. More precisely, we have

$$\begin{aligned} L(t, T_k) &= L(0, T_k) \exp \left( \int_0^t b^L(s, T_k) ds + \int_0^t \sqrt{c_s} \sigma(s, T_k) dW_s^{T_{k+1}} \right. \\ &\quad \left. + \int_0^t \int_{\mathbb{R}^{d+1}} \langle (\sigma^1(s, T_k), \dots, \sigma^d(s, T_k)), (y^1, \dots, y^d) \rangle (\mu - \nu^{T_{k+1}})(ds, dy) \right). \end{aligned}$$

Recalling (i) and (ii) of the local characteristics of  $X$ , combined with (20) and (22), we note that the risk-free Libor rates are actually driven by the time-inhomogeneous Lévy process  $\tilde{X}$  given in Proposition 3.1, i.e.

$$\begin{aligned} L(t, T_k) &= L(0, T_k) \exp \left( \int_0^t b^L(s, T_k) ds \right. \\ &\quad \left. + \int_0^t (\sigma^1(s, T_k), \dots, \sigma^d(s, T_k)) d\tilde{X}_s^{T_{k+1}} \right), \end{aligned}$$

where  $\tilde{X}^{T_{k+1}}$  is obtained from  $\tilde{X}$  in the same way as  $X^{T_{k+1}}$  is obtained from  $X$ . This is precisely the Lévy Libor model of Eberlein and Özkan (2005).

For each  $k = 1, \dots, n$ , we denote by  $(0, c^{\tilde{X}}, F^{\tilde{X}, T_k})$  the  $\mathbb{Q}_{T_k}$ -local characteristics of  $\tilde{X}^{T_{k+1}}$ , where

$$F_t^{\tilde{X}, T_k}(E) = F_t^{T_k}(E \times \mathbb{R}), \quad E \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}), \quad (24)$$

and by  $\nu^{A, T_k}$  the  $\mathbb{Q}_{T_k}$ -compensator of the loss process  $A$ , where  $\nu^{A, T_k}(dt, dy) = F_t^{A, T_k}(dy)dt$  with

$$F_t^{A, T_k}(E) = F_t^{T_k}(\mathbb{R}^d \times E), \quad E \in \mathcal{B}(\mathbb{R} \setminus \{0\}). \quad (25)$$

The intensity process of the indicator process  $\mathbf{1}_{\{A_t \leq x\}}$  under  $\mathbb{Q}_{T_k}$  is denoted by  $\lambda^{T_k}(\cdot, x)$ , i.e.

$$M_t^{x, T_k} = \mathbf{1}_{\{A_t \leq x\}} + \int_0^t \mathbf{1}_{\{A_s \leq x\}} \lambda^{T_k}(s, x) ds \quad (26)$$

is a  $\mathbb{Q}_{T_k}$ -martingale.

The above construction guarantees that processes  $\frac{P(\cdot, T_j)}{P(\cdot, T_k)}$  are  $\mathbb{Q}_{T_k}$ -martingales for all  $j, k = 1, \dots, n$  and therefore, the model is free of arbitrage. Moreover, the price at time  $t$  of a contingent claim with payoff  $Y$  at maturity  $T_k$  equals  $P(t, T_k) \mathbb{E}_{\mathbb{Q}_{T_k}}[Y | \mathcal{G}_t]$ .

## 5. CONSTRUCTION OF $(T_k, x)$ -LIBOR RATES AND ABSENCE OF ARBITRAGE

The goal of market modeling is to identify a suitable quantity and to impose tractable dynamics for it which is compatible with absence of arbitrage. It turns out that in our case this quantity is the credit spread  $H$ . We assume that for all  $(T_k, x) \in \bar{T} \times \mathcal{I}$  and  $t \leq T_k$

$$H(t, T_k, x) = \mathbf{1}_{\{A_t \leq x\}} h(t, T_k, x), \quad (27)$$

where  $h(\cdot, T_k, x)$  is positive and is called the *pre-default credit spread*. From (6) we obtain on  $\{A_t \leq x\}$

$$1 + \delta_k L(t, T_k, x) = (1 + \delta_k L(t, T_k))(1 + \delta_k h(t, T_k, x))$$

and consequently,

$$L(t, T_k, x) = \mathbf{1}_{\{A_t \leq x\}} \frac{1}{\delta_k} \left( (1 + \delta_k L(t, T_k))(1 + \delta_k h(t, T_k, x)) - 1 \right). \quad (28)$$

In other words, every forward  $(T_k, x)$ -Libor rate can be obtained from the risk-free forward Libor rate with the same maturity and the corresponding pre-default credit spread. Note that

$$L(t, T_k, x) > L(t, T_k) \iff h(t, T_k, x) > 0$$

on  $\{A_t \leq x\}$ . Hence,  $h(\cdot, T_k, x) > 0$  ensures that the defaultable forward  $(T_k, x)$ -Libor rates are higher than their risk-free counterparts, an important property in practice.

The aim of the following part of this section is to present a construction of the forward default intensities  $H$  such that the  $(T_k, x)$ -bond market is free of arbitrage. To this, we impose some assumptions, where  $\mathcal{O}$  denotes the optional  $\sigma$ -algebra on  $(\Omega \times [0, T^*])$ .

- (A4) For all  $T_k$  there is a deterministic,  $\mathbb{R}_+^{d+1}$ -valued function  $\gamma(s, T_k, x)$ , which as a function of  $(s, x) \mapsto \gamma(s, T_k, x)$  is  $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathcal{I})$ -measurable. Moreover,

$$\gamma^{d+1}(s, T_k, x) = 0 \quad \text{and} \quad \sum_{k=1}^{n-1} (\sigma^j(s, T_k) + \gamma^j(s, T_k, x)) \leq C,$$

for all  $s \in [0, T^*]$  and every coordinate  $j \in \{1, \dots, d+1\}$ , where  $C$  is the constant from (A1). If  $s > T_k$ , then  $\gamma(s, T_k, x) = 0$ .

**(A5)** For all  $T_k$  there is an  $\mathbb{R}$ -valued function  $c(s, T_k, x; y)$ , which is called the *contagion* parameter and which as a function of  $(s, x, y) \mapsto c(s, T_k, x; y)$  is  $\mathcal{P} \otimes \mathcal{B}(\mathcal{I}) \otimes \mathcal{B}(\mathcal{I})$ -measurable. We also assume

$$\sup_{s \leq T_k, x, y \in \mathcal{I}, \omega \in \Omega} |c(s, T_k, x; y)| < \infty$$

and  $c(s, T_k, x; y) = 0$  for  $s > T_k$ .

**(A6)** The initial term structure  $P(0, T_k, x)$  is strictly positive, strictly decreasing in  $k$  and satisfies

$$F(0, T_k, x) = \frac{P(0, T_k, x)}{P(0, T_k)} \geq \frac{P(0, T_{k+1}, x)}{P(0, T_{k+1})} = F(0, T_{k+1}, x).$$

Here  $\gamma(\cdot, T_k, x)$  represents the *volatility* of the credit spread  $H(\cdot, T_k, x)$ . Moreover, we denote

$$\tilde{\gamma}(s, T_k, x) := (\gamma^1(s, T_k, x), \dots, \gamma^d(s, T_k, x)).$$

We assume that the pre-default credit spread  $h$  (see (27)) follows a semi-martingale of the following form:

**(A7)** For every  $t \leq T_k$

$$\begin{aligned} h(t, T_k, x) = & h(0, T_k, x) \exp \left( \int_0^t b(s, T_k, x) ds + \int_0^t \tilde{\gamma}(s, T_k, x) d\tilde{X}_s^{T_{k+1}} \right. \\ & \left. + \int_0^t \int_{\mathcal{I}} c(s, T_k, x; y) (\mu^A - \nu^{A, T_{k+1}})(ds, dy) \right) \end{aligned}$$

with the initial condition

$$h(0, T_k, x) = \frac{1}{\delta_k} \left( \frac{F(0, T_k, x)}{F(0, T_{k+1}, x)} - 1 \right).$$

The drift term  $b(\cdot, T_k, \cdot)$  is an  $\mathbb{R}$ -valued,  $\mathcal{O} \otimes \mathcal{B}(\mathcal{I})$ -measurable process with  $b(s, T_k, x) = 0$ , for  $s > T_k$ , that will be specified later.

Here  $\tilde{X}^{T_{k+1}}$  is the special semimartingale obtained from the time-inhomogeneous Lévy process  $\tilde{X}$  in Remark 4.2 and  $\nu^{A, T_{k+1}}$  is the  $\mathbb{Q}_{T_{k+1}}$ -compensator of  $\mu^A$  defined via (25). Note that  $b(s, T_k, x) = 0$ , for  $s > T_k$  implies that  $h(t, T_k, x) = h(T_k, T_k, x)$  for  $t \geq T_k$ .

**Remark 5.1.** Specifying the dynamics of  $h(\cdot, T_k, x)$  in this way, we allow for two kinds of jumps: the jumps caused by market forces, represented by the time-inhomogeneous Lévy process  $\tilde{X}$ , and the jumps caused by defaults in the portfolio, represented through the aggregate loss process  $A$ , which allows for contagion effects.

We introduce the following notation

$$\varrho(s, T_k, x; y) := \langle \tilde{\gamma}(s, T_k, x), (y^1, \dots, y^d) \rangle + c(s, T_k, x; y^{d+1}), \quad (29)$$

for every  $T_k \in \bar{\mathcal{T}}$ ,  $x \in \mathcal{I}$  and  $y \in \mathbb{R}^{d+1}$ . Then **(A7)** can be equivalently written as:

(A7') For every  $t \leq T_k$

$$h(t, T_k, x) = h(0, T_k, x) \exp \left( \int_0^t b(s, T_k, x) ds + \int_0^t \sqrt{c_s} \gamma(s, T_k, x) dW_s^{T_{k+1}} + \int_0^t \int_{\mathbb{R}^{d+1}} \varrho(s, T_k, x; y) (\mu - \nu^{T_{k+1}})(ds, dy) \right);$$

here  $W^{T_{k+1}}$  is a  $\mathbb{Q}_{T_{k+1}}$ -standard  $(d+1)$ -dimensional Brownian motion defined in (20), and  $\nu^{T_{k+1}}$  is the  $\mathbb{Q}_{T_{k+1}}$ -compensator of  $\mu$  defined in (22).

Let us prove that (A7') is equivalent to (A7). We have

$$\begin{aligned} h(t, T_k, x) &= h(0, T_k, x) \exp \left( \int_0^t b(s, T_k, x) ds + \int_0^t \sqrt{c_s} \gamma(s, T_k, x) dW_s^{T_{k+1}} \right. \\ &\quad \left. + \int_0^t \int_{\mathbb{R}^{d+1}} \left( \langle \tilde{\gamma}(s, T_k, x), (y^1, \dots, y^d) \rangle + c(s, T_k, x; y^{d+1}) \right) \right. \\ &\quad \left. \times (\mu - \nu^{T_{k+1}})(ds, dy) \right) \\ &= h(0, T_k, x) \exp \left( \int_0^t b(s, T_k, x) ds + \int_0^t \sqrt{c_s} \gamma(s, T_k, x) dW_s^{T_{k+1}} \right. \\ &\quad \left. + \int_0^t \int_{\mathbb{R}^d} \langle \tilde{\gamma}(s, T_k, x), (y^1, \dots, y^d) \rangle (\mu^{\tilde{X}} - \nu^{\tilde{X}, T_{k+1}})(ds, d(y^1, \dots, y^d)) \right. \\ &\quad \left. + \int_0^t \int_{\mathbb{R}} c(s, T_k, x; y^{d+1}) (\mu^A - \nu^{A, T_{k+1}})(ds, dy^{d+1}) \right) \\ &= h(0, T_k, x) \exp \left( \int_0^t b(s, T_k, x) ds + \int_0^t \tilde{\gamma}(s, T_k, x) d\tilde{X}_s^{T_{k+1}} \right. \\ &\quad \left. + \int_0^t \int_{\mathcal{I}} c(s, T_k, x; y) (\mu^A - \nu^{A, T_{k+1}})(ds, dy) \right), \end{aligned}$$

where we have used the property (i) of  $c$  stated after equation (10) and equations (24) and (25).

**Remark 5.2.** Note that the dynamics of the forward  $(T_k, x)$ -Libor rate under the measure  $\mathbb{Q}_{T_k}$  can now be derived by means of stochastic calculus for semimartingales starting from relation (28) (compare with Theorem 3.7 in Grbac (2010), where a similar calculation is done for the rating-dependent Libor rates in the rating based Lévy Libor model).

Let us now explore conditions that ensure the absence of arbitrage. It is well-known that the model is free of arbitrage if for each  $i, k = 1, \dots, n$  the

bond price process

$$\left( \frac{P(t, T_k, x)}{P(t, T_i)} \right)_{0 \leq t \leq T_i \wedge T_k}$$

is a local martingale with respect to the corresponding forward measure  $\mathbb{Q}_{T_i}$ . This is equivalent to the claim that for each  $k$  the forward bond price process

$$\left( \frac{P(t, T_k, x)}{P(t, T_k)} \right)_{0 \leq t \leq T_k}$$

is a  $\mathbb{Q}_{T_k}$ -local martingale, as the following lemma shows.

**Lemma 5.3.** *The following are equivalent:*

(a) *For each  $k = 1, \dots, n$  the process*

$$\left( \frac{P(t, T_k, x)}{P(t, T_k)} \right)_{0 \leq t \leq T_k}$$

*is a  $\mathbb{Q}_{T_k}$ -local martingale.*

(b) *For each  $k, i = 1, \dots, n$  the process*

$$\left( \frac{P(t, T_k, x)}{P(t, T_i)} \right)_{0 \leq t \leq T_i \wedge T_k}$$

*is a  $\mathbb{Q}_{T_i}$ -local martingale.*

*Proof:* It suffices to note that for fixed  $i, k \in \{1, \dots, n\}$  such that  $i \geq k$  (the other case is treated in the same way) we have

$$\frac{P(t, T_k, x)}{P(t, T_i)} = F(t, T_k, x) \frac{P(t, T_k)}{P(t, T_i)},$$

where  $F(\cdot, T_k, x) = \frac{P(\cdot, T_k, x)}{P(\cdot, T_k)}$  is a  $\mathbb{Q}_{T_k}$ -local martingale by (a) and  $\frac{P(\cdot, T_k)}{P(\cdot, T_i)}$  is the density process of the measure  $\mathbb{Q}_{T_k}$  relative to  $\mathbb{Q}_{T_i}$ , up to a norming constant (cf. equation (19)). Then  $\frac{P(\cdot, T_k, x)}{P(\cdot, T_i)}$  is a  $\mathbb{Q}_{T_i}$ -local martingale by Proposition III.3.8 in Jacod and Shiryaev (2003). The implication (a)  $\Rightarrow$  (b) is thus shown.

The implication (b)  $\Rightarrow$  (a) is obvious. □

Let us denote for each  $t \leq T_k$ ,  $k = 1, \dots, n-1$ , and  $x \in \mathcal{I}$

$$y(t, T_k, x) := \frac{1}{1 + \delta_k h(t, T_k, x)}; \tag{30}$$

where (7) ensures that the denominator does not vanish. In the following lemma we deduce the connection between the forward  $(T_k, x)$ -bond price processes and  $y$ .

**Lemma 5.4.** *Consider  $t \in (0, T_{k-1}]$ , where  $t \in (T_{l-1}, T_l]$  for some  $l \in \{1, \dots, k-1\}$ . Then*

$$F(t, T_k, x) = \left( \prod_{i=l}^{k-1} y(t, T_i, x) \right) F(t, T_l, x). \tag{31}$$

*Proof:* The proof relies on the relation between the forward default intensities and the bond prices given in (7). First, consider the case when  $A_t \leq x$ . With (30) and (7),

$$F(t, T_k, x) = y(t, T_{k-1}, x)F(t, T_{k-1}, x). \quad (32)$$

Since  $t \in (T_{l-1}, T_l]$ , we deduce recursively

$$F(t, T_k, x) = \left( \prod_{i=l}^{k-1} y(t, T_i, x) \right) F(t, T_l, x),$$

which establishes (31) on  $\{A_t \leq x\}$ . On  $\{A_t > x\}$ , (31) follows trivially by (2) and (5).  $\square$

Consequently, as soon as the pre-default intensities  $h(\cdot, T_k, x)$  are specified, the forward  $(T_k, x)$ -bond price process is also partially specified. More precisely, according to the previous lemma, in order to describe completely the dynamics of  $F(\cdot, T_k, x)$ , it remains to specify for each  $l = 1, \dots, n$  the dynamics of the process  $F(\cdot, T_l, x)$  on the interval  $(T_{l-1}, T_l]$ . This can be done in different ways; the specification below being an obvious and simple choice.

**(A8)** For every  $t \leq T_k$  and  $x \in \mathcal{I}$

$$\frac{p(t, T_k, x)}{P(t, T_k)} = \left( \prod_{i=0}^{k-1} y(t, T_i, x) \right) e^{\int_0^t b^P(s, T_k, x) ds},$$

where  $b^P(\cdot, T_k, \cdot)$  is an  $\mathbb{R}$ -valued,  $\mathcal{O} \otimes \mathcal{B}(\mathcal{I})$ -measurable, locally integrable process. Recall that  $y(t, T_i, x) = y(T_i, T_i, x)$ , for  $t \geq T_i$ , by assumption **(A7)**.

Then the forward  $(T_k, x)$ -bond price is given by

$$F(t, T_k, x) = \frac{p(t, T_k, x)}{P(t, T_k)} \mathbf{1}_{\{A_t \leq x\}} = \left( \prod_{i=0}^{k-1} y(t, T_i, x) \right) e^{\int_0^t b^P(s, T_k, x) ds} \mathbf{1}_{\{A_t \leq x\}}. \quad (33)$$

**Remark 5.5.** To ease notation we work with a continuous, finite variation process  $e^{\int_0^t b^P(s, T_k, x) ds}$  in **(A8)**. A more general specification with an exponential of some special semimartingale is possible and the occurring calculations can be done in the same way.

Before stating the main theorem of the section, which provides necessary and sufficient conditions for the forward  $(T_k, x)$ -bond price process (33) to be a  $\mathbb{Q}_{T_k}$ -local martingale, we need some auxiliary results.

**Lemma 5.6.** *Assume **(A1)**–**(A7')** and let  $\tilde{Y}(\cdot, T_k, x) := \prod_{i=0}^{k-1} y(\cdot, T_i, x)$ . Then*

$$\begin{aligned} d\tilde{Y}(t, T_k, x) = & \tilde{Y}(t-, T_k, x) \left[ D(t, T_k, x) dt - \sum_{i=1}^{k-1} g(t-, T_i, x) \sqrt{c_t} \gamma(t, T_i, x) dW_t^{T_k} \right. \\ & \left. + \int_{\mathbb{R}^{d+1}} \left( \prod_{i=1}^{k-1} \left( 1 + g(t-, T_i, x) (e^{\varrho(t, T_i, x; y)} - 1) \right)^{-1} - 1 \right) (\mu - \nu^{T_k})(dt, dy) \right], \end{aligned}$$

where

$$g(t, T_i, x) := \frac{\delta_i h(t, T_i, x)}{1 + \delta_i h(t, T_i, x)} \quad (34)$$

and

$$\begin{aligned} D(t, T_k, x) &:= - \sum_{i=1}^{k-1} g(t-, T_i, x) b(t, T_i, x) \\ &+ \sum_{i=1}^{k-1} g(t-, T_i, x) \left\langle \gamma(t, T_i, x), c_t \sum_{j=i+1}^{k-1} \alpha(t, T_j) \right\rangle \\ &- \sum_{i=1}^{k-1} \frac{1}{2} (g(t-, T_i, x) - g(t-, T_i, x)^2) \|\sqrt{c_t} \gamma(t, T_i, x)\|^2 \\ &+ \frac{1}{2} \left\| \sum_{i=1}^{k-1} g(t-, T_i, x) \sqrt{c_t} \gamma(t, T_i, x) \right\|^2 \\ &+ \int_{\mathbb{R}^{d+1}} \left[ \prod_{i=1}^{k-1} \left( 1 + g(t-, T_i, x) (e^{\varrho(t, T_i, x; y)} - 1) \right)^{-1} - 1 \right. \\ &\quad \left. + \sum_{i=1}^{k-1} g(t-, T_i, x) \varrho(t, T_i, x; y) \right. \\ &\quad \left. \times \left( \prod_{j=i+1}^{k-1} \beta(t, T_j, y) \right) \right] F_t^{T_k}(\mathrm{d}y), \end{aligned} \quad (35)$$

with  $\alpha(t, T_j)$  and  $\beta(t, T_j, y)$  defined in (21) and (23) respectively.

Note that on  $\{A_t \leq x\}$

$$g(t, T_i, x) = 1 - \frac{F(t, T_{i+1}, x)}{F(t, T_i, x)}.$$

*Proof:* The proof is deferred to the appendix. □

**Lemma 5.7.** *Assume (A1)–(A8). The dynamics of the process  $\frac{p(\cdot, T_k, x)}{P(\cdot, T_k)}$  under the forward measure  $\mathbb{Q}_{T_k}$  is given by*

$$\begin{aligned} \mathrm{d} \left( \frac{p(t, T_k, x)}{P(t, T_k)} \right) &= \frac{p(t-, T_k, x)}{P(t-, T_k)} \left( \left( b^P(t, T_k, x) + D(t, T_k, x) \right) \mathrm{d}t \right. \\ &\quad \left. - \sum_{i=1}^{k-1} g(t-, T_i, x) \sqrt{c_t} \gamma(t, T_i, x) \mathrm{d}W_t^{T_k} \right. \\ &\quad \left. + \int_{\mathbb{R}^{d+1}} \left( \prod_{i=1}^{k-1} \left( 1 + g(t-, T_i, x) (e^{\varrho(t, T_i, x; y)} - 1) \right)^{-1} - 1 \right) \right. \\ &\quad \left. \times (\mu - \nu^{T_k})(\mathrm{d}t, \mathrm{d}y) \right), \end{aligned}$$

where  $g(t, T_i, x)$  is given in (34) and  $D(t, T_k, x)$  in (35).

*Proof:* The proof is deferred to the appendix.  $\square$

Let us now state the main result of this section. It provides necessary and sufficient conditions for the forward  $(T_k, x)$ -bond price process  $F(\cdot, T_k, x)$  being a local martingale under the forward measure  $\mathbb{Q}_{T_k}$ , which ensures the absence of arbitrage in the market.

**Theorem 5.8.** *Assume that (A1)–(A8) are in force. Then the forward bond price process  $(F(t, T_k, x))_{0 \leq t \leq T_k}$  is a  $\mathbb{Q}_{T_k}$ -local martingale if and only if*

$$\begin{aligned} D(t, T_k, x) &= \lambda^{T_k}(t, x) - b^P(t, T_k, x) \\ &+ \int_{\mathbb{R}^{d+1}} \left( \prod_{i=1}^{k-1} \left( 1 + g(t-, T_i, x)(e^{\varrho(t, T_i, x; y)} - 1) \right)^{-1} - 1 \right) \\ &\quad \times \mathbf{1}_{\{A_t + y^{d+1} > x\}} F_t^{T_k}(dy), \end{aligned} \quad (36)$$

on the set  $\{A_t \leq x\}$ , for every  $t \in [0, T_k]$   $dt \times \mathbb{Q}_{T_k}$ -a.s.

It has been long acknowledged<sup>2</sup> that in HJM-models with default risk essentially two drift conditions appear. The first one fixes the short forward rate to the risk-free rate plus the default intensity and the other is the classical HJM drift condition. To our knowledge, Theorem 5.8 is the first result which allows for this distinguishing in market modeling.

*Proof:* Recall that  $F(t, T_k, x) = \frac{p(t, T_k, x)}{P(t, T_k)} \mathbf{1}_{\{A_t \leq x\}}$ . Using integration by parts yields

$$\begin{aligned} dF(t, T_k, x) &= \frac{p(t-, T_k, x)}{P(t-, T_k)} d\mathbf{1}_{\{A_t \leq x\}} + \mathbf{1}_{\{A_{t-} \leq x\}} d\left(\frac{p(t, T_k, x)}{P(t, T_k)}\right) \\ &+ d\left[\frac{p(\cdot, T_k, x)}{P(\cdot, T_k)}, \mathbf{1}_{\{A_{\cdot} \leq x\}}\right]_t \\ &=: (1') + (2') + (3'). \end{aligned}$$

We deal separately with each of the above three summands. For (1') recall (26), which leads to

$$\begin{aligned} d\mathbf{1}_{\{A_t \leq x\}} &= dM_t^{x, T_k} - \mathbf{1}_{\{A_t \leq x\}} \lambda^{T_k}(t, x) dt \\ &= \mathbf{1}_{\{A_{t-} \leq x\}} dM_t^{x, T_k} - \mathbf{1}_{\{A_{t-} \leq x\}} \lambda^{T_k}(t, x) dt \\ &= \mathbf{1}_{\{A_{t-} \leq x\}} \left( dM_t^{x, T_k} - \lambda^{T_k}(t, x) dt \right), \end{aligned}$$

since  $dM_t^{x, T_k} = \mathbf{1}_{\{A_{t-} \leq x\}} dM_t^{x, T_k}$ . Hence,

$$\begin{aligned} (1') &= \frac{p(t-, T_k, x)}{P(t-, T_k)} \mathbf{1}_{\{A_{t-} \leq x\}} \left( dM_t^{x, T_k} - \lambda^{T_k}(t, x) dt \right) \\ &= \frac{P(t-, T_k, x)}{P(t-, T_k)} \left( dM_t^{x, T_k} - \lambda^{T_k}(t, x) dt \right). \end{aligned}$$

<sup>2</sup>See Duffie and Singleton (1999), Schönbucher (1998) or Özkan and Schmidt (2005).

To calculate (2') we make use of Lemma 5.7, where the expression for the dynamics of  $\frac{p(\cdot, T_k, x)}{P(\cdot, T_k)}$  was derived and obtain

$$(2') = \mathbf{1}_{\{A_{t-} \leq x\}} \frac{p(t-, T_k, x)}{P(t-, T_k)} \left( \left( D(t, T_k, x) + b^P(t, T_k, x) \right) dt - \sum_{i=1}^{k-1} g(t-, T_i, x) \sqrt{c_t} \gamma(t, T_i, x) dW_t^{T_k} + \int_{\mathbb{R}^{d+1}} \left( \prod_{i=1}^{k-1} \left( 1 + g(t-, T_i, x) \left( e^{\varrho(t, T_i, x; y)} - 1 \right) \right)^{-1} - 1 \right) (\mu - \nu^{T_k})(dt, dy) \right).$$

It remains to calculate the covariation part (3'). Since  $\mathbf{1}_{\{A_t \leq x\}}$  has finite variation, we have

$$\left[ \frac{p(\cdot, T_k, x)}{P(\cdot, T_k)}, \mathbf{1}_{\{A_\cdot \leq x\}} \right]_t = \sum_{s \leq t} \Delta \left( \frac{p(s, T_k, x)}{P(s, T_k)} \right) \Delta \mathbf{1}_{\{A_s \leq x\}}$$

(see formula (26.9.4) in (Métivier 1982, Section V.26.9)). Moreover,

$$\begin{aligned} \Delta \mathbf{1}_{\{A_s \leq x\}}(\omega) &= \mathbf{1}_{\{A_s \leq x\}}(\omega) - \mathbf{1}_{\{A_{s-} \leq x\}}(\omega) \\ &= \begin{cases} -1; & \text{if } A_{s-}(\omega) \leq x \text{ and } A_s(\omega) > x \\ 0; & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore,

$$\Delta \mathbf{1}_{\{A_s \leq x\}} = -\mathbf{1}_{\{A_{s-} \leq x, A_s > x\}} = -\mathbf{1}_{\{A_{s-} \leq x, A_{s-} + \Delta A_s > x\}}$$

and it follows

$$\Delta \mathbf{1}_{\{A_s \leq x\}} = \int_{\mathbb{R}} z \mu^{\mathbf{1}_{\{A_\cdot \leq x\}}}(\{s\}, dz) = - \int_{\mathcal{I}} \mathbf{1}_{\{A_{s-} \leq x\}} \mathbf{1}_{\{A_{s-} + y > x\}} \mu^A(\{s\}, dy).$$

Combining this with Lemma 5.7 we deduce

$$(3') = \frac{p(t-, T_k, x)}{P(t-, T_k)} \mathbf{1}_{\{A_{t-} \leq x\}} \int_{\mathbb{R}^{d+1}} - \left( \prod_{i=1}^{k-1} \left( 1 + g(t-, T_i, x) \left( e^{\varrho(t, T_i, x; y)} - 1 \right) \right)^{-1} - 1 \right) \times \mathbf{1}_{\{A_{t-} + y^{d+1} > x\}} \mu(dt, dy).$$

Summing up the calculations, we obtain

$$\begin{aligned}
dF(t, T_k, x) = & F(t-, T_k, x) \left( dM_t^{x, T_k} - \lambda^{T_k}(t, x) dt \right. \\
& + \left( D(t, T_k, x) + b^P(t, T_k, x) \right) dt \\
& - \sum_{i=1}^{k-1} g(t-, T_i, x) \sqrt{c_t} \gamma(t, T_i, x) dW_t^{T_k} \\
& + \int_{\mathbb{R}^{d+1}} \left( \prod_{i=1}^{k-1} \left( 1 + g(t-, T_i, x) \left( e^{\varrho(t, T_i, x; y)} - 1 \right) \right)^{-1} - 1 \right) \\
& \qquad \qquad \qquad \times (\mu - \nu^{T_k})(dt, dy) \\
& - \int_{\mathbb{R}^{d+1}} \left( \prod_{i=1}^{k-1} \left( 1 + g(t-, T_i, x) \left( e^{\varrho(t, T_i, x; y)} - 1 \right) \right)^{-1} - 1 \right) \\
& \qquad \qquad \qquad \times \mathbf{1}_{\{A_{t-} + y^{d+1} > x\}} (\mu - \nu^{T_k})(dt, dy) \\
& - \int_{\mathbb{R}^{d+1}} \left( \prod_{i=1}^{k-1} \left( 1 + g(t-, T_i, x) \left( e^{\varrho(t, T_i, x; y)} - 1 \right) \right)^{-1} - 1 \right) \\
& \qquad \qquad \qquad \times \mathbf{1}_{\{A_t + y^{d+1} > x\}} \nu^{T_k}(dt, dy) \left. \right).
\end{aligned}$$

Therefore, we conclude that  $F(\cdot, T_k, x)$  is a  $\mathbb{Q}_{T_k}$ -local martingale if and only if (36) holds.  $\square$

**Example 5.9 (The EKS Lévy Libor model with default risk).** The Lévy Libor model with default risk by Eberlein, Kluge, and Schönbucher (2006) is a special case of our model when the credit portfolio consists of only one defaultable bond. More precisely, this model is a special case of our model in the doubly stochastic setting with no contagion, i.e.  $c(\cdot, T_k; y) = 0$ , for all  $T_k$ . Note that we can suppress  $x$  from the notation since in this case  $\mathcal{I} = \{0\}$  and

$$\mathbf{1}_{\{A_t \leq 0\}} = \mathbf{1}_{\{\tau > t\}},$$

where  $\tau$  is the default time of the considered defaultable bond.

The doubly stochastic setting means that the filtration  $\mathbb{G}$  is given as  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ , where  $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T^*}$  is the *background filtration* (or the *reference filtration*) and the filtration  $\mathbb{H} := (\mathcal{H}_t)$  is generated by the default time, i.e.  $\mathcal{H}_t := \sigma(\mathbf{1}_{\{\tau \leq s\}}; 0 \leq s \leq t)$ . Moreover, the default time  $\tau$  is modeled as the first jump of the Cox process with hazard process denoted by  $\Gamma$ , i.e.  $\Gamma$  is an  $\mathbb{F}$ -adapted, right-continuous, increasing process such that  $\Gamma_0 = 0$  and for every  $t \leq T^*$

$$\mathbb{Q}_{T_n}(\tau > t | \mathcal{F}_t) = e^{-\Gamma_t}.$$

Let us assume that the hazard process has an intensity, i.e.  $\Gamma_t = \int_0^t \lambda_s ds$ , for some non-negative, integrable  $\mathbb{F}$ -adapted process  $\lambda$ . In the doubly stochastic setting,  $\lambda$  remains the  $\mathbb{F}$ -intensity process of  $\tau$  under *all* forward measures

$\mathbb{Q}_{T_k}$  (cf. Lemma 2 in Eberlein, Kluge, and Schönbucher (2006)). In Eberlein, Kluge, and Schönbucher (2006) the pre-default value  $\bar{B}(\cdot, T_k)$  of the defaultable bond is specified as follows

$$\frac{\bar{B}(t, T_k)}{B(t, T_k)} = \prod_{i=0}^{k-1} \frac{1}{1 + \delta_i h(t, T_i)} e^{\Gamma_t}, \quad (37)$$

where  $B(\cdot, T_k)$  is the default-free bond price process and where

$$h(t, T_k) = h(0, T_k) \exp \left( \int_0^t b^H(s, T_k) ds + \int_0^t \sqrt{c_s} \tilde{\gamma}(s, T_k) d\tilde{X}_s^{T_{k+1}} \right);$$

$\tilde{X}^{T_{k+1}}$  being the  $d$ -dimensional special semimartingale obtained from the time-inhomogeneous Lévy process  $\tilde{X}$  by changing from  $\mathbb{Q}_{T_n}$  to the forward measure  $\mathbb{Q}_{T_{k+1}}$ . By assumption,  $\tilde{X}$  is  $\mathbb{F}$ -adapted and so is  $h(\cdot, T_k)$ .

The no-arbitrage condition of Eberlein, Kluge, and Schönbucher (2006) is obtained as a special case of Theorem 5.8, stated in the following corollary.

**Corollary 5.10.** *The forward defaultable bond price process  $\frac{\bar{B}(\cdot, T_k)}{B(\cdot, T_k)} \mathbf{1}_{\{\tau > t\}}$  with specification (37) is a  $(\mathbb{G}, \mathbb{Q}_{T_k})$ -local martingale if and only if*

$$D(t, T_k) = 0,$$

for almost all  $t \in [0, T_k]$ , or equivalently, if and only if the process

$$\prod_{i=0}^{k-1} y(t, T_i) = \prod_{i=0}^{k-1} \frac{1}{1 + \delta_i h(t, T_i)}, \quad t \leq T_k,$$

is an  $(\mathbb{F}, \mathbb{Q}_{T_k})$ -local martingale.

*Proof:* The claim follows directly from Theorem 5.8 by inserting  $b^P(t, T_k) = \lambda_t$  (cf. equation (37)) and noting that the covariation part vanishes if the contagion parameter  $c = 0$ .

For the second equivalence note that, by Lemma 5.6,  $D(\cdot, T_k)$  is exactly the drift term in the dynamics of  $\prod_{i=0}^{k-1} y(\cdot, T_i)$ .  $\square$

## 6. PRICING OF STCDOs

In this section we show how derivative valuation can be facilitated by using appropriate defaultable forward measures. We illustrate this procedure by considering a standard instrument for investing in a credit pool, a so-called single tranche CDO. A *single tranche CDO* (STCDO) is specified by:

- a collection of future dates  $T_1 < T_2 < \dots < T_m$ ,
- *lower* and *upper detachment points*  $x_1 < x_2$  in  $[0, 1]$
- a fixed spread  $S$ .

The STCDO offers premium in exchanges for payments at defaults: the *premium leg* (received by the investor) consists of a series of payments equal to

$$S[(x_2 - A_{T_k})^+ - (x_1 - A_{T_k})^+], \quad (38)$$

received at  $T_k$ ,  $k = 1, \dots, m - 1$ . Letting

$$f(x) := (x_2 - x)^+ - (x_1 - x)^+ = \int_{x_1}^{x_2} \mathbf{1}_{\{x \leq y\}} dy,$$

we have that (38) =  $Sf(A_{T_k})$ .

The *default leg* (paid by the investor) consists of a series of payments at times  $T_{k+1}$ ,  $k = 1, \dots, m - 1$ , given by

$$f(A_{T_k}) - f(A_{T_{k+1}}). \quad (39)$$

This payment is non-zero only if  $\Delta A_t \neq 0$  for some  $t \in (T_k, T_{k+1}]$ . In the literature alternative payment schemes can be found as well (see Filipović, Overbeck, and Schmidt (2009), for example). For simplicity, we neglect the interest of the default payments from a default in  $(T_k, T_{k+1}]$  until  $T_{k+1}$ .

We have

$$(39) = \int_{x_1}^{x_2} \left[ \mathbf{1}_{\{A_{T_k} \leq y\}} - \mathbf{1}_{\{A_{T_{k+1}} \leq y\}} \right] dy = \int_{x_1}^{x_2} \mathbf{1}_{\{A_{T_k} \leq y, A_{T_{k+1}} > y\}} dy.$$

To calculate the time- $t$  value of the payment  $\mathbf{1}_{\{A_{T_k} \leq x, A_{T_{k+1}} > x\}}$  done at  $T_{k+1}$  it is convenient to change from the measure  $\mathbb{Q}_{T_{k+1}}$  to a new measure defined below. We make the following assumption:

**(A9)** For every  $(T_k, x) \in \mathcal{T} \times \mathcal{I}$  the forward bond price process  $F(\cdot, T_k, x)$  is a *true*  $\mathbb{Q}_{T_k}$ -martingale.

**Remark 6.1.** If the drift condition from Theorem 5.8 is satisfied, then  $F(\cdot, T_k, x)$  is a local  $\mathbb{Q}_{T_k}$ -martingale. Moreover, if  $|b^P(s, T_k, x)| \leq B$ , for every  $s \leq T_k$ , with some random variable  $B$  such that  $\mathbb{E}_{\mathbb{Q}_{T_k}}(e^{T_k B}) < \infty$ , then  $F(\cdot, T_k, x)$  is a true martingale. This follows from Proposition I.1.47(c) in Jacod and Shiryaev (2003) as  $F(\cdot, T_k, x)$  is with this condition a process of class (D) (recall (33) and note that  $\prod_{i=0}^{k-1} y(\cdot, T_i, x)$  is bounded by 0 and 1).

Let  $x \in [0, 1]$  and  $k \in \{1, \dots, m - 1\}$ . We define the  $(T_{k+1}, x)$ -forward measure  $\mathbb{Q}_{T_{k+1}, x}$  on  $(\Omega, \mathcal{G}_{T_{k+1}})$  by its Radon–Nikodym derivative

$$\frac{d\mathbb{Q}_{T_{k+1}, x}}{d\mathbb{Q}_{T_{k+1}}} := \frac{F(T_k, T_{k+1}, x)}{\mathbb{E}_{\mathbb{Q}_{T_{k+1}}}[F(T_k, T_{k+1}, x)]} = \frac{F(T_k, T_{k+1}, x)}{F(0, T_{k+1}, x)},$$

where the last equality follows under **(A9)**. The density process is then

$$\left. \frac{d\mathbb{Q}_{T_{k+1}, x}}{d\mathbb{Q}_{T_{k+1}}} \right|_{\mathcal{G}_t} = \frac{F(t, T_{k+1}, x)}{F(0, T_{k+1}, x)}.$$

Note that  $\mathbb{Q}_{T_{k+1}, x}$  is not equivalent to  $\mathbb{Q}_{T_{k+1}}$  if  $\mathbb{Q}_{T_{k+1}}(A_{T_k} > x) > 0$ .

Let us denote by  $e(t, T_{k+1}, x)$  the value at time  $t$  of the payment given by  $\mathbf{1}_{\{A_{T_k} \leq x, A_{T_{k+1}} > x\}}$  at the tenor date  $T_{k+1}$ .

**Lemma 6.2.** *Assume (A9). Let  $x \in \mathcal{I}$  and  $k \in \{1, \dots, m - 1\}$ . Then, for every  $t \leq T_k$ ,*

$$e(t, T_{k+1}, x) = \delta_k P(t, T_{k+1}, x) \mathbb{E}_{\mathbb{Q}_{T_{k+1}, x}}(h(T_k, T_k, x) | \mathcal{G}_t).$$

*Proof:* The price at time  $t$  of a contingent claim with payoff

$$e(T_{k+1}, T_{k+1}, x) = \mathbf{1}_{\{A_{T_k} \leq x\}} - \mathbf{1}_{\{A_{T_{k+1}} \leq x\}}$$

at  $T_{k+1}$  equals

$$e(t, T_{k+1}, x) = P(t, T_{k+1}) \mathbb{E}_{\mathbb{Q}_{T_{k+1}}} \left( \mathbf{1}_{\{A_{T_k} \leq x\}} - \mathbf{1}_{\{A_{T_{k+1}} \leq x\}} \middle| \mathcal{G}_t \right). \quad (40)$$

Regarding the second term, observe that

$$P(t, T_{k+1}) \mathbb{E}_{\mathbb{Q}_{T_{k+1}}} \left( \mathbf{1}_{\{A_{T_{k+1}} \leq x\}} \middle| \mathcal{G}_t \right) = P(t, T_{k+1}, x) \quad (41)$$

by (A9). For the first term, (7) with  $t = T_k$  yields on  $\{A_{T_k} \leq x\}$

$$1 + \delta_k h(T_k, T_k, x) = \left( \frac{p(T_k, T_{k+1}, x)}{P(T_k, T_{k+1})} \right)^{-1}$$

and thus,

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}_{T_{k+1}}} \left( \mathbf{1}_{\{A_{T_k} \leq x\}} \middle| \mathcal{G}_t \right) \\ &= \mathbb{E}_{\mathbb{Q}_{T_{k+1}}} \left( \mathbf{1}_{\{A_{T_k} \leq x\}} \frac{p(T_k, T_{k+1}, x)}{P(T_k, T_{k+1})} (1 + \delta_k h(T_k, T_k, x)) \middle| \mathcal{G}_t \right) \\ &= \mathbb{E}_{\mathbb{Q}_{T_{k+1}}} \left( F(T_k, T_{k+1}, x) (1 + \delta_k h(T_k, T_k, x)) \middle| \mathcal{G}_t \right). \end{aligned}$$

Changing to the measure to  $\mathbb{Q}_{T_{k+1}, x}$  yields

$$\mathbb{E}_{\mathbb{Q}_{T_{k+1}}} \left( \mathbf{1}_{\{A_{T_k} \leq x\}} \middle| \mathcal{G}_t \right) = F(t, T_{k+1}, x) \mathbb{E}_{\mathbb{Q}_{T_{k+1}, x}} \left( 1 + \delta_k h(T_k, T_k, x) \middle| \mathcal{G}_t \right).$$

Therefore,

$$\begin{aligned} e(t, T_{k+1}, x) &= P(t, T_{k+1}) \frac{P(t, T_{k+1}, x)}{P(t, T_{k+1})} \left( 1 + \delta_k \mathbb{E}_{\mathbb{Q}_{T_{k+1}, x}} (h(T_k, T_k, x) \middle| \mathcal{G}_t) \right) \\ &\quad - P(t, T_{k+1}, x) \\ &= \delta_k P(t, T_{k+1}, x) \mathbb{E}_{\mathbb{Q}_{T_{k+1}, x}} (h(T_k, T_k, x) \middle| \mathcal{G}_t) \end{aligned}$$

and the lemma is proved. □

**Proposition 6.3.** *Assume (A9). Then the value of the STCDO at time  $t \leq T_1$  is*

$$\pi^{STCDO}(t, S) = \int_{x_1}^{x_2} \left( S \sum_{k=1}^{m-1} P(t, T_k, y) - \sum_{k=1}^{m-1} e(t, T_{k+1}, y) \right) dy. \quad (42)$$

Recall that we have assumed that the interest  $Sf(A_{T_k})$  is paid at times  $T_1, \dots, T_{m-1}$ , while the default payments are due at time points  $T_2, \dots, T_m$ .

*Proof:* The value of the premium leg at time  $t \leq T_1$  equals

$$\begin{aligned} \sum_{k=1}^{m-1} P(t, T_k) \mathbb{E}_{\mathbb{Q}_{T_k}} (Sf(A_{T_k}) | \mathcal{G}_t) &= \sum_{k=1}^{m-1} SP(t, T_k) \int_{x_1}^{x_2} \mathbb{E}_{\mathbb{Q}_{T_k}} (\mathbf{1}_{\{A_{T_k} \leq y\}} | \mathcal{G}_t) dy \\ &= S \sum_{k=1}^{m-1} \int_{x_1}^{x_2} P(t, T_k, y) dy, \end{aligned}$$

where we have used (41).

On the other side, the default payment at time  $T_{k+1}$  is given by  $f(A_{T_k}) - f(A_{T_{k+1}})$ . Its value at time  $t$  is equal to

$$\begin{aligned} P(t, T_{k+1}) \mathbb{E}_{\mathbb{Q}_{T_{k+1}}} (f(A_{T_k}) - f(A_{T_{k+1}}) | \mathcal{G}_t) & \quad (43) \\ &= P(t, T_{k+1}) \mathbb{E}_{\mathbb{Q}_{T_{k+1}}} \left( \int_{x_1}^{x_2} \mathbf{1}_{\{A_{T_k} \leq y, A_{T_{k+1}} > y\}} dy \middle| \mathcal{G}_t \right) \\ &= \int_{x_1}^{x_2} P(t, T_{k+1}) \mathbb{E}_{\mathbb{Q}_{T_{k+1}}} (\mathbf{1}_{\{A_{T_k} \leq y, A_{T_{k+1}} > y\}} | \mathcal{G}_t) dy \\ &= \int_{x_1}^{x_2} e(t, T_{k+1}, y) dy. \end{aligned}$$

Hence, the value of the default leg at time  $t \leq T_1$  is given by

$$\sum_{k=1}^{m-1} \int_{x_1}^{x_2} e(t, T_{k+1}, y) dy.$$

Finally, the value of the STCDO is the difference of these two values and thus we obtain (42).  $\square$

The forward STCDO spread  $S_t^*$  at time  $t \leq T_1$  is the spread which makes the value of the STCDO equal to 0, i.e. one has to solve  $\pi^{STCDO}(t, S) = 0$ . The previous proposition yields

$$S_t^* = \frac{\int_{x_1}^{x_2} \sum_{k=1}^{m-1} e(t, T_{k+1}, y) dy}{\int_{x_1}^{x_2} \sum_{k=1}^{m-1} P(t, T_k, y) dy}. \quad (44)$$

**6.1. The relation of STCDOs and Libor rates.** In this section we describe how to extract the Libor rates from the observed STCDO prices. We impose in addition a certain form of independence between the loss process and the risk-free market by assuming that

$$P(t, T_k, x) = P(t, T_k) \mathbb{E}_{\mathbb{Q}_{T_{k+1}}} (\mathbf{1}_{\{A_{T_k} \leq x\}} | \mathcal{G}_t), \quad (45)$$

for all  $(T_k, x) \in \mathcal{T} \times \mathcal{I}$  and  $0 \leq t \leq T_k$ . Note that (45) is equivalent to

$$\mathbb{E}_{\mathbb{Q}_{T_{k+1}}} (\mathbf{1}_{\{A_{T_k} \leq x\}} | \mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}_{T_k}} (\mathbf{1}_{\{A_{T_k} \leq x\}} | \mathcal{G}_t).$$

**Lemma 6.4.** *Assume (A9) and (45). Then*

$$e(t, T_{k+1}, x) = \frac{P(t, T_{k+1})}{P(t, T_k)} P(t, T_k, x) - P(t, T_{k+1}, x)$$

for  $k = 1, \dots, m-1$  and  $x \in \mathcal{I}$ .

*Proof:* The claim follows from (40) by inserting (41) and (45). □

Assume now that risk-free Libor rates and STCDO prices at time  $t$  are observed for maturities  $T_1, \dots, T_m$  and levels  $(x_{i-1}, x_i)$  with  $i = 1, \dots, n$  where  $x_0 = 0$  and  $x_n = 1$ . We assume throughout that (45) is in force.

- **Step 1** STCDOs with maturity  $T_1$  do not have any future spread payments. Hence, the (observed) values can be expressed, by (40), in the form

$$\int_{x_{i-1}}^{x_i} (P(t, T_1) - P(t, T_1, y)) dy.$$

This allows directly to compute

$$P(t, T_1, x_{i-1}, x_i) := \int_{x_{i-1}}^{x_i} P(t, T_1, y) dy$$

for all  $i = 1, \dots, n$ .

- **j**  $\rightarrow$  **j+1** Assume that the values  $P(t, T_k, x_{i-1}, x_i)$  are given for all  $i = 1, \dots, n$  and  $k = 1, \dots, j \leq m-1$ . A STCDO with maturity  $T_{j+1}$  satisfies according to (44) and Lemma 6.4

$$S(t, T_{j+1}, x_{i-1}, x_i) = \frac{\sum_{k=1}^j \left( \frac{P(t, T_{k+1})}{P(t, T_k)} P(t, T_k, x_{i-1}, x_i) - P(t, T_{k+1}, x_{i-1}, x_i) \right)}{\sum_{k=1}^j P(t, T_k, x_{i-1}, x_i)}.$$

The denominator is given as a sum of quantities which have been computed in the previous  $j$  steps. The numerator equals

$$\begin{aligned} & \frac{P(t, T_{j+1})}{P(t, T_j)} P(t, T_j, x_{i-1}, x_i) - P(t, T_{j+1}, x_{i-1}, x_i) \\ & + \sum_{k=1}^{j-1} \left( \frac{P(t, T_{k+1})}{P(t, T_k)} P(t, T_k, x_{i-1}, x_i) - P(t, T_{k+1}, x_{i-1}, x_i) \right) \end{aligned}$$

and therefore  $P(t, T_{j+1}, x_{i-1}, x_i)$  equals

$$\begin{aligned} & \frac{P(t, T_{j+1})}{P(t, T_j)} P(t, T_j, x_{i-1}, x_i) \tag{46} \\ & + \sum_{k=1}^{j-1} \left( \frac{P(t, T_{k+1})}{P(t, T_k)} P(t, T_k, x_{i-1}, x_i) - P(t, T_{k+1}, x_{i-1}, x_i) \right) \\ & - S(t, T_{j+1}, x_{i-1}, x_i) \sum_{k=1}^j P(t, T_k, x_{i-1}, x_i) \end{aligned}$$

and this step is completed.

In this way one is able to extract  $(T_k, x)$ -rates from STCDO prices. A detailed empirical study, however, is beyond the scope of this paper and will be presented elsewhere.

**Remark 6.5.** If we assume in addition zero risk-free interest rates, we obtain the following formula for the default leg of the STCDO:

$$\sum_{k=1}^{m-1} \int_{x_{i-1}}^{x_i} e(t, T_{k+1}, y) dy = \int_{x_{i-1}}^{x_i} (P(t, T_1, y) - P(t, T_m, y)) dy.$$

Moreover, in the above algorithm (46) simplifies to

$$\begin{aligned} P(t, T_{j+1}, x_{i-1}, x_i) &= P(t, T_1, x_{i-1}, x_i) \\ &\quad - S(t, T_{j+1}, x_{i-1}, x_i) \sum_{k=1}^j P(t, T_k, x_{i-1}, x_i). \end{aligned}$$

#### APPENDIX A. AUXILIARY RESULTS AND PROOFS

Assume that a complete stochastic basis  $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{0 \leq t \leq T^*}, \mathbb{Q}_{T^*})$  is given and all processes we consider are defined on this basis.

**Lemma A.1.** *Let  $U$  be a real-valued special semimartingale such that  $U_0 = 0$  with canonical representation given by*

$$U = U^c + x * (\mu^U - \nu^U) + A,$$

where  $U^c$  is the continuous martingale part,  $\mu^U$  is the random measure of jumps of  $U$  with compensator  $\nu^U$  and  $A$  is the predictable, finite-variation process.

Set  $V_t := V_0 \exp U_t$  and let  $\delta > 0$  be a real number. Define a semimartingale  $Z$  by setting

$$Z_t := 1 + \delta V_t = 1 + \delta V_0 \exp U_t, \quad t \geq 0.$$

Then

$$\begin{aligned} Z_t &= Z_0 \exp \left( (v_- \cdot U^c)_t + \ln(1 + v_-(e^x - 1)) * (\mu^U - \nu^U)_t + (v_- \cdot A)_t \right. \\ &\quad \left. + \left( \frac{1}{2} (v_- - (v_-)^2) \cdot \langle U^c, U^c \rangle \right)_t + [\ln(1 + v_-(e^x - 1)) - v_- x] * \nu_t^U \right), \end{aligned}$$

where

$$v_t := \frac{\delta V_t}{1 + \delta V_t}, \quad t \geq 0.$$

*Proof:* To prove the lemma we make use of Lemma 2.6 in Kallsen and Shiryaev (2002), which provides a link between the stochastic and the ordinary exponential of a semimartingale.

First, we have

$$\begin{aligned} V_t &= V_0 \mathcal{E} \left( U + \frac{1}{2} \langle U^c, U^c \rangle + (e^x - 1 - x) * \mu^U \right)_t \\ &= V_0 \mathcal{E} \left( U^c + (e^x - 1) * (\mu^U - \nu^U) + A + \frac{1}{2} \langle U^c, U^c \rangle \right. \\ &\quad \left. + (e^x - 1 - x) * \nu^U \right)_t. \end{aligned} \tag{47}$$

Since  $Z_t = 1 + \delta V_t$ , it follows that  $dZ_t = \delta dV_t$  and hence

$$\frac{dZ_t}{Z_{t-}} = \frac{\delta dV_t}{\delta V_{t-}} \frac{\delta V_{t-}}{Z_{t-}} = \frac{dV_t}{V_{t-}} \frac{\delta V_{t-}}{1 + \delta V_{t-}} = \frac{dV_t}{V_{t-}} v_{t-}, \quad (48)$$

where we denote

$$v_t := \frac{\delta V_t}{1 + \delta V_t}, \quad t \geq 0.$$

Combining (47) with (48), we obtain

$$\begin{aligned} Z_t &= Z_0 \mathcal{E} \left( v_- \cdot \left( U^c + (e^x - 1) * (\mu^U - \nu^U) + A + \frac{1}{2} \langle U^c, U^c \rangle \right. \right. \\ &\quad \left. \left. + (e^x - 1 - x) * \nu^U \right) \right)_t. \end{aligned}$$

Alternatively, an application of Itô's formula to the function  $f(x) = 1 + \delta V_0 e^x$  yields the same result.

Now we apply again Lemma 2.6 in Kallsen and Shiryaev (2002) and obtain

$$\begin{aligned} Z_t &= Z_0 \exp \left( (v_- \cdot U^c)_t + v_-(e^x - 1) * (\mu^U - \nu^U)_t + (v_- \cdot A)_t \right. \\ &\quad \left. + \frac{1}{2} v_- \cdot \langle U^c, U^c \rangle_t + v_-(e^x - 1 - x) * \nu_t^U \right. \\ &\quad \left. - \frac{1}{2} (v_-)^2 \cdot \langle U^c, U^c \rangle_t \right. \\ &\quad \left. + [\ln(1 + v_-(e^x - 1)) - v_-(e^x - 1)] * \mu_t^U \right) \\ &= Z_0 \exp \left( (v_- \cdot U^c)_t + \ln(1 + v_-(e^x - 1)) * (\mu^U - \nu^U)_t + (v_- \cdot A)_t \right. \\ &\quad \left. + \frac{1}{2} (v_- - (v_-)^2) \cdot \langle U^c, U^c \rangle_t + [\ln(1 + v_-(e^x - 1)) - v_- x] * \nu_t^U \right). \end{aligned}$$

□

**Lemma A.2.** Denote by  $W$  a standard  $d$ -dimensional Brownian motion and by  $\mu$  the random measure of jumps of some semimartingale  $X$  with compensator  $\nu(ds, dx) = F_s(dx)ds$ . Fix an  $m \in \mathbb{N}$  and let  $V^k$ ,  $k = 1, \dots, m$ , be special semimartingales given by

$$V_t^k = V_0^k \exp \left( \int_0^t b^k(s) ds + \int_0^t \sigma^k(s) dW_s + \int_0^t \int_{\mathbb{R}^d} S^k(s, x) (\mu - \nu)(ds, dx) \right),$$

for some  $\sigma^k \in L(W)$  and  $S^k \in G_{\text{loc}}(\mu)$ . Further let  $\delta_k > 0$  be real numbers, for  $k = 1, \dots, m$ . Then

$$\begin{aligned} \prod_{k=1}^m \frac{1}{1 + \delta_k V_t^k} &= \left( \prod_{k=1}^m \frac{1}{1 + \delta_k V_0^k} \right) \exp \left( - \int_0^t \sum_{k=1}^m a^k(s) ds - \int_0^t \sum_{k=1}^m v_{s-}^k \sigma^k(s) dW_s \right. \\ &\quad \left. - \int_0^t \int_{\mathbb{R}^d} \ln \prod_{k=1}^m \left( 1 + v_{s-}^k (e^{S^k(s, x)} - 1) \right) (\mu - \nu)(ds, dx) \right), \end{aligned}$$

where

$$v_s^k := \frac{\delta_k V_s^k}{1 + \delta_k V_s^k}$$

and

$$\begin{aligned} a^k(s) &:= v_{s-}^k b^k(s) + \frac{1}{2}(v_{s-}^k - (v_{s-}^k)^2) \|\sigma^k(s)\|^2 \\ &\quad + \int_{\mathbb{R}^d} \left( \ln \left( 1 + v_{s-}^k (e^{S^k(s,x)} - 1) \right) - v_{s-}^k S^k(s,x) \right) F_s(dx). \end{aligned}$$

*Proof:* To prove the lemma, it suffices to express each process  $1 + \delta_k V^k$ ,  $k = 1, \dots, m$ , as an ordinary exponential. For a fixed  $k$ , we apply Lemma A.1 to the semimartingale  $V^k = V_0^k \exp U^k$  with

$$U_t^k = \int_0^t b^k(s) ds + \int_0^t \sigma^k(s) dW_s + \int_0^t \int_{\mathbb{R}^d} S^k(s,x) (\mu - \nu)(ds, dx)$$

and obtain

$$\begin{aligned} 1 + \delta_k V_t^k &= (1 + \delta_k V_0^k) \exp \left( \int_0^t a^k(s) ds + \int_0^t v_{s-}^k \sigma^k(s) dW_s \right. \\ &\quad \left. + \int_0^t \int_{\mathbb{R}^d} \ln \left( 1 + v_{s-}^k (e^{S^k(s,x)} - 1) \right) (\mu - \nu)(ds, dx) \right), \end{aligned}$$

where we denote

$$v_s^k := \frac{\delta_k V_s^k}{1 + \delta_k V_s^k}$$

and

$$\begin{aligned} a^k(s) &:= v_{s-}^k b^k(s) + \frac{1}{2}(v_{s-}^k - (v_{s-}^k)^2) \|\sigma^k(s)\|^2 \\ &\quad + \int_{\mathbb{R}^d} \left( \ln \left( 1 + v_{s-}^k (e^{S^k(s,x)} - 1) \right) - v_{s-}^k S^k(s,x) \right) F_s(dx). \end{aligned}$$

Now we simply calculate the product, which yields

$$\begin{aligned} \prod_{k=1}^m \frac{1}{1 + \delta_k V_t^k} &= \left( \prod_{k=1}^m \frac{1}{1 + \delta_k V_0^k} \right) \exp \left( - \int_0^t \sum_{k=1}^m a^k(s) ds - \int_0^t \sum_{k=1}^m v_{s-}^k \sigma^k(s) dW_s \right. \\ &\quad \left. - \int_0^t \int_{\mathbb{R}^d} \ln \prod_{k=1}^m \left( 1 + v_{s-}^k (e^{S^k(s,x)} - 1) \right) (\mu - \nu)(ds, dx) \right). \end{aligned}$$

□

Let us prove Lemma 5.6 and Lemma 5.7.

*Proof of Lemma 5.6:* For every  $i = 1, \dots, k-1$  we begin by expressing the dynamics of  $h(\cdot, T_i, x)$  under the measure  $\mathbb{Q}_{T_k}$ . Recall that

$$W_s^{T_{i+1}} = W_s^{T_k} - \int_0^s \sqrt{c_u} \left( \sum_{j=i+1}^{k-1} \alpha(u, T_j) \right) du$$

with  $\alpha(u, T_j)$  defined in (21) and

$$\nu^{T_{i+1}}(ds, dy) = \left( \prod_{j=i+1}^{k-1} \beta(s, T_j, y) \right) F_s^{T_k}(dy) ds,$$

with  $\beta(s, T_j, y)$  defined in (23). Therefore, for each  $i = 1, \dots, k-1$  and under the measure  $\mathbb{Q}_{T_k}$ , **(A7')** becomes

$$\begin{aligned} h(t, T_i, x) = & h(0, T_i, x) \exp \left( \int_0^t b(s, T_i, T_k, x) ds + \int_0^t \sqrt{c_s} \gamma(s, T_i, x) dW_s^{T_k} \right. \\ & \left. + \int_0^t \int_{\mathbb{R}^{d+1}} \varrho(s, T_i, x; y) (\mu - \nu^{T_k})(ds, dy) \right), \end{aligned} \quad (49)$$

where

$$\begin{aligned} b(s, T_i, T_k, x) := & b(s, T_i, x) - \left\langle \gamma(s, T_i, x), c_s \sum_{j=i+1}^{k-1} \alpha(s, T_j) \right\rangle \\ & - \int_{\mathbb{R}^{d+1}} \varrho(s, T_i, x; y) \left( \prod_{j=i+1}^{k-1} \beta(s, T_j, y) - 1 \right) F_s^{T_k}(dy). \end{aligned} \quad (50)$$

Applying Lemma A.2 to the special semimartingales  $h(\cdot, T_i, x)$  given by (49) (and adapting the notation correspondingly), we obtain

$$\begin{aligned} \prod_{i=1}^{k-1} y(t, T_i, x) = & \left( \prod_{i=1}^{k-1} y(0, T_i, x) \right) \exp \left( - \int_0^t \sum_{i=1}^{k-1} a(s, T_i, T_k, x) ds \right. \\ & - \int_0^t \sum_{i=1}^{k-1} g(s-, T_i, x) \sqrt{c_s} \gamma(s, T_i, x) dW_s^{T_k} \\ & - \int_0^t \int_{\mathbb{R}^{d+1}} \ln \prod_{i=1}^{k-1} \left( 1 + g(s-, T_i, x) (e^{\varrho(s, T_i, x; y)} - 1) \right) \\ & \left. \times (\mu - \nu^{T_k})(ds, dy) \right), \end{aligned}$$

with  $g(s, T_i, x)$  defined in (34) and

$$\begin{aligned} a(s, T_i, T_k, x) := & g(s-, T_i, x) b(s, T_i, T_k, x) \\ & + \frac{1}{2} (g(s-, T_i, x) - g(s-, T_i, x)^2) \|\sqrt{c_s} \gamma(s, T_i, x)\|^2 \\ & + \int_{\mathbb{R}^{d+1}} \left( \ln \left( 1 + g(s-, T_i, x) (e^{\varrho(s, T_i, x; y)} - 1) \right) \right. \\ & \left. - g(s-, T_i, x) \varrho(s, T_i, x; y) \right) F_s^{T_k}(dy). \end{aligned} \quad (51)$$

Finally, using the connection between the ordinary and the stochastic exponential given in (Kallsen and Shiryaev 2002, Lemma 2.6), we deduce

$$\begin{aligned} d \left( \prod_{i=1}^{k-1} y(t, T_i, x) \right) &= \left( \prod_{i=1}^{k-1} y(t-, T_i, x) \right) \left( D(t, T_k, x) dt \right. \\ &\quad - \sum_{i=1}^{k-1} g(t-, T_i, x) \sqrt{c_t} \gamma(t, T_i, x) dW_t^{T_k} \\ &\quad \left. + \int_{\mathbb{R}^{d+1}} \left( \prod_{i=1}^{k-1} \left( 1 + g(t-, T_i, x) (e^{\varrho(t, T_i, x; y)} - 1) \right)^{-1} - 1 \right) \right. \\ &\quad \left. \times (\mu - \nu^{T_k})(dt, dy) \right), \end{aligned}$$

where  $D(t, T_k, x)$  is defined in (35) and obtained by plugging in the expressions for  $b(t, T_i, T_k, x)$  and  $a(t, T_i, T_k, x)$  given in (50) and (51) respectively.

Recall that for  $i = 0$ , we have  $T_0 = 0$  and thus,  $y(t, T_0, x) = y(T_0, T_0, x) = \frac{1}{1 + \delta_0 H(0, 0, x)} = \frac{P(0, T_1, x)}{P(0, T_1)}$ , for every  $t$ . We multiply both sides in the above equation by this initial value and the lemma is proved.  $\square$

*Proof of Lemma 5.7:* Recall that by **(A8)**

$$\frac{p(t, T_k, x)}{P(t, T_k)} = \tilde{Y}(t, T_k, x) e^{\int_0^t b^P(s, T_k, x) ds},$$

where  $\tilde{Y}(t, T_k, x) = \prod_{i=0}^{k-1} y(t, T_i, x)$ . Apply the integration by parts to obtain

$$\begin{aligned} d \left( \frac{p(t, T_k, x)}{P(t, T_k)} \right) &= \tilde{Y}(t-, T_k, x) d \left( e^{\int_0^t b^P(s, T_k, x) ds} \right) \\ &\quad + e^{\int_0^t b^P(s, T_k, x) ds} d\tilde{Y}(t, T_k, x) \\ &= \tilde{Y}(t-, T_k, x) e^{\int_0^t b^P(s, T_k, x) ds} b^P(t, T_k, x) dt \\ &\quad + e^{\int_0^t b^P(s, T_k, x) ds} d\tilde{Y}(t, T_k, x). \end{aligned}$$

Note that the covariation part in the above calculation vanishes since the process  $e^{\int_0^t b^P(s, T_k, x) ds}$  is continuous and has finite variation. Finally, inserting the expression for  $d\tilde{Y}(t, T_k, x)$  from Lemma 5.6 delivers the desired result.  $\square$

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