Optimal Stopping beyond the Free Boundary Approach

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June 16, 2011
A Parking Problem

\[ S_0 = -Q \]
\[ S_n = \sum_{i=1}^{n} X_i - Q \]

\( X_i \) i.i.d. geometric (\( p \))
\( p \): probability of empty spot

Park as near as possible at ’’0’’!

Find a stopping time \( T^* \) of \( S_i, i \geq 0 \) with

\[ E | S_{T^*} | = \min_T E | S_T | . \]

Solution: \( T^* = \min\{n \geq 1 | S_n > -s_0\} \)

with \( s_0 = \min\{s \in \mathbb{N} | 1 - 2(1 - p)^s > 0\} \)

Chow, Robbins, Siegmund (1971): Great Expectations, p. 45
Generalized Parking Problem (GPP)

Let $g$ be a convex nonnegative function with a unique minimum at $x^* > 0$.
Assume $X_i$ i.i.d. with $EX_i > 0$,

$$S_n = \sum_{i=1}^{n} X_i, \quad S_0 = 0.$$ 

Find a stopping time $T^*$ with

$$E_g(S_{T^*}) = \min_T E_g(S_T).$$

Solution (Woodroofe, Lerche, Keener ’94):

$$T^* = \min\{n \geq 0 \mid S_n \geq a\}$$

with $a = \sup\{x \mid H^+ g(x) < g(x)\}$ where $H^+$ is the ladder-height distribution of $S_n$; $n \geq 1$ and $H^+ g(x) = \int g(x + y)H(dy)$.
Lorden’s Result on the SPRT

Let

\[ R(T) = P_0(T < \infty) + cIE_1T, \]

where \( I = E_1 \log \frac{dP_1}{dP_0}(X) \).

Let

\[ S_n = \log \frac{dP_1^n}{dP_0^n}. \]

Then by Wald’ identity

\[ R(T) = \int g(S_T) dP_1 \]

with \( g(x) = e^{-x} + cx \).

\( g \) is a nonnegative convex function with a unique minimum at \( \log \frac{1}{c} \).

Then \( T^* = \min\{n \geq 1 | S_n \geq \log(\frac{\kappa}{c})\} \)

where \( \kappa = \lim_{a \to \infty} E_1 \exp(-S_{\tau_a} - a) \)

and \( \tau_a = \min\{n \geq 1 | S_n \geq a\} \).

Lorden(AS 1977)
The Repeated Significance Test as Bayes Test (RST)

$W_t$, $t \geq 0$ Brownian motion with drift $\theta$

**Testing sequentially:** $H_0 : \theta < 0$ versus $H_1 : \theta > 0$

Prior: $G = N(\mu, r^{-1})$

$$R(T, \delta) = \int_{-\infty}^{0} \left( P_{\theta}\{\delta \text{ rejects } H_0\} + \frac{c}{2} \theta^2 E_{\theta} T \right) G(d\theta)$$

$$+ \int_{0}^{\infty} \left( P_{\theta}\{\delta \text{ rejects } H_1\} + \frac{c}{2} \theta^2 E_{\theta} T \right) G(d\theta)$$

Find $(T^*, \delta^*)$ with $R(T^*, \delta^*) = \min_{(T, \delta)} R(T, \delta)$.

$$\delta^* = \delta_{T^*}^* = 1_{\{W_{T^*+r\mu}>0\}} \quad T^* = ?$$

PNAS, 83 (1986)
Representation of the risk:

\[ R(T, \delta_T^*) = \int g \left( \frac{(W_T + r\mu)^2}{T + r} \right) dQ \]

with \( g(x) = \Phi(-\sqrt{x}) + cx/2 \), \( Q = \int P_\theta G(d\theta) \), \( G = N(0, r^{-1}) \).

\( g \) is convex with unique minimum \( x^* \) and

\[ R(T, \delta_T^*) = \int g \left( \frac{(W_T + r\mu)^2}{T + r} \right) dQ \geq g(x^*) \]

Let \( T^* = \min\{t > 0 \mid W_t^2 / (t + r) = x^* \} \).

Since \( Q\{T^* < \infty\} = 1 \) it follows \( R(T^*, \delta_T^{**}) = g(x^*) \) if \( r\mu^2 \leq x^* \).
Let $r \mu^2 > x^\ast$. We show that for any stopping time $T$ such that $Q(T < \infty) = 1$, it holds $R(T, \delta^\ast) \geq g(r \mu^2)$.

Consider the $Q$-Martingale

$$N_t = \frac{dP_0}{dQ} \bigg|_{\mathcal{F}_t} = \sqrt{\frac{t + r}{r}} \exp \left( - \frac{(W_t + r \mu)^2}{2(t + r)} + \frac{r \mu^2}{2} \right),$$

where $P_0$ is the measure of B.M. without drift.

Since $h(x) = 2 \log(1/x)$ is convex

$$h(N_t) \quad \text{and} \quad Z_t = \frac{(W_t + r \mu)^2}{t + r}$$

is a $Q$-submartingale.

Since $g$ is convex on $[0, \infty)$ and increasing on $[x^\ast, \infty)$ for any bounded stopping time $T$ it holds,

$$R(T, \sigma^\ast) = E_Q g(Z_T) \geq g(E_Q Z_t) \geq g(Z_0) = g(r \mu^2).$$
Disruption Problem

Shiryaev (1961) studied the following problem.

Observations: \( W_t = B_t + \theta(t - \tau)^+ \) with
\( B_t, \ t \geq 0 \) standard Brownian motion,
\( \theta > 0 \) fixed

Filtration: \( \mathcal{F}_t = \sigma(W_t; 0 \leq s \leq t) \)

Change-point: \( \tau \) random time, independent of \( B \)
with distribution \( \pi = p\delta_0 + (1 - p)F, \)
where \( F(t) = 1 - e^{-\lambda t} \)

Risk: \( R(T) = \mathbb{P}_\pi(T < \tau) + c\mathbb{E}_\pi(T - \tau)^+ \)
Find \( T^* \) with \( R(T^*) = \min_T R(T) \).
Let $\pi_t = P(\tau \leq t \mid \mathcal{F}_t)$. Then

$$
\pi_t = \frac{\varphi_t}{e^{-\lambda t} + \varphi_t}
$$

where

$$
\varphi_t = \frac{p}{1-p} L_t + \int_0^t \frac{L_t}{L_s} \lambda e^{-\lambda s} \, ds
$$

with

$$
L_t = \exp(\theta W_t - \theta^2 t/2).
$$
\( \pi_t \) is a diffusion with
\[
d\pi_t = \lambda(1 - \pi_t)dt + \theta \pi_t(1 - \pi_t)d\overline{W}_t \text{ with } \overline{W}_t \text{ is a standard Brownian motion. Itô's formula yields:}
\]
\[
dG(\pi_t) = G'(\pi_t)d\pi_t + \frac{1}{2} G''(\pi_t)(d\pi_t)^2
\]
\[
= G'(\pi_t) \left[ \lambda(1 - \pi_t)dt + \theta \pi_t(1 - \pi_t)d\overline{W}_t \right]
\]
\[
+ \frac{1}{2} G''(\pi_t)\theta^2 \pi_t^2 (1 - \pi_t)^2 dt
\]

If \( G \) satisfies the equation
\[
\frac{\theta^2}{2} x^2 (1 - x)^2 G''(x) + \lambda (1 - x) G'(x) = cx
\]
and behaves well at 0, then
\[
G(\pi_t) - G(\pi_0) = c \int_0^t \pi_s ds + c \int_0^t \theta \pi_s (1 - \pi_s) d\overline{W}_s
\]
\[
\Rightarrow E \left[ G(\pi_T) - G(\pi_0) \right] = c E \int_0^T \pi_s ds
\]
Then one obtains

\[ R(T) = P(T < \tau) + cE(T - \tau)^+ \]

\[ = E \left[ (1 - \pi_T) + c \int_0^T \pi_s ds \right] \]

with \( g(x) = (1 - x) + G(x) \)

\[ R(T) = \int g(\pi_T)dP - g(p) \]

g is convex with a unique minimum at \( p^* \).

**Theorem**

\[ T^* = \min\{t > 0 \mid \pi_t \geq p^*\} \text{ with } \pi_t = P(\tau \leq t \mid \mathcal{F}_t) \]

Here \( p^* \) is the unique solution in \((0, 1)\) of \( G'(p) = 1 \), where \( G \) is the (finite at 0) solution of

\[ \frac{\theta}{2} x^2 (1 - x^2) G''(x) + \lambda (1 - x) G'(x) = cx. \]
The Basic Idea: OS as GPP

Let \((Z_t, \mathcal{F}_t; t \geq 0)\) denote a continuous stochastic process on a probability space \((\Omega, \mathcal{F}, P)\).

Find a stopping time \(T^*\) with

\[
E_P(Z_{T^*} 1_{\{T^*<\infty\}}) = \max_T E_P(Z_T 1_{\{T<\infty\}}).
\]

**Idea:**

Find a process \((X_t, \mathcal{F}_t; t \geq 0)\), a nonnegative martingale \((M_t, \mathcal{F}_t; t \geq 0)\) with \(EM_0 = 1\) and a function \(g\) with unique maximum at \(x^*\) such that

\[Z_t = g(X_t)M_t.\]

Then

\[
EZ_T 1_{\{T<\infty\}} = E(g(X_T)M_T 1_{\{T<\infty\}}) \leq g(x^*)EM_T 1_{\{T<\infty\}} \leq g(x^*)
\]

With \(T^* = \min\{t \geq 0 \mid X_t = x^*\}\) the inequalities become equalities, if \(EM_{T^*} 1_{\{T^*<\infty\}} = 1\).
Perpetual American Put Option

Samuelson (1965), McKean (1965)

\[ X_t = \sigma B_t + \mu t, \quad t \geq 0 \]

Brownian Motion with drift \( \mu \) and variance \( \sigma^2 \).

Find a stopping time \( T^* \) which maximizes

\[
E_P e^{-rT} (K - e^{X_T})^+ 1_{\{T < \infty\}}.
\]

Idea:

Find \( M \) and \( g \) with

\[
E_P e^{-rT} (K - e^{X_T})^+ 1_{\{T < \infty\}} = E_P g(X_T) M_T 1_{\{T < \infty\}},
\]

where \( g \) has a unique maximum at \( x^* \).

Then

\[
T^* = \min\{ t \geq 0 \mid X_t = x^* \} \text{ if } E_P M_{T^*} = 1.
\]
Let \( f(x) = (K - e^x)^+ \). How to find \( M_t \)?

It holds for all \( \alpha \in \mathbb{R} \)

\[
f(X_T) e^{-rT} = f(X_T)(e^{X_T})^{-\alpha} (e^{X_T})^\alpha e^{-rT}.\]

Choose \( g(x) = f(x)e^{-\alpha x} \) and \( \alpha \) such that \( M_t = e^{\alpha X_t} e^{-rt} \) is a martingale.

This holds when

\[
M_t = \exp[\alpha(\sigma B_t) + t(\alpha \mu - r)]
\]

\[
= \exp[(\alpha \sigma)B_t - t(\alpha \sigma)^2/2].
\]

\( M_t \) is a positive marginale with \( M_0 = 1 \) iff \( (\alpha \sigma)^2/2 + \alpha \mu - r = 0 \)

\( \alpha^\pm = -\frac{\mu}{\sigma^2} \pm \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}} \) are the two possible values.
Then we have

\[ E_P e^{-rT} \left( K - e^{X_T} \right)^+ 1_{\{T<\infty\}} = E_Q g(X_T) 1_{\{T<\infty\}} \]

with \( g(x) = \frac{f(x)}{e^{\alpha-x}} \) and \( \frac{dQ_t}{dP_t} = M_t \).

Let \( K < 1 + (-\alpha)^{-1} \). Then \( g \) has a unique maximum at

\[ x^* = \log \frac{\alpha-K}{\alpha-1} < 0. \]

Under \( Q \), \( X \) is Brownian motion with drift

\[ \alpha^{-\sigma^2} + \mu = -\sigma^2 \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}} < 0. \]

This yields \( Q(T^* < \infty) = 1 \) for \( T^* = \inf\{t > 0 \mid X_t = x^*\} \).

Then

\[ \sup_T E_P \left( e^{-rT} (K - e^{X_T})^+ 1_{\{T<\infty\}} \right) = E_Q g(X_{T^*}) = C^* \]

with \( C^* = \frac{(K - e^{x^*})}{e^{\alpha-x^*}} \).
Put Options with strike $\max_{0 \leq s \leq t} X_s$

$X_t = \exp (\sigma B_t + (\mu - \sigma^2/2) t)$, $t > 0$ geometric Brownian motion with

$\sigma > 0$, $\mu \in \mathbb{R}$, $S_t = \max_{0 \leq u \leq t} X_u$.

Find a stopping time $T^*$ which maximizes

$$E \left( e^{-rT} (S_T - X_T) \right)$$

when $\mu < r$ and $r > 0$.

Let $\gamma_{1,2} = -\left( \frac{\mu}{\sigma^2} - \frac{1}{2} \right) \mp \sqrt{\frac{2r}{\sigma^2} + \left( \frac{\mu}{\sigma^2} - \frac{1}{2} \right)^2}$ and $\alpha = \left( \frac{1 - 1/\gamma_1}{1 - 1/\gamma_2} \right)^{1/(\gamma_2 - \gamma_1)}$.

Let $h(x) = \frac{1}{\gamma_2 \alpha^{\gamma_1 - \gamma_1} \alpha^{\gamma_2}} \left( \gamma_2 (\alpha x)^{\gamma_1} - \gamma_1 (\alpha x)^{\gamma_2} \right)$.

Then $M_t = e^{-rt} S_t h \left( \frac{X_t}{S_t} \right)$ is a nonnegative local martingale and

$$E e^{-rT} (S_T - X_T) = E \frac{1 - X_T/S_T}{h(X_T/S_T)} M_T \leq \frac{1 - x^*}{h(x^*)},$$

where $x^* = \arg\max\{(1 - x)/h(x) \mid x \in (0, 1)\}$.

Then $T^* = \inf\{t > 0 \mid X_t/S_t \leq x^*\}$ is optimal.
Optimality of Parabolic Boundaries

Let $X_t = B_t + x_0$, $t \geq 0$ with $B_t$ standard Brownian motion. For a measurable function $g$ find a stopping time $T$ that maximizes

$$E \left( (T + 1)^{-\beta} g \left( \frac{X_T}{\sqrt{T+1}} \right) \right). \quad (\text{Moerbeke (1974)})$$

Let $H(x) = \int_0^\infty e^{ux-u^2/2} u^{2\beta-1} du$ with $\beta > 0$

and assume that there exists a unique point $x^*$ with

$$\sup_{x \in \mathbb{R}} \frac{g(x)}{H(x)} = \frac{g(x^*)}{H(x^*)} = C^* \quad \text{and} \quad 0 < C^* < \infty$$

Let $x_0 < x^*$. Then

$$\sup_T E \left\{ (T + 1)^{-\beta} g \left( \frac{X_T}{\sqrt{T+1}} \right) \right\} = E \left\{ (T^* + 1)^{-\beta} g \left( \frac{X_{T^*}}{\sqrt{T^*+1}} \right) \right\}$$

$$= H(x_0) C^*$$

where $T^* = \inf \{ t > 0 \mid \frac{X_t}{\sqrt{t+1}} = x^* \}$. 
\[(t + 1)^{-\beta} H\left(\frac{X_t}{\sqrt{t + 1}}\right) = \int_0^\infty e^{uX_t - \frac{u^2}{2} t} \left(e^{-\frac{u^2}{2}} u^{2\beta - 1}\right) du\]

is a positive martingale with starting value \(H(x_0)\).

Thus \(M_t = (t + 1)^{-\beta} H\left(\frac{X_t}{\sqrt{t + 1}}\right)/H(x_0)\) is a positive martingale with \(EM_0 = 1\).

Then

\[E_P \left(\left(T + 1\right)^{-\beta} g\left(\frac{X_T}{\sqrt{T + 1}}\right)\right) = H(x_0) E_P g\left(\frac{X_T}{\sqrt{T + 1}}\right) M_T\]

\[\leq H(x_0) C^*\]

But for \(E_P M_{T^*} = 1\)

\[T^* = \inf \left\{ t > 0 \mid \frac{X_t}{\sqrt{t + 1}} = x^* \right\} .\]
Special case:

\[ g(x) = x, \quad x_0 = 0, \quad \beta = \frac{1}{2} \]

\[ E(X_T/(T + 1)) = \max \quad \text{with} \]

\[ T^* = \min \left\{ t > 0 \mid \frac{X_t}{\sqrt{t+1}} = x^* \right\} \]

\[ x^* \text{ is solution of } \quad x = (1 - x^2) \int_0^{\infty} e^{ux-u^2/2} \, du. \quad \text{(Shepp 1969)} \]
Two-Sided Boundaries

Let $g$ be measurable, $X_t = \sigma B_t + \mu t$ Brownian motion with drift $\mu$ and variance $\sigma^2$. $X_0 = 0$ find a stopping time $T^*$ which maximizes

$$E e^{-rT} g(X_T) 1_{\{T < \infty\}}.$$

Let $\alpha_{1,2} = -\frac{\mu}{\sigma^2} \pm \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}}$ ($\alpha_2 < 0 < \alpha_1$).

Then $M_t^{(i)} = e^{-rt} e^{\alpha_i X_t}$, $i = 1, 2$ are positive martingales.

We consider boundaries of the type

1.) $g(x) = x^2$

2.) $g(x) = \max\{(L - e^x)^+, (e^x - K)^+\}$

Let $p \in [0, 1]$. Let $M_t = pM_t^{(1)} + (1 - p)M_t^{(2)}$. Then

$$E e^{-rT} g(X_t) = EM_T \frac{g(X_T)}{pe^{\alpha_1 X_T} + (1 - p)e^{\alpha_2 X_T}}.$$
Let $g(x)$ be nonnegative and measurable with

\[
\begin{align*}
\text{a)} & \quad \sup_{x \leq 0} (e^{-\alpha_1 x} g(x)) > \sup_{x \geq 0} (e^{-\alpha_1 x} g(x)) > 0 \\
\text{b)} & \quad \sup_{x \geq 0} (e^{-\alpha_2 x} g(x)) > \sup_{x \leq 0} (e^{-\alpha_2 x} g(x)) > 0.
\end{align*}
\]

**Lemma**

*If a) and b) holds, there exists a $p^* \in (0, 1)$ with $\sup_{x \geq 0} G_{p^*}(x) = \sup_{x \leq 0} G_{p^*}(x)$, where*

\[
G_{p}(x) = \frac{g(x)}{pe^{\alpha_1 x} + (1 - p)e^{\alpha_2 x}}.
\]

**Theorem**

*Let $C^* = \sup_{x \in \mathbb{R}} G_{p^*}(x)$. If there exists points $x_1 > 0$ and $x_2 < 0$ with $G_{p^*}(x_1) = C^* = G_{p^*}(x_2)$. Then*

\[
\sup_{T} E e^{-rT} g(X_T) = C^*
\]

*and*

\[T^* = \inf\{t > 0 \mid X_t = x_1 \text{ or } X_t = x_2\}.
\]
Stopping of Diffusions with Random Exponential Discounting

$X$ diffusion with $X_0 = x$ and $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$

and $B$ standard Brownian motion,

$g : \mathbb{R} \rightarrow \mathbb{R}_+$ a continuous function.

Find a stopping time $T^*$ of $X$ with

$$E_x \left( e^{-A(T)} g(X_T) 1\{T < \infty\} \right) = \max.$$

$A(s)$: additive continuous stochastic process adapted to $\mathcal{F}^X$

$$A(s + t) = A(s) + A(t) \circ \theta_s$$

Example:

$$E_x \left( \exp \left\{ - \int_0^T B_t^2 dt \right\} (B_T^+)^{\alpha} 1\{T < \infty\} \right) = \max$$
How to choose the martingales?

\[
\psi_+(x) = \begin{cases} 
E_x \left( e^{-A(T_{x_0})} 1_{\{T_{x_0} < \infty\}} \right) & \text{for } x \leq x_0 \\
\left[ E_{x_0} \left( e^{-A(T_x)} 1_{\{T_x < \infty\}} \right) \right]^{-1} & \text{for } x \geq x_0 
\end{cases}
\]

\[
\psi_-(x) = \begin{cases} 
E_{x_0} \left( e^{-A(T_{x_0})} 1_{\{T_{x_0} < \infty\}} \right) & \text{for } x \leq x_0 \\
E_x \left( e^{-A(T_{x_0})} 1_{\{T_{x_0} < \infty\}} \right) & \text{for } x \geq x_0.
\end{cases}
\]

\[
M_t^{(+)} = e^{-A(t)} \psi_+(X_t)
\]
are u.i. martingales with

\[
M_t^{(-)} = e^{-A(t)} \psi_-(X_t)
\]

\[
E_x \left( M_t^{(+)} 1_{\{T_b < \infty\}} \right) = \psi_+(x) \quad \text{for } b \geq x \text{ on } 0 \leq t \leq T_b
\]

\[
E_x \left( M_t^{(-)} 1_{\{T_a < \infty\}} \right) = \psi_-(x) \quad \text{for } x \geq a \text{ on } 0 \leq t \leq T_a.
\]

**Note:**

If \( A(t) = \int_0^t r(X_s) ds \) with \( r(x) \geq 0 \), then \( \psi_\pm(x) \) are the solutions of

\[
D\psi = r \cdot \psi
\]
with

\[
D = \mu(x) \frac{\partial}{\partial x} + \frac{1}{2} \sigma(x) \frac{\partial^2}{\partial x^2}.
\]
Example 1:

\( r(x) = rx^2 \) with \( r > 0, \ x = 0 \).

Let \( \psi(t) = e^{-x^2/2} \frac{2^{5/4}}{\Gamma(1/2)} \int_0^\infty e^{xt} - t^2/2 \frac{1}{\sqrt{t}} \ dt \)

Then \( \psi_+(x) = \psi \left( \sqrt{8/r \ x} \right) \) is a solution of

\[
\frac{1}{2} \psi''(x) = rx^2 \psi(x) \quad \text{with} \quad \psi(0) = 1.
\]

Then \( \exp \left( -r \int_0^t X_s^2 \ ds \right) \psi_+(X_t) \) is a local martingale and

\[
\sup_{x \in \mathbb{R}} \left[ (x^+)^\alpha / \psi_+(x) \right] = \sup_{x \geq 0} \left[ (x^+)^\alpha / \psi_+(x) \right] < \infty.
\]

\( T^* = \inf\{ t > 0 \mid X_t = x^* \} \) with \( x^* = \arg \max_x \left[ (x^+)^\alpha / \psi_+(x) \right] > 0 \)
Distinguish the following cases for state space $I$:

1) $\sup_{x \geq x_0, x \in I} (g(x)/\psi_+(x)) = \infty$

2) $\sup_{x \leq x_0, x \in I} (g(x)/\psi_-(x)) = \infty$

3) $0 < C^* = \sup_{x \in I} \frac{g(x)}{\psi_+(x)} = \sup_{x \geq x_0, x \in I} \frac{g(x)}{\psi_+(x)}$

4) $0 < C^* = \sup_{x \in I} \frac{g(x)}{\psi_-(x)} = \sup_{x \leq x_0, x \in I} \frac{g(x)}{\psi_-(x)}$

5) $0 < \sup_{x \geq x_0, x \in I} (g(x)/\psi_+(x)) < \infty$

$0 < \sup_{x \leq x_0, x \in I} (g(x)/\psi_-(x)) < \infty$

and

$\sup_{x \leq x_0, x \in I} \frac{g(x)}{\psi_+(x)} > \sup_{x \geq x_0, x \in I} \frac{g(x)}{\psi_+(x)}$ and $\sup_{x \geq x_0, x \in I} \frac{g(x)}{\psi_-(x)} > \sup_{x \leq x_0, x \in I} \frac{g(x)}{\psi_-(x)}$

In case 5) there exists a $p^* \in (0, 1)$ such that

$$\sup_{x \geq x_0, x \in I} \frac{g(x)}{p^*\psi_+(x) + (1 - p^*)\psi_-(x)} = \sup_{x \leq x_0, x \in I} \frac{g(x)}{p^*\psi_+(x) + (1 - p^*)\psi_-(x)}.$$
Case 5)

**Theorem “5”:**

Let $x_0$ be such that $\psi_+(x_0) = 1 = \psi_-(x_0)$

1) Let $p^*$ be such that

$$0 < \sup_{x \geq x_0, x \in I} \frac{g(x)}{p^* \psi_+(x) + (1 - p^*) \psi_-(x)} = \sup_{x \leq x_0, x \in I} \frac{g(x)}{p^* \psi_+(x) + (1 - p^*) \psi_-(x)}.$$

Then $\sup_T E_{x_0} \left( e^{-AT} g(X_T) 1_{\{T < \infty\}} \right) = C^*$.

2) If there exist points $x_1 > x_0$ and $x_2 < x_0$ such that

$$\frac{g(x_1)}{p^* \psi_+(x_1) + (1 - p^*) \psi_-(x_1)} = \frac{g(x_2)}{p^* \psi_+(x_2) + (1 - p^*) \psi_-(x_2)} = C^*,$$

then the supremum is attained for $T^* = \inf \{ t > 0 \mid X_t = x_1 \text{ or } X_t = x_2 \}$.

3) Let $x_1 \leq x \leq x_2$. Then

$$\sup_T E_x e^{-AT} g(X_T) = E_x e^{AT^*} g(X_{T^*}) = C^* (p^* \psi_+(x) + (1 - p) \psi_-(x)).$$
Case \( r(x) = r \)

Remarks:

1) When \( g \) is twice continuously differentiable up to a finite number of points and when \( T^* = \inf\{ t > 0 \mid X_t \notin (x_1^*, x_2^*) \} \) is optimal. Then the FB-approach and the BL-approach yield the same value-function in the continuation set.

2) A complete characterization for all points of the stopping set has been given by Christensen in his dissertation (2010). He showed by using a Choquet-representation result for \( r \)-harmonic funtions that the optimal stopping set \( S^* \) can be characterized as

\[
S^* = \left\{ x \mid \exists f \text{ } r\text{-harmonic with } x = \arg \max_y \frac{g(y)}{f(y)} \right\}.
\]
### Multiplicative Minimax Characterization

Let \((Z_t, \mathcal{F}_t; t \geq 0)\) denote a right-continuous process of class \(D\).

#### Theorem (Jamshidian (2007))

1) Let \(m > 0\) and \(Z_m \geq 0\) a.s. Then

\[
\sup_{T \leq m} EZ_T = \inf_{M \in \mathcal{M}_+} E \left( M_m \sup_{0 \leq t \leq m} \frac{Z_t}{M_t} \right)
\]

\(\mathcal{M}_+\) denotes the class of positive \(\mathcal{F}\)-adapted martingales with \(M_0 = 1\). If \(Z_m > 0\) a.s. then the infimum is attained.

2) Let \(Z_\infty := \lim_{t \to \infty} Z_t \geq 0\). Then

\[
\sup_{0 \leq T < \infty} EZ_T = \inf_{M \in \mathcal{C}_+} E \left( M_\infty \sup_{0 \leq t < \infty} \frac{Z_t}{M_t} \right)
\]

\(\mathcal{C}_+\) denotes the subclass of \(\mathcal{M}_+\) consisting of the uniformly integrable martingales \(M\) with strictly positive limit.
Relation to the BL-approach

Let $Z_t = g(X_t) \cdot M_t$, with $g$ a continuous function and $X$ a continuous stochastic process and $M$ a continuous positive martingale. Assume $g$ has a unique maximum at $x^*$.

Then

$$\sup_{0 \leq t < \infty} \frac{Z_t}{M_t} \leq g(x^*).$$

Let $T^* = \inf\{t > 0 \mid X_t = x^*\} < \infty$ and $EM_{T^*} = 1$. Then

(E1) \hspace{1cm} EZ_{T^*} = EM_{T^*} g(X_{T^*}) = g(x^*).

But $M_{T^* \land t}$, $t \geq 0$ is not necessary a minimizing martingale in the sense of Jamshidian, namely:

$$\sup_T EZ_T = g(x^*) < EM_{T^*} \sup_{0 \leq t < \infty} \frac{Z_t}{M_{T^* \land t}}.$$
Note that for many examples of option pricing $M$ is not uniformly integrable and

$$P \left( \sup_{t > T^*} (g(X_t)M_t) > g(x^*)M_{T^*} \right) > 0$$

holds.

Nevertheless $M_{m^\wedge t}$ will approach the infimum in many cases when

$m \to \infty$. 
A Modified Minimax Duality

Theorem (Lerche-Urusov(2010))

Let the equation (E1) hold and let the process \((Z_{T*∧t})\) belong to class \(D\). Then it holds

\[
(E2) \quad \sup_T EZ_T = g(x^*) = \inf_{N \in C^+} E \left( N_{T*} \sup_{t \geq 0} \frac{Z_{T*∧t}}{N_{T*∧t}} \right)
\]

Furthermore the sequence \(N^{(n)} = M_{T*} + \frac{1}{n} \in C^+\) is a minimizing sequence of (E2). If \(M_{T*} > 0\) a.s., then \((M_{T*∧t}, t \geq 0) \in C^+\) and it is a minimizing martingale. Finally \((g(x^*)M_{T*∧t}, t \geq 0)\) is the Snell-envelope of \(Z\) on \(\{T^* > 0\}\).
Let $B$ be standard Brownian motion.

Observations:

a) $W_t = B_t$ $\forall t > 0$

or

b) $W_t = B_t + D 1_{\{t > \tau\}}$ with $D = \begin{cases} +1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2} \end{cases}$

$\tau$ has distribution $\varrho$.

$B$, $D$, $\tau$ are independent.

$P_\infty$: measure of $W = B$,

$Q$: measure of $W$ for case b)

Consider

$$R_c(T) = P_\infty(T < \infty) + cE_Q(T - \tau)^+.$$ 

Find a stopping time $T_c^*$, which minimizes $R_c(T)$ for $c > 0$. 
Then
\[
R_c(T) = E_Q(g_c(L_T) + c/I(\theta) V_T) \text{ with } g_c(x) = e^{-x} + c/I(\theta) x,
\]
\[l(\theta) = \theta^2/2 \text{ and } L_t = \log \left. \frac{dQ}{dP}\right|_{F_t}, \text{ with } F_t = \sigma(W_s; 0 \leq s \leq t) \text{ and}
\]
\[(V_t; t \geq 0) \text{ is a nonnegative increasing process.}
\]
This is a non-Markovian stopping problem since
\[
L_t = \int_0^\infty \exp \left(-\theta^2/2 (t - s)^+\right) \cosh(\theta(W_t - W_{t\land s})) \varrho(ds)
\]
g_c has a minimum at \(x^* = l(\theta)/c\).

Remark:

For \(\tau = 0\):
\[
V_t = \frac{1}{2} \int_0^t \frac{\theta^2}{\cosh^2(\theta W_s)} ds
\]

Theorem

Let \(S_b = \inf \{t > 0 \mid L_t \geq b\}\).

Then
\[
\inf_T R_c(T) = R_c(S_{l(\theta)/c}) + o(c) \text{ if } c \to 0.
\]
Appendix
Generalized Parking Problem: Discrete Case, Details

\( X_1, X_2, \ldots \) i.i.d. with \( EX_1 > 0 \), \( S_n = \sum_{i=1}^{n} X_i \), \( S_0 = 0 \).

Find a stopping time \( T^* \) with \( Eg(S_{T^*}) = \min_T Eg(S_T) \), where \( g \) is nonnegative convex with a unique minimum at \( b \).

Solution:

\( T^* = \min\{n \geq 0 \mid S_n \geq a\} \) with \( a = \sup\{x \mid H^+g(x) < g(x)\} \).

\[
H^+g(x) := \int g(x + y)H^+(dy)
\]

\[
H^+(y) := P(S_\eta \leq y)
\]

\( \eta := \min\{n > 0 \mid S_n > 0\} \)

\( H^+ \): the distribution of the first ladder height \( S_\eta \).
Let $K(z) = \int_0^z \frac{1 - H^+(y)}{\nu_1} dy$ 

with $\nu_i = \int y^i dH^+(y), i \in \mathbb{N}$ and $\nu_1 = E(X_1) \exp \left( \sum_{i=1}^{\infty} \frac{1}{n} P[S_n < 0] \right)$.

**Theorem**

If $Kg(x) < \infty$ for all $0 \leq x < \infty$, then $Kg(x)$ is minimized at $x = a$.

**Example 1:**

If $g(x) = |x - b|$ for $x \in \mathbb{R} \implies a = b - \text{med}(K)$

**Example 2:**

If $g(x) = (x - b)^2$ for $x \in \mathbb{R} \implies a = b - \frac{\nu_2}{\nu_1}$
Example 3:

If \( g(x) = e^{-x} + cx \) for \( x \in \mathbb{R} \), with \( 0 < c < 1 \) \( \Rightarrow \) \( b = \log \left( \frac{1}{c} \right) \).

If \( \int x^2 H^+(dx) < \infty \) and if \( \kappa := \int_0^\infty e^{-x} K(dx) \)

\( \Rightarrow Kg(x) = \kappa e^{-x} + c \left( x + \frac{\nu_2}{2\nu_1} \right) \)

and is minimized when \( a = \log \left( \frac{\kappa}{c} \right) = b - \log \left( \frac{1}{\kappa} \right) \).

\[ Kg_c(b) = c \left[ \left( 1 + \log \frac{1}{c} \right) + \log \kappa + \frac{\nu_2}{2\nu_1} \right] \]
Nonlinear Parking Problem: Discrete Case

$Z_1, Z_2, \ldots$ a perturbed random walk, say

$$Z_n = S_n + \xi_n \quad \text{for } n = 0, 1, 2, \ldots,$$

where

$$S_n = \sum_{i=1}^{n} X_i, \quad n \geq 1$$

with

$$X_1, X_2, \ldots \text{ i.i.d. with } EX_1 > 0,$$

having a non-arithmetic distribution.

$\xi_n$ are slowly changing in the sense of “Woodroofe, SIAM, 1982”.

Let $g_c, 0 < c \leq 1$ denote convex functions. Find $T_c^*$ with

$$E g_c(Z_{T_c^*}) = \min_T E g_c(Z_T).$$
For each $0 < c \leq 1$ let $g_c$ be a convex function with a unique minimum at $b = b_c \geq 0$. Assume $\lim_{c \downarrow 0} b_c = \infty$ and there exists a convex function $h_0 : \mathbb{R} \rightarrow \mathbb{R}$ with minimum at zero and with

$$h_c(x) := \frac{g_c(b + x) - g_c(b)}{c} \rightarrow h_0(x) < \infty.$$  

Let $K(y) = \int_0^y \frac{1 - H(x)}{\gamma_1} \, dx$ for $S_n; n \geq 1$ and $H$ as in the Generalized Parking Problem.

**Theorem**

Let $\gamma = \arg \min_x Kh(-x)$ and $T_{b-\gamma} = \min\{n \geq 1 \mid Z_n \geq b - \gamma\}$.

Then as $c \rightarrow 0$

$$\inf_T E_{g_c}(Z_T) = E_{g_c}(Z_{T_{b-\gamma}}) + o(c)$$

$$= g_c(b) + cK_{h_0}(-\gamma) + o(c).$$
Repeated Significance Test Again

\[ R_c(T) = \int g_c \left( \frac{S_T^2}{T + r} \right) dQ, \]

with \[ G_c(z) = \Phi(-z) + cz/2, \]

\[ Q = \int P_\theta G(d\theta) \text{ with } G = N(0, r^{-1}), \]

\[ b_c = \arg \min_z g_c(z). \]

Then

\[ \inf_T R_c(T) \geq g_c(b_c) + c \left( \int_{-\infty}^{\infty} \inf_T E_\theta h(Z_T - b_c) G(d\theta) + o(1) \right), \]

where \[ Z_n = \frac{S_n^2}{2(n + r)} \text{ and } h(z) = z + e^{-z} - 1. \]
Then by the *Theorem* above

\[
\inf_T E \theta h(Z_T - b_c) \geq K^\theta h(-\gamma(\theta)) + o(1)
\]

\[
= \log(\kappa(\theta)) + \frac{\nu_2(\theta)}{2\nu_1(\theta)} + o(1).
\]

Note that

\[
Z_n = \left( \theta S_n - \frac{n\theta^2}{2} \right) + \xi_n
\]

Finally:

\[
\inf_T R_c(T) \geq g_c(b_c) + c \int_{-\infty}^{\infty} \left( \log(\kappa(\theta)) + \frac{\nu_2(\theta)}{2\nu_1(\theta)} \right) G(d\theta) + o(1).
\]
Related Literature

The Main Idea

Sequential Statistics

The Discrete Case