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APPROXIMATE EXIT PROBABILITIES FOR A BROWNIAN BRIDGE ON A SHORT TIME INTERVAL, AND APPLICATIONS

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To Henry Daniels on his 75th birthday

Abstract

Let T be the first exit time of Brownian motion $W(t)$ from a region $\mathcal{R}$ in d-dimensional Euclidean space having a smooth boundary. Given points ξ_0 and ξ_1 in $\mathcal{R}$, ordinary and large-deviation approximations are given for $\Pr\{T < \varepsilon \mid W(0) = \xi_0, W(\varepsilon) = \xi_1\}$ as $\varepsilon \to 0$. Applications are given to hearing the shape of a drum and approximating the second virial coefficient.

FIRST PASSAGE; HEARING THE SHAPE OF A DRUM

1. Introduction

Let $W(t)$, $0 \leqq t < \infty$, denote Brownian motion in $\mathbb{R}^d$ with $W(0) = \xi_0$. For $t > 0$ and events A in the σ-algebra generated by $W(s)$, $0 \leqq s \leqq t$, let

$$P^{(t)}_{\xi_0, \xi_1}(A) = \Pr(A \mid W(0) = \xi_0, W(t) = \xi_1).$$

Assume that ξ_0 and ξ_1 belong to some region $\mathcal{R}$ with a smooth boundary $\partial\mathcal{R}$, and let T denote the time W first leaves $\mathcal{R}$, i.e., $T = \inf\{t : W(t) \in \partial\mathcal{R}\}$. The principal subject of this paper is the asymptotic behavior of

$$P^{(t)}_{\xi_0, \xi_1}\{T < t\} \tag{1.1}$$

as $t \to 0$ and the ξ_i are at a distance $O(t^{\frac{1}{2}})$ from each other and from $\partial\mathcal{R}$. A secondary consideration is the case where the distances of the ξ_i to the boundary and each other are fixed as $t \to 0$.

This problem for $d = 2$ and $\xi_0 = \xi_1$ arises naturally in the beautiful paper of Kac (1966), who was interested in the relation between the geometry of $\mathcal{R}$ and the

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eigenvalues of the Laplacian, and was led to consider the behavior for small t of

$$\sum \exp(-\lambda_k t), \tag{1.2}$$

where the λ_k are eigenvalues of the Laplacian acting on functions having domain $\mathcal{R}$ and vanishing on $\partial\mathcal{R}$. For $\xi_0, \xi_1 \in \mathcal{R}$ define $p(t, \xi_0, \xi_1)$ by

$$p(t, \xi_0, \xi_1)\, d\xi_1 = \Pr(T > t,\ W(t) \in d\xi_1 \mid W(0) = \xi_0),$$

and observe that

$$p(t, \xi_0, \xi_0) = (2\pi t)^{-d/2}[1 - P^{(t)}_{\xi_0,\xi_0}\{T < t\}]. \tag{1.3}$$

In order to study (1.2) in a bounded region $\mathcal{R}$ in $\mathbb{R}^2$ Kac starts from the representation

$$\sum \exp(-\lambda_k t) = \iint_{\mathcal{R}} p(t, \xi_0, \xi_0)\, d\xi_0,$$

which by (1.3) equals

$$(2\pi t)^{-1}\left[|\mathcal{R}| - \iint_{\mathcal{R}} P^{(t)}_{\xi_0,\xi_0}\{T < t\}\, d\xi_0\right], \tag{1.4}$$

where $|\mathcal{R}|$ denotes the area of $\mathcal{R}$. He then in effect obtains the asymptotic relation

$$P^{(t)}_{\xi_0,\xi_0}\{T < t\} \sim \exp(-2y_0^2/t) \quad (t, y_0 \to 0), \tag{1.5}$$

where y_0 is the distance of ξ_0 to the boundary $\partial\mathcal{R}$. (This relation has the interpretation that to a first approximation the conditional probability of leaving $\mathcal{R}$ during a short time interval equals the conditional probability of crossing the tangent to $\partial\mathcal{R}$ at the point closest to ξ_0.) From (1.4) and (1.5) Kac obtains

$$\sum \exp(-\lambda_k t) = (2\pi t)^{-1}|\mathcal{R}| - [4(2\pi t)^{\frac{1}{2}}]^{-1}|\partial\mathcal{R}| + o(t^{-\frac{1}{2}}) \tag{1.6}$$

as $t \to 0$, where $|\partial\mathcal{R}|$ denotes the length of $\partial\mathcal{R}$. By a piecewise linear approximation to $\partial\mathcal{R}$ combined with a substantial calculation Kac argues heuristically that the next term in the expansion (1.6) should be $(1-h)/6$, where h is the number of holes in $\mathcal{R}$. For various complements to these results see Louchard (1968), McKean and Singer (1967), Stewartson and Waechter (1971), and Smith (1981). Of these, only Louchard attempts to carry out Kac's program of a probabilistic analysis, and his argument appears to contain an improper use of the Markov property.

Starting from the physical problem of evaluating the second virial coefficient of a hard sphere gas, Handelsman and Keller (1966) arrive at essentially the same mathematical problem as Kac, for the case $d = 3$, $\xi_0 = \xi_1$, and $\mathcal{R}$ the region exterior to a sphere. They derive what in Kac's problem corresponds to the next two terms in (1.6). Although their method does not seem capable of being turned into a rigorous proof, minor modifications appear to produce correct answers under much more general conditions.

In this paper we use methods borrowed from sequential analysis to obtain in Theorem 1 the first term in an Edgeworth-type expansion of (1.1) as $t \to 0$ with the ξ_i at a distance $O(t^{\frac{1}{2}})$ from $\partial\mathscr{R}$ and from each other. Theorem 2 gives a large-deviation approximation as $t \to 0$ while ξ_0 and ξ_1 remain fixed. Theorem 3 is concerned with the substantially more complicated second Edgeworth term. For computational simplicity only the case $\xi_0 = \xi_1$ is considered, but this is the case which arises in both the Kac and Handelsman–Keller problems. These results make it possible to complete Kac's program and obtain by probabilistic methods the expansion

$$\sum \exp(-\lambda_k t) = (2\pi t)^{-\frac{1}{2}} |\mathscr{R}| - [4(2\pi t)^{\frac{1}{2}}]^{-1} |\partial\mathscr{R}| + (1-h)/6 + 2^{-8}(2\pi)^{-\frac{1}{2}} t^{\frac{1}{2}} \int_{\partial\mathscr{R}} c^2(\sigma)\, d\sigma + o(t^{\frac{1}{2}}), \tag{1.7}$$

where $|\mathscr{R}|$, $|\partial\mathscr{R}|$, and h are as defined above, σ denotes arc length on $\partial\mathscr{R}$, and $c(\sigma)$ is the curvature of $\partial\mathscr{R}$ at σ. One can similarly obtain the Handelsman–Keller expansion. See Section 4.

We begin in Section 2 by collecting together a number of technical results which will be used later. The reader may wish to proceed directly to Section 3 and refer back to Section 2 when needed.

2. Preliminaries

This section summarizes a number of basic results which are used in what follows. Lemmas 1, 2, and 4 are well known. The notation used here is not always consistent with the rest of the paper.

Let $W(t) = (W_1(t), \cdots, W_d(t))$, $0 \leqq t < \infty$, be standard Brownian motion in $\mathbb{R}^d$, and let $\mathscr{F}(t)$ denote the σ-algebra generated by $W(s)$, $s \leqq t$. Let

$$P_\xi^{(t)}(A) = \Pr(A \mid W(t) = \xi) \quad (A \in \mathscr{F}(t)).$$

In Sections 3–5 we shall want to assume $W(0) = \xi_0 \neq 0$. The results given below are easily adapted to that case by a translation of the origin.

Lemma 1. Let $t > 0$ and $\xi \neq \xi'$. For all $s < t$ the restriction of $P_\xi^{(t)}$ to $\mathscr{F}(s)$ is absolutely continuous with respect to $P_{\xi'}^{(t)}$ so restricted and has likelihood ratio (Radon–Nikodym derivative) given by

$$l(t, s; \xi, \xi') = \exp\{[\langle \xi - \xi', W(t)\rangle - s(\|\xi\|^2 - \|\xi'\|^2)/2t]/(t-s)\}.$$

Hence for any stopping time T and $A \in \mathscr{F}(T)$

$$P_\xi^{(t)}[A \cap \{T < t\}] = E_{\xi'}^{(t)}[l(t, T; \xi, \xi'); A \cap \{T < t\}].$$

Proof. This is a version of Wald's fundamental identity of sequential analysis. See Siegmund (1985), p. 39.

Let $b>0$ and define $\tau=\tau_b=\inf\{t: W_1(t)\geqq b\}$.

Lemma 2. For arbitrary $t>0$ and $\xi=(\xi^1,\cdots,\xi^d)\in\mathbb{R}^d$

$$P_\xi^{(t)}\{\tau<t\}=\min(1,\exp[-2b(b-\xi^1)/t]).$$

For $|\xi^1|<b$

$$P_\xi^{(t)}\left\{\max_{s\leqq t}|W_1(s)|\geqq b\right\}\leqq\exp[-2b(b-\xi^1)/t]+\exp[-2b(b+\xi^1)/t].$$

Proof. The equality (in the non-trivial case $\xi^1<b$) follows immediately from Lemma 1 if one puts $\xi'=(2b-\xi^1,\xi^2,\cdots,\xi^d)$ and observes that $2b-\xi^1>b$, so $P_{\xi'}^{(t)}\{\tau<t\}=1$. The inequality follows from the equality.

Lemma 3. For $0<t<1$ and $\xi=(\xi^1,\cdots,\xi^d)\in\mathbb{R}^d$ the $P_\xi^{(1)}$ density of τ is given by

$$\frac{b}{[t^3(1-t)]^{\frac{1}{2}}}\varphi\left\{b\left(\frac{1-t}{t}\right)^{\frac{1}{2}}-(\xi^1-b)\left(\frac{t}{1-t}\right)^{\frac{1}{2}}\right\}. \tag{2.1}$$

As $t\to0$, $P_\xi^{(1)}\{\tau<t\}=O[\exp(-b^2/4t)]$. For $\xi^1>b$

$$E_\xi^{(1)}(\tau)=b\Phi(-\xi^1)/\varphi(\xi^1)$$

and

$$E_\xi^{(1)}[\tau^2/(1-\tau)]=b/(\xi^1-b)-b\Phi(-\xi^1)/\varphi(\xi^1).$$

Here φ and Φ are the standard normal probability density and distribution function, respectively.

Proof. We have

$$P_\xi^{(1)}\{\tau<t\}=E_\xi^{(1)}[P_\xi^{(1)}\{\tau<t\mid W(t)\}].$$

The conditional probability equals $P_{W(t)}^{(t)}\{\tau<t\}$ and is given by Lemma 2. Integration with respect to the $P_\xi^{(1)}$ distribution of $W(t)$ followed by differentiation with respect to t yields (2.1). The asymptotic behavior of $P_\xi^{(1)}\{\tau\leqq t\}$ as $t\to0$ is easily obtained from (2.1).

From (2.1) one also obtains

$$E_\xi^{(1)}[\tau^2/(1-\tau)]=b\int_0^\infty s^{-\frac{3}{2}}(1+s)^{-1}\varphi[(\xi^1-b)/s^{\frac{1}{2}}-bs^{\frac{1}{2}}]\,ds.$$

Writing $(1+s)^{-1}=\int_0^\infty\exp(-\alpha(1+s))\,d\alpha$, interchanging orders of integration and using the well-known equality

$$\int_0^\infty e^{-\alpha s}as^{-\frac{3}{2}}\varphi(as^{-\frac{1}{2}}-\mu s^{\frac{1}{2}})\,ds=\exp\{-a[(2\alpha+\mu^2)^{\frac{1}{2}}-\mu]\}$$

$(a>0,\ \alpha>0,\ -\infty<\mu<\infty)$ (e.g. Siegmund (1985), (3.16) and Problem 3.1), one obtains the indicated expression for $E_\xi^{(1)}[\tau^2/(1-\tau)]$. A similar calculation applies to $E_\xi^{(1)}(\tau)$.

Lemma 4. Let $(\Omega, \mathcal{G}, P)$ be a probability space and $\{\mathcal{G}(t), 0 \leqq t < t_1\}$ a family of sub-σ-algebras of $\mathcal{G}$. (i) If $E|Z| < \infty$, then $\{E(Z \mid \mathcal{G}(t)), 0 \leqq t < t_1\}$ is uniformly integrable. (ii) Suppose $\{Z(t), \mathcal{G}(t), 0 \leqq t < t_1\}$ is a right-continuous submartingale and that S and T are stopping times. If $P\{T < t_1\} = 1$, $E|Z(T)| < \infty$, and

$$\liminf_{t \to t_1} E(Z^+(t); T > t) = 0,$$

then $E(Z(T) \mid \mathcal{G}(S)) \geqq Z(S)$ a.e. on $\{T > S\}$.

Proof. These results are essentially standard martingale theory. Although we could not find them in exactly the form we require, almost any discussion of martingale theory contains similar results and the essential ideas for a proof. Part (ii) is known as Doob's optional sampling theorem (cf. Loève (1963), pp. 530–535).

Lemma 5. For each $t > 0$ and $-\infty < \xi^i < \infty$, $\{[W_i(s) - t^{-1}s\xi^i]/(t-s), \mathcal{F}(s), 0 \leqq s < t\}$ and $\{[W_i(s) - t^{-1}s\xi^i]^2/(t-s)^2 - t^{-1}s/(t-s), \mathcal{F}(s), 0 \leqq s < t\}$ are $P_\xi^{(t)}$-martingales.

Proof. Let $0 < s_1 < s_2 < t$. It is easy to show that the conditional distribution of $W_i(s_2)$ given $\mathcal{F}(s_1)$ is normal with mean $(s_2 - s_1)[\xi^i - W_i(s_1)]/(t - s_1)$ and variance $(s_2 - s_1)[1 - (s_2 - s_1)/(t - s_1)]$, from which the lemma follows by direct calculation.

Lemma 6. Let $\xi^1 > b$. Then $E_\xi^{(1)}[\tau/(1-\tau)] = b/(\xi^1 - b)$.

Proof. It is possible to obtain this result by a direct calculation, starting from (2.1). More simply, one can use Lemmas 4 and 5 to obtain

$$\xi^1 E_\xi^{(1)}[\tau/(1-\tau)] = E^{(1)}[W_1(\tau)/(1-\tau)] = bE_\xi^{(1)}[1/(1-\tau)] = b + bE_\xi^{(1)}[\tau/(1-\tau)].$$

Lemma 7. Let $\xi_1 > b$, so $P_\xi^{(t)}\{\tau < t\} = 1$. Given $W_1(s)$, $0 \leqq s \leqq \tau$, the $P_\xi^{(t)}$ conditional joint distribution of $W_i(\tau)$, $2 \leqq i \leqq d$, is that of $[\tau(1 - \tau/t)]^{\frac{1}{2}} Z_i + \tau\xi^i/t$, where $Z_2, \cdots, Z_d$ are independent standard normal variables.

Proof. For fixed $s < t$, the $P_\xi^{(t)}$ joint distribution of $W_i(s)$, $2 \leqq i \leqq d$, is that of $[s(1 - s/t)]^{\frac{1}{2}} Z_i + s\xi^i/t$, where the Z_i are independent standard normal and independent of $W_1(\cdot)$. Since τ is defined in terms of $W_1(\cdot)$, the desired result follows immediately.

Lemma 8. Assume $b \to \infty$, $\xi^1 \to \infty$ in such a way that b/ξ^1 converges to some fixed value in $(0, 1)$. Then

$$P_\xi^{(1)}\{\tau - b/\xi^1 \leqq x[(b/(\xi^1)^3)(1 - b/\xi^1)]^{\frac{1}{2}}\} \to \Phi(x),$$

where Φ denotes the standard normal distribution function.

Proof. One can prove this result by a direct computation starting from (2.1). A more general argument, which will be useful in the proof of Theorem 2 is given by Siegmund (1968).

3. Approximations to $P^{(t)}_{\xi_0,\xi_1}\{T<t\}$

For ease of exposition we consider in detail only the case $d=2$ and indicate in some remarks how the results are modified for general d.

Given $\xi_0,\ \xi_1 \in \mathcal{R} \subset \mathbb{R}^2$, close to $\partial\mathcal{R}$ and to each other, assume there exists a unique point on $\partial\mathcal{R}$ the sum of whose distances to the given points is a minimum. Consider the Cartesian coordinate system which has this point as its origin, the x-axis as tangent and the y-axis as outward normal to $\mathcal{R}$. There exists a function $y=f(x)$ such that locally near $(0,0)$ $\partial\mathcal{R}$ is given by the graph $(x, f(x))$. Let ξ_i have coordinates (x_i, y_i) in this coordinate system, and assume that $y_i<0$ $(i=0,1)$. It is easily seen that ξ_0 and ξ_1 satisfy $-x_0/|y_0| = x_1/|y_1|$. (A ray of light emanating from ξ_0 and reflecting off the x-axis at the origin passes through ξ_1). Let $W(t)$ denote Brownian motion starting from $W(0)=\xi_0$, and define

$$T = \inf\{t : W_2(t) \geqq f(W_1(t))\}. \tag{3.1}$$

In general, T is not the exit time of W from $\mathcal{R}$, but for ξ_0 close to $(0,0)$ it is with probability close to 1. (A more precise estimate is given below.)

In order to study $P^{(\varepsilon)}_{\xi_0,\xi_1}\{T<\varepsilon\}$ it is convenient to use Brownian scaling to replace the given problem on the time interval $[0, \varepsilon]$ by an equivalent one on $[0,1]$. Since $W(\varepsilon t)/\varepsilon^{\frac{1}{2}}$ is Brownian motion starting from $\tilde{\xi}_0 = \xi_0/\varepsilon^{\frac{1}{2}}$, it is easy to see that

$$P^{(\varepsilon)}_{\xi_0,\xi_1}\{T<\varepsilon\} = P^{(1)}_{\tilde{\xi}_0,\tilde{\xi}_1}\{\tilde{T}<1\}, \tag{3.2}$$

where $\tilde{\xi}_i = \xi_i/\varepsilon^{\frac{1}{2}}$ $(i=0,1)$ and

$$\tilde{T} = \tilde{T}_\varepsilon = \inf\{t : W_2(t) \geqq \varepsilon^{-\frac{1}{2}} f(\varepsilon^{\frac{1}{2}} W_1(t))\}. \tag{3.3}$$

To give a precise statement of our first result it is convenient to change our viewpoint slightly and regard f as given and the points ξ_i as variable.

Theorem 1. Assume f is twice continuously differentiable, $f(0)=f'(0)=0$, and $f''(0)\neq 0$. Suppose $\xi_i=(x_i, y_i)$ $(i=0,1)$ satisfy $y_i<0$ $(i=0,1)$ and $-x_0/|y_0| = x_1/|y_1|$, and converge to $(0,0)$ as $\varepsilon\to 0$ in such a way that $\tilde{\xi}_i = \xi_i/\varepsilon^{\frac{1}{2}}$ are fixed $(i=0,1)$. Then for T defined by (3.1)

$$\begin{aligned} P^{(\varepsilon)}_{\xi_0,\xi_1}\{T<\varepsilon\} &= \exp(-2y_0y_1/\varepsilon)\Big\{1 - f''(0)\Big[\varepsilon^{-\frac{1}{2}}|y_0y_1| \\ &\quad\times \frac{\Phi[(y_0+y_1)/\varepsilon^{\frac{1}{2}}]}{\varphi[(y_0+y_1)/\varepsilon^{\frac{1}{2}}]}(1-\varepsilon^{-1}(x_1-x_0)^2) + \varepsilon^{-1}(x_0^2|y_1| + x_1^2|y_0|) + o(\varepsilon^{\frac{1}{2}})\Big]\Big\}, \end{aligned} \tag{3.4}$$

where Φ and φ are the standard normal distribution and density function respectively.

Proof. By virtue of (3.2) it suffices to consider the standardized problem on the time interval $[0,1]$, with fixed initial and terminal points $\tilde{\xi}_0=(\tilde{x}_0, \tilde{y}_0)$ and $\tilde{\xi}_1 = (\tilde{x}_1, \tilde{y}_1)$, and $\tilde{T}$ defined by (3.3). To simplify the notation we consider only this standardized problem and omit the tildes for the rest of the proof. In this new notation, where all variables have tildes, but the tildes are omitted, (3.4) becomes

$$
\begin{aligned}
(3.5)\qquad P^{(1)}_{\xi_0,\xi_1}\{T_\varepsilon < 1\} &= \exp(-2y_0y_1)\Big\{1 - \varepsilon^{\frac{1}{2}} f''(0)\Big[\,|y_0y_1| \\
&\quad \times \frac{\Phi(y_0+y_1)}{\varphi(y_0+y_1)}(1-(x_1-x_0)^2) + (x_0^2\,|y_1| + x_1^2\,|y_0|) + o(1)\Big]\Big\},
\end{aligned}
$$

with T_ε defined by (3.3).

We begin with an informal calculation to convey the main ideas and provide a justification later. The argument proceeds from a suitable likelihood ratio identity. Let $\xi_1' = (x_1, |y_1|)$. The likelihood ratio of $W(s)$, $s \leqq t$, under $P^{(1)}_{\xi_0,\xi_1}$ relative to $P^{(1)}_{\xi_0,\xi_1'}$ is easily calculated to be (cf. Lemma 1)

$$\exp(-2\,|y_0y_1|)\exp[-2\,|y_1|\,W_2(t)/(1-t)].$$

Thus since $W_2(t) = \varepsilon^{-\frac{1}{2}} f(\varepsilon^{\frac{1}{2}} W_1(T))$, we have, again by Lemma 1,

$$(3.6)\qquad P^{(1)}_{\xi_0,\xi_1}\{T<1\}\exp(2\,|y_0y_1|) = E^{(1)}_{\xi_0,\xi_1'}\left\{\exp\left[\frac{-2\,|y_1|\,f(\varepsilon^{\frac{1}{2}}W_1(T))}{\varepsilon^{\frac{1}{2}}(1-T)}\right]; T<1\right\}.$$

Since

$$(3.7)\qquad \varepsilon^{-\frac{1}{2}} f(\varepsilon^{\frac{1}{2}}x) \sim \varepsilon^{\frac{1}{2}} f''(0)x^2/2 \to 0 \quad (\varepsilon \to 0),$$

for all sufficiently small ε, $P^{(1)}_{\xi_0,\xi_1'}\{T<1\} = 1$, and the right-hand side of (3.6) can be expanded to become

$$(3.8)\qquad 1 - |y_1|\,\varepsilon^{\frac{1}{2}} f''(0) E^{(1)}_{\xi_0,\xi_1'}[W_1^2(T)/(1-T)] + \cdots.$$

Define

$$(3.9)\qquad \tau = \inf\{t : W_2(t) \geqq 0\}.$$

From (3.7) follows $P^{(1)}_{\xi_0,\xi_1'}\{T_\varepsilon \to \tau\} = 1$, and hence (one expects that)

$$(3.10)\qquad E^{(1)}_{\xi_0,\xi_1'}[W_1^2(T)/(1-T)] \to E^{(1)}_{\xi_0,\xi_1'}[W_1^2(\tau)/(1-\tau)].$$

By Lemma 7 conditional on $W_2(t)$, $t \leqq \tau$, $W_1(\tau)$ is distributed as $[\tau(1-\tau)]^{\frac{1}{2}} Z + x_0 + (x_1 - x_0)\tau$, where Z has a standard normal distribution. Hence

$$(3.11)\qquad E^{(1)}_{\xi_0,\xi_1'}[W_1^2(\tau)/(1-\tau)] = x_0^2 + (1 + 2x_0x_1 - x_0^2)E^{(1)}_{\xi_0,\xi_1'}(\tau) + x_1^2 E^{(1)}_{\xi_0,\xi_1'}[\tau^2/(1-\tau)].$$

Equation (3.5) follows from (3.6), (3.8), (3.10), (3.11), and the evaluations given in Lemma 3.

To make the preceding manipulations into a proof, one must consider the remainders in (3.7) and (3.8), and justify the convergence indicated in (3.10).

Let $\zeta = (\log \varepsilon)^2$ and $A = \{\max_{t \leqq T} |W_1(t)| \leqq \zeta\}$. From the inequality in Lemma 2 one obtains

$$P^{(1)}_{\xi_0,\xi_1}(A^c) + P^{(1)}_{\xi_0,\xi_1'}(A^c) = o(\varepsilon^2).$$

Let $0<\eta<|y_1|$ and $t_1=1-\zeta^{-7}$. Also let

$$\tau_b=\inf\{t: W_2(t)\geqq b\}. \tag{3.12}$$

(Note that (3.9) is the special case $b=0$). For all ε sufficiently small $|f(\varepsilon^{\frac{1}{2}}x)|/\varepsilon^{\frac{1}{2}}\leqq\eta$ for all $|x|\leqq\zeta$. Hence by Lemma 3

$$\begin{aligned}P^{(1)}_{\xi_0,\xi_1}[\{t_1<T<1\}\cap A] &\leqq P^{(1)}_{\xi_0,\xi_1}\Big\{\max_{t_1<t<1} W_2(t)>-\eta\Big\}\\ &\leqq P^{(1)}_{\xi_1,\xi_0}\{\tau_{-\eta}<\zeta^{-7}\}\\ &=O(\exp[-(|y_1|-\eta)^2\zeta^7/4])\\ &=o(\varepsilon^2).\end{aligned}$$

Similarly, $P^{(1)}_{\xi_0,\xi_1'}[\{t_1<T<1\}\cap A]=o(\varepsilon^2)$. Hence (3.6) can be replaced by

$$\begin{aligned}&P^{(1)}_{\xi_0,\xi_1}\{T<1\}\exp(-2|y_0y_1|)\\ &\quad=E^{(1)}_{\xi_0,\xi_1'}\Big\{\exp\Big[\frac{-2|y_1|f(\varepsilon^{\frac{1}{2}}W_1(T))}{\varepsilon^{\frac{1}{2}}(1-T)}\Big];\{T<t_1\}\cap A\Big\}+o(\varepsilon^2).\end{aligned} \tag{3.13}$$

Let $\delta>0$. By two applications of Taylor's theorem with remainder along the lines suggested in (3.7) and (3.8) one can obtain upper and lower bounds for the right-hand side of (3.13) in the form

$$1-|y_1|\,\varepsilon^{\frac{1}{2}}[f''(0)\pm\delta]E^{(1)}_{\xi_0,\xi_1'}[W_1^2(T)/(1-T);\{T<t_1\}\cap A]+o(\varepsilon^2). \tag{3.14}$$

Since $\delta>0$ is arbitrary, by (3.11) and Lemma 3 it suffices to show (cf. (3.10))

$$E^{(1)}_{\xi_0,\xi_1'}[W_1^2(T)/(1-T);\{T<t_1\}\cap A]\to E^{(1)}_{\xi_0,\xi_1'}[W_1^2(\tau)/(1-\tau)], \tag{3.15}$$

where τ is defined by (3.9). By (3.7) $P^{(1)}_{\xi_0,\xi_1'}\{T_\varepsilon\to\tau\}=1$, and $P_{\xi_0,\xi_1'}(\{T<t_1\}\cap A)\to 1$. Hence to prove (3.15) it is sufficient (and necessary) to show that

$$\{1_{\{T<t_1\}\cap A}W_1^2(T)/(1-T);\ \varepsilon>0\} \tag{3.16}$$

is uniformly integrable.

For all sufficiently small ε, since $f(\varepsilon^{\frac{1}{2}}x)/\varepsilon^{\frac{1}{2}}<\eta$ for all $|x|<\zeta$, $A\subset\{T<\tau_\eta\}$. By Lemma 5 $[W_1(t)-x_0-(x_1-x_0)t]^2/(1-t)^2-t/(1-t)$, $0\leqq t<1$, is a $P^{(1)}_{\xi_0,\xi_1'}$-martingale and since $t\mapsto t/(1-t)$ is increasing, $[W_1(t)-x_0-(x_1-x_0)t]^2/(1-t)^2$, $0\leqq t<1$, is a submartingale. By Lemma 7

$$E^{(1)}_{\xi_0,\xi_1'}\{[W_1(\tau_\eta)-x_0-(x_1-x_0)\tau_\eta]^2/(1-\tau_\eta)^2\}=E^{(1)}_{\xi_0,\xi_1'}[\tau_\eta/(1-\tau_\eta)],$$

which is finite by Lemma 3. Also

$$E^{(1)}_{\xi_0,\xi_1'}\{[W_1(t)-x_0-(x_1-x_0)t]^2/(1-t)^2;\ \tau_\eta>t\}=t(1-t)^{-1}P^{(1)}_{\xi_0,\xi_1'}\{\tau_\eta>t\}\to 0$$

as $t\to 1$, again by Lemma 3. It follows from Lemma 4(ii) that on $\{T<\tau_\eta\}$

$$\begin{aligned}&[W_1(T)-x_0-(x_1-x_0)T]^2/(1-T)^2\\ &\quad\leqq E^{(1)}_{\xi_0,\xi_1'}\{[W_1(\tau_\eta)-x_0-(x_1-x_0)\tau_\eta]^2/(1-\tau_\eta)^2\mid W(t),t\leqq T\}.\end{aligned}$$

Hence by Lemma 4(i) $\{1_{\{T<\tau_\eta\}}[W_1(T)-x_0-(x_1-x_0)T]^2/(1-T)^2,\ \varepsilon>0\}$ is uniformly integrable. The uniform integrability of (3.16) now follows from the relation $A\subset\{T<\tau_\eta\}$, the inequality $(a+b)^2\leqq 2(a^2+b^2)$, and Lemma 3.

Remarks. (i) As observed above, the boundary of $\mathscr{R}$ can be defined locally near $(0,0)$ by a function $y=f(x)$, but in general it cannot be so defined globally. However, for ε sufficiently small, on $\{\max_{0\leqq t\leqq\varepsilon}|W_1(t)|<\zeta\varepsilon^{\frac{1}{2}}\}T$ defined by (3.1) and the exit time from $\mathscr{R}$ coincide, so there is no loss of generality restricting attention to stopping times of this form. (ii) In higher dimensions, f'' in (3.4) becomes the Laplacian Δf, and x_i^2 $(i=0,1)$ and $(x_1-x_0)^2$ become Euclidean distances $\|x_i\|^2$ $(i=0,1)$ and $\|x_1-x_0\|^2$. The proof is essentially unchanged.

In Theorem 1 ξ_0 and ξ_1 are at a distance $O(\varepsilon^{\frac{1}{2}})$ from the boundary of $\mathscr{R}$ and from each other, and consequently $P^{(\varepsilon)}_{\xi_0,\xi_1}\{T<\varepsilon\}$ converges to a limit between 0 and 1. Theorem 2 is concerned with the case that ξ_0 and ξ_1 are fixed as $\varepsilon\to 0$, so $P^{(\varepsilon)}_{\xi_0,\xi_1}\{T<\varepsilon\}\to 0$.

As above, for given ξ_0, $\xi_1\in\mathscr{R}$ suppose there exists a unique point on $\mathscr{R}$, the sum of whose distances from ξ_0 and ξ_1 is a minimum, and consider the tangent-normal coordinate system through this point. Let ξ_i have coordinates (x_i,y_i) $(i=0,1)$, and let $\partial\mathscr{R}$ be given by the graph of $(x,f(x))$ in some neighborhood of $(0,0)$, so $f(0)=f'(0)=0$.

Theorem 2. Assume f is twice continuously differentiable, $y_0y_1>0$, and

$$2y_0y_1f''(0)[1+(x_1/y_1)^2]/|y_0+y_1|>-1. \tag{3.17}$$

Let $T=\inf\{t: W(t)\in\partial\mathscr{R}\}$. Then as $\varepsilon\to 0$

$$P^{(\varepsilon)}_{\xi_0,\xi_1}\{T<\varepsilon\}\sim\frac{\exp(-2\varepsilon^{-1}y_0y_1)}{\{1+2y_0y_1f''(0)[1+(x_1/y_1)^2]/|y_0+y_1|\}^{\frac{1}{2}}}. \tag{3.18}$$

One can prove Theorem 2 along the lines of the proof of Theorem 1, but the details are rather different. To keep this paper to a reasonable length the proof is only sketched. See Siegmund (1982) for a somewhat similar argument. An example comparing the numerical accuracy of (3.18) and (3.4) is given in Section 5.

An interesting case which fails to satisfy the conditions of Theorem 2 is $\mathscr{R}$ a disk with $\xi_0=\xi_1$ at the center. In this case, the nearest point on $\partial\mathscr{R}$ is not unique and (3.17) is not satisfied. For an approximation in this case, which leans heavily on rotational symmetry, see Siegmund (1985), Problem 11.1. An exact expression involving infinite series of Bessel functions has been obtained by Kiefer (1959).

Sketch of the Proof of Theorem 2. We again consider the equivalent standardized problem on the time interval $[0,1]$, but in this case it is helpful to distinguish between the original variables $\xi_i=(x_i,y_i)$ and the scaled variables $\tilde{\xi}_i=\xi_i/\varepsilon^{\frac{1}{2}}$, as well as between T defined by (3.1) and $\tilde{T}$ defined by (3.3). Our starting point is again

(3.6) which we write as

$$(3.19)\quad P^{(\varepsilon)}_{\xi_0,\xi_1}\{T<\varepsilon\}\exp(2\,|y_0y_1|/\varepsilon)=E^{(1)}_{\xi_0,\xi_1'}\left\{\exp\left[\frac{-2\,|y_1|f(\varepsilon^{\frac12}W_1(\tilde T))}{\varepsilon(1-\tilde T)}\right];\ \tilde T<1\right\},$$

where $\xi_1'=(x_1/\varepsilon^{\frac12},\ |y_1|/\varepsilon^{\frac12})$.

Let $t^*=|y_0|/|y_0+y_1|$ and observe that $E^{(1)}_{\xi_0,\xi_1'}[W(t^*)]=(0,0)$. Let τ be defined as in (3.9). By Lemma 8, in $P^{(1)}_{\xi_0,\xi_1'}$-probability $\tau-t^*=O_p(\varepsilon^{\frac12})$. By an argument similar to that given in Siegmund (1968) and cited in the proof of Lemma 8, we also have $\tilde T-t^*=O_p(\varepsilon^{\frac12})$. Since $W_1(t)$ drifts at a rate no greater than $\varepsilon^{-\frac12}$, we have $W_1(\tilde T)=W_1(t^*)+W_1(\tilde T)-W_1(t^*)=O_p(1)$. Hence by (3.7) the right-hand side of (3.19) has the same limiting value as

$$(3.20)\qquad E^{(1)}_{\xi_0,\xi_1'}\{\exp[-|y_1|f''(0)W_1^2(\tilde T)/(1-\tilde T)];\ \tilde T<1\}.$$

Since $W_2(t)$ drifts at rate $\varepsilon^{-\frac12}$, it follows from (3.7) that $\tilde T-\tau=O_p(\varepsilon)$. Hence $W_1(\tilde T)-W_1(\tau)=O_p(\varepsilon^{-\frac12})O_p(\varepsilon)=O_p(\varepsilon^{\frac12})$. In particular the limiting behavior of (3.20) is the same as that of

$$(3.21)\qquad E^{(1)}_{\xi_0,\xi_1'}\{\exp[-|y_1|f''(0)W_1^2(\tau)/(1-\tau)]\}.$$

Conditioning on τ and appealing to Lemma 7, one finds that (3.21) equals

$$E^{(1)}_{\xi_0,\xi_1'}\left\{[1+2\,|y_1|f''(0)\tau]^{-\frac12}\exp\left[\frac{-|y_1|f''(0)\tau(x_1\{\tau/(1-\tau)\}^{\frac12}+x_0\{(1-\tau)/\tau\}^{\frac12})^2}{\varepsilon\{1+2\,|y_1|f''(0)\tau\}}\right]\right\}.$$

A Taylor series expansion of this function of τ about $t^*=|y_0|/(|y_1|+|y_0|)$, the observation that $x_1(y_0/y_1)^{\frac12}+x_1(y_1/y_0)^{\frac12}=0$, an application of Lemma 8, and some messy calculations yield (3.18).

Remarks. (i) Although the proof of Theorem 2 shows that both τ and $\tilde T$ converge in probability to t^*, and $W_1(\tau)$ and $W_1(\tilde T)$ are asymptotically equivalent in law, their common limiting distribution is not that of $W_1(t^*)$ unless $\xi_0=\xi_1$.

(ii) A related but somewhat more complicated problem than that discussed in Theorem 1 is to approximate the joint distribution of $(T,W_1(T))$, which can be attacked via the characteristic function

$$(3.22)\qquad E^{(\varepsilon)}_{\xi_0,\xi_1}[\exp\{i\lambda_1W_1(T)/\varepsilon^{\frac12}+i\lambda_2T/\varepsilon\};\ T<\varepsilon].$$

Expansion of (3.22) to the precision of Theorem 1 seems to require more complicated calculations, which turn out to be very similar to those given in the following section in order to obtain the term of order ε in the expansion of $P^{(\varepsilon)}_{\xi_0,\xi_1}\{T<\varepsilon\}$.

(iii) It seems possible to obtain the results of this section by the methods of Jennen and Lerche (1981), (1982), but the computations appear to be somewhat more complicated. If one is interested in the joint behavior of T and $W_1(T)$, their method might turn out to be the simpler one.

4. The term of order ε and applications

Calculation of higher-order terms in the expansion (3.4) rapidly becomes very complicated in detail. In this section we see what is involved by examining the term of order ε. (See Equation (4.10).) To simplify the algebra we suppose that $\xi_0 = \xi_1$. This special case suffices for applications to the problems of Kac (1966) and Handelsman and Keller (1966), which are discussed below. We proceed informally as in the first part of the proof of Theorem 1. The localization and uniform integrability arguments necessary for a rigorous proof are similar to those in Theorem 1 and have been omitted.

Let $\xi_0 = \xi_1$. In the notation of Section 2 for the standardized problem on the time interval $[0, 1]$, (3.6) becomes

$$(4.1) \qquad P^{(1)}_{\xi_0,\xi_0}\{T<1\}\exp(2y_0^2) = E^{(1)}_{\xi_0,\xi_0'}\left\{\exp\left[\frac{-2\,|y_0|\,f(\varepsilon^{\frac{1}{2}}W_1(T))}{\varepsilon^{\frac{1}{2}}(1-T)}\right]\right\},$$

where $\xi_0 = (0, y_0)$, $\xi_0' = (0, |y_0|)$, and $T = T_\varepsilon$ is defined by (3.3). Assuming that f is three times continuously differentiable, we have

$$\varepsilon^{-\frac{1}{2}}f(\varepsilon^{\frac{1}{2}}x) = \varepsilon^{\frac{1}{2}}f''(0)x^2/2 + \varepsilon f'''(0)x^3/6 + o(\varepsilon);$$

and hence the right-hand side of (4.1) becomes

$$(4.2) \qquad \begin{aligned} &1 - \varepsilon^{\frac{1}{2}}\,|y_0|\,f''(0)E^{(1)}_{\xi_0,\xi_0'}[W_1^2(T)/(1-T)] - \tfrac{1}{3}\varepsilon\,|y_0|\,f'''(0)E^{(1)}_{\xi_0,\xi_0'}[W_1^3(T)/(1-T)] \\ &+ \tfrac{1}{2}\varepsilon y_0^2[f''(0)]^2E^{(1)}_{\xi_0,\xi_0'}[W_1^4(T)/(1-T)^2] + o(\varepsilon). \end{aligned}$$

Until further notice, we shall write P and E for $P^{(1)}_{\xi_0,\xi_0'}$ and $E^{(1)}_{\xi_0,\xi_0'}$. Recall the definition of τ given in (3.9) and note that by Lemma 7 the conditional distribution of $W_1(\tau)$ given τ is normal with mean 0 and variance $\tau(1-\tau)$. By (3.7) $P\{T_\varepsilon \to \tau\} = 1$ $(\varepsilon \to 0)$, and hence

$$(4.3) \qquad E[W_1^3(T)/(1-T)] \to 0$$

and

$$(4.4) \qquad E[W_1^4(T)/(1-T)^2] \to 3E\tau^2.$$

Also

$$(4.5) \qquad E[W_1^2(T)/(1-T)] = E\tau + \{E[W_1^2(T)/(1-T)] - E[W_1^2(\tau)/(1-\tau)]\};$$

and the final contribution to the term of order ε in (4.2) comes from the difference on the right-hand side of (4.5), which is itself of order $\varepsilon^{\frac{1}{2}}$.

First suppose that $f''(0) < 0$ and to simplify some details that $f(x) \leqq 0$ for all x. The case $f''(0) > 0$ involves a similar argument with slightly more complicated calculations. Let $\mathcal{F}_t$ denote the σ-algebra generated by $W(s)$, $s \leqq t$. Since $T \leqq \tau$, we have

$$E[W_1^2(\tau)/(1-\tau)] = E[(W_1(T) + W_1(\tau) - W_1(T))^2/(1-\tau)]$$
$$= E\left\{\frac{W_1^2(T)}{1-\tau} + \frac{2W_1(T)}{1-\tau} E[W_1(\tau) - W_1(T)]\mathscr{F}_T, \tau]\right. \tag{4.6}$$
$$\left. + (1-\tau)^{-1} E[(W_1(\tau) - W_1(T))^2 \mid \mathscr{F}_T, \tau]\right\}.$$

Conditional on $\mathscr{F}_T$, and τ, by Lemma 7 $W_1(\tau) - W_1(T)$ is normally distributed with mean $-W_1(T)(\tau - T)/(1-T)$ and variance $(\tau - T)(1-\tau)/(1-T)$. Hence after some algebra one obtains

$$\begin{aligned} &E[W_1^2(T)/(1-T)] - E[W_1^2(\tau)/(1-\tau)] \\ &\quad = E[(1-T)^{-2} W_1^2(T) E(\tau - T \mid \mathscr{F}_T)] - E[(1-T)^{-1} E(\tau - T \mid \mathscr{F}_T)]. \end{aligned} \tag{4.7}$$

By Lemmas 4 and 5

$$E\{(1-\tau)^{-1}[W_2(\tau) - W_2(T) - (\tau - T)(|y_0| - W_2(T))/(1-T)] \mid \mathscr{F}_T\} = 0$$

and hence with probability 1 as $\varepsilon \to 0$

$$\begin{aligned} E(\tau - T \mid \mathscr{F}_T) &\sim \frac{\frac{1}{2}\varepsilon^{\frac{1}{2}} |f''(0)|\, W_1^2(T)(1-T)}{|y_0| + \frac{1}{2}\varepsilon^{\frac{1}{2}} |f''(0)|\, W_1^2(T)} \\ &\sim \tfrac{1}{2}\varepsilon^{\frac{1}{2}} |y_0|^{-1} |f''(0)|\, W_1^2(\tau)(1-\tau). \end{aligned} \tag{4.8}$$

Substitution of (4.8) into (4.7) yields

$$\begin{aligned} &E[W_1^2(T)/(1-T)] - E[W_1^2(\tau)/(1-\tau)] \\ &\qquad \sim \tfrac{1}{2}\varepsilon^{\frac{1}{2}} |y_0|^{-1} |f''(0)| \{E[W_1^4(\tau)/(1-\tau)] - EW_1^2(\tau)\} \\ &\qquad = \tfrac{1}{2}\varepsilon^{\frac{1}{2}} |y_0|^{-1} |f''(0)| \{3E[\tau^2(1-\tau)] - E[\tau(1-\tau)]\}. \end{aligned} \tag{4.9}$$

From (4.1)–(4.5) and (4.9) we obtain the following expansion in the case $f''(0) < 0$.

Theorem 3. If f is three times continuously differentiable and $f(0) = f'(0) = 0$, then

$$\begin{aligned} P^{(1)}_{\xi_0,\xi_0}\{T < 1\} = \exp(-2y_0^2)\{1 &- \varepsilon^{\frac{1}{2}} |y_0| f''(0) E^{(1)}_{\xi_0,\xi_0'}(\tau) \\ &+ \tfrac{1}{2}\varepsilon [f''(0)]^2 E^{(1)}_{\xi_0,\xi_0'}[3\tau^2(1-\tau) - \tau(1-\tau) + 3y_0^2\tau^2] + o(\varepsilon)\}, \end{aligned} \tag{4.10}$$

where T is defined by (3.3), τ by (3.9) $\xi_0 = (0, y_0)$, and $\xi_0' = (0, |y_0|)$.

The proof of (4.10) is slighly different when $f''(0) > 0$. In this case $\tau \leqq T$, so (4.6) must be replaced by

$$\begin{aligned} E[W_1^2(T)/(1-T) &= E[W_1^2(\tau)/(1-T)] \\ &+ 2E\left\{W_1(\tau) E\left[\frac{W_1(T) - W_1(\tau)}{1-T} \,\Big|\, \mathscr{F}_\tau\right]\right\} + E\left\{E\left[\frac{(W_1(T) - W_1(\tau))^2}{1-T} \,\Big|\, \mathscr{F}_\tau\right]\right\}. \end{aligned}$$

By Lemmas 4 and 5

$$E\left[\frac{W_1(T)-W_1(\tau)}{1-T}\;\Big|\;\mathcal{F}_\tau\right]=-(1-\tau)^{-1}W_1(\tau)E[(T-\tau)/(1-T)\mid\mathcal{F}_\tau],$$

and it may be shown that

$$E\{[W_1(T)-W_1(\tau)]^2/(1-T)\mid\mathcal{F}_\tau\}\sim E[(T-\tau)/(1-T)\mid\mathcal{F}_\tau].$$

Hence in place of the equality (4.7) one obtains

$$\begin{aligned}&E[W_1^2(\tau)/(1-\tau)]-E[W_1^2(T)/(1-T)]\\&\quad\sim E[(1-\tau)^{-2}W_1^2(\tau)E(T-\tau\mid\mathcal{F}_\tau)]-E[(1-\tau)^{-1}E(T-\tau\mid\mathcal{F}_\tau)].\end{aligned}$$

A result similar to (4.8) holds for $E(T-\tau\mid\mathcal{F}_\tau)$, and the rest follows as before.

Remarks. (i) By the method of Lemma 3 one can evaluate the moments appearing on the right-hand side of (4.10). However, for the applications given below, which in effect involve an integration of (4.10) over ξ_0, the computations are considerably simpler if one interchanges the order of the two integrations and integrates over ξ_0 first. (ii) In higher dimensions the relation of $\partial\mathcal{R}$ to its tangent planes can be more complicated than in two dimensions. In general, one must condition on $\mathcal{F}_{T\wedge\tau}$ and consider the two cases $\{T\leqq\tau\}$ and $\{T>\tau\}$. Whereas the term of order $\varepsilon^{\frac{1}{2}}$ involves only the Laplacian of f, i.e., the mean curvature of $\partial\mathcal{R}$, the term of order ε involves a quadratic function of the mean and Gaussian curvatures. For the problem studied by Handelsman and Keller (1966), where $\mathcal{R}$ is the region exterior to a sphere in $\mathbb{R}^3$ one does not encounter these complications.

Now let T denote the first exit time of W from $\mathcal{R}$, and for $\xi_0,\xi_1\in\mathcal{R}$ define $p(t,\xi_0,\xi_1)$ by

$$p(t,\xi_0,\xi_1)\,d\xi_1=\Pr(T>t,\;W(t)\in d\xi_1\mid W(0)=\xi_0).$$

Recall from Section 1 that Kac's (1966) problem involves the behavior for small t of

$$\sum\exp(-\lambda_k t)=(2\pi t)^{-1}\left[|\mathcal{R}|-\iint_{\mathcal{R}}P^{(t)}_{\xi_0,\xi_0}\{T<t\}\,d\xi_0\right],\tag{4.11}$$

where $|\mathcal{R}|$ denotes the area of $\mathcal{R}$. Handelsman and Keller (1966) are interested in $\mathbb{R}^3$ and the integral

$$\iiint_{\mathcal{R}}[1-(2\pi t)^{\frac{3}{2}}p(t,\xi_0,\xi_0)]\,d\xi_0,$$

which by (1.3) equals

$$\iiint_{\mathcal{R}}P^{(t)}_{\xi_0,\xi_0}\{T<t\}\,d\xi_0.\tag{4.12}$$

In order to analyse the integral in (4.11) it is convenient to make a change of variables to obtain

$$\iint_{\mathscr{R}} P^{(t)}_{\xi_0,\xi_0}\{T<t\}\,d\xi_0=\int_{\partial\mathscr{R}}\int_0^{\delta} P^{(t)}_{\xi_0,\xi_0}\{T<t\}[1-|y_0|\,c(\sigma)]\,d\,|y_0|\,d\sigma+O(e^{-\delta^2/t}),\tag{4.13}$$

where σ denotes arc length on $\partial\mathscr{R}$, $c(\cdot)$ is the (signed) curvature of $\partial\mathscr{R}$, and ξ_0 has coordinates $(0, y_0)$ in the tangent–normal coordinate system with its origin at the point σ of $\partial\mathscr{R}$, so $|y_0|$ is the distance from ξ_0 to $\partial\mathscr{R}$ and $c(\sigma)=-f''(0)$. (The change of variables from ξ_0 to (σ, y_0) has Jacobian $1+y_0c(\sigma)$ under our convention that the positive y_0 axis is an outer normal to $\partial\mathscr{R}$, provided δ is small enough that $\delta\max_\sigma|c(\sigma)|<1$. The derivation is a straightforward application of the Frenet–Serret equations for plane curves. See, for example, Millman and Parker (1977), pp. 30 and 52.)

Keeping (2.1) and (3.2) in mind, one can substitute (4.10) into (4.13), integrate with respect to $|y_0|$, then with respect to σ, to obtain an expansion of $\sum\exp(-\lambda_k t)$. Since $c(\sigma)=d\theta/d\sigma$, where θ is the angle between the tangent to $\partial\mathscr{R}$ and some fixed direction, it follows that $\int_{\partial\mathscr{R}}c(\sigma)\,d\sigma=2\pi(1-h)$, where h is the number of holes in $\mathscr{R}$. (This is a special case of the turning tangents theorem of plane differential geometry. See Millman and Parker (1977), Section 3.2.) The result is (1.7).

The expansion (1.7) agrees with those given by Stewartson and Waechter (1971) and Smith (1981), both of whom used analytical methods and obtained additional terms. The term of order $t^{\frac{1}{2}}$ disagrees with that given by Louchard (1968), whose argument appears to contain an improper use of the Markov property.

Since (1.7) involves termwise integration of (4.10), some additional justification is required to claim that (1.7) has been proved rigorously. (Previous authors are about equally divided between those who concern themselves with this justification and those who do not.) In order to indicate the general nature of the argument, we sketch a justification of (1.7) with a remainder of $o(1)$, but we have not attempted the substantially more technical calculation required to include the term of order $t^{\frac{1}{2}}$. We summarize the result as follows.

Theorem 4. Let $\mathscr{R}$ be a bounded region in $\mathbb{R}^2$ with a twice continuously differentiable boundary $\partial\mathscr{R}$. Let $\{\lambda_k\}$ denote the eigenvalues of the Laplacian operating on functions with domain $\bar{\mathscr{R}}$ which vanish on $\partial\mathscr{R}$. Then as $t\to 0$

$$\sum\exp(-\lambda_k t)=(2\pi t)^{-1}|\mathscr{R}|-[4(2\pi t)^{\frac{1}{2}}]^{-1}|\partial\mathscr{R}|+(1-h)/6+o(1),$$

where $|\mathscr{R}|$ is the area of $\mathscr{R}$, $|\partial\mathscr{R}|$ is the length of $\partial\mathscr{R}$, and h is the number of holes in $\mathscr{R}$.

The change of variables $\tilde{y}_0=y_0/t^{\frac{1}{2}}$ in the integral on the right-hand side of (4.13) yields

$$t^{\frac{1}{2}}\int_{\partial\mathcal{R}}\int_0^{\delta t^{-\frac{1}{2}}} P^{(1)}_{\xi_0,\xi_0}\{\tilde{T}_t<1\}[1-t^{\frac{1}{2}}\,|\tilde{y}_0|\,c(\sigma)]\,d\,|\tilde{y}_0|\,d\sigma,$$

where $\tilde{T}_\varepsilon$ is defined by (3.3). As in the proof of Theorem 1, we henceforth omit the tildes from our notation and write ε instead of t. Hence by (4.11) and (4.13) our problem reduces to that of evaluating

$$\int_{\partial\mathcal{R}}\int_0^{\delta\varepsilon^{-\frac{1}{2}}} P^{(1)}_{\xi_0,\xi_0}\{T<1\}[1-\varepsilon^{\frac{1}{2}}\,|y_0|\,c(\sigma)]\,d\,|y_0|\,d\sigma \tag{4.14}$$

with a remainder equal to $o(\varepsilon^{\frac{1}{2}})$ as $\varepsilon\to 0$. For this it would be more than enough for the expansion of Theorem 1 to hold uniformly in ξ_0 throughout the range of integration. It seems difficult to justify such uniformity for y_0 close to 0. We shall indicate the proofs of three weaker results which are easily seen to suffice. Recall that $\zeta=(\log\varepsilon)^2$.

Lemma 9. (i) For arbitrary $\gamma>0$, the expansion (3.5) with $\xi_0=\xi_1$ holds uniformly in ξ_0 such that $\log\varepsilon<y_0<-\gamma$. (ii) Uniformly in ξ_0 for which $\delta\varepsilon^{-\frac{1}{2}}<y_0<-\zeta^{-3}$

$$P^{(1)}_{\xi_0,\xi_0}\{T<1\}=\exp(-2y_0^2)[1+O(\varepsilon^{\frac{1}{2}}\,|y_0|)]. \tag{4.15}$$

(iii) Uniformly in ξ_0 for which $-\zeta^{-3}\leqq y_0<0$,

$$P^{(1)}_{\xi_0,\xi_0}\{T<1\}=\exp(-2y_0^2)[1+o(\varepsilon^{\frac{1}{2}})].$$

Proof. Let $A_1=\{\max_{0<t<1}W_1(t)<\zeta\}$ and observe that by Lemma 2 $P^{(1)}_{\xi_0,\xi_0}\{T<1\}=P^{(1)}_{\xi_0,\xi_0}[\{T<1\}\cap A_1]+o(\varepsilon^2)$, uniformly in ξ_0. Let $K=\max_\sigma|c(\sigma)|$. By Taylor's theorem with remainder, uniformly in ξ_0

$$|f(\varepsilon^{\frac{1}{2}}x)|/\varepsilon^{\frac{1}{2}}\leqq K\varepsilon^{\frac{1}{2}}x^2\leqq K\varepsilon^{\frac{1}{2}}\zeta^2 \quad\text{for}\quad |x|\leqq\zeta. \tag{4.16}$$

Let $\eta=K\varepsilon^{\frac{1}{2}}\zeta^2$. By Lemma 2 again

$$\begin{aligned} P^{(1)}_{\xi_0,\xi_0}[\{T<1\}\cap A_1] &\geqq P^{(1)}_{\xi_0,\xi_0}[\{\tau_\eta<1\}\cap A_1]\\ &\geqq P^{(1)}_{\xi_0,\xi_0}\{\tau_\eta<1\}+o(\varepsilon^2)\\ &=\exp[-2(|y_0|+\eta)^2]+o(\varepsilon^2)\\ &=\exp[-2y_0^2](1+o(\varepsilon^{\frac{1}{2}})) \end{aligned}$$

uniformly in $|y_0|\leqq\zeta^{-3}$. A similar calculation gives an upper bound to complete the proof of (iii).

The uniformity asserted with regard to (4.15) is easily inferred from a careful reading of the proof of Theorem 1. With the aid of (4.16) one easily sees that (3.13) holds uniformly in ξ_0 in the indicated range. (Note, however, that the reason for the choice $t_1=1-\zeta^{-7}$ is now apparent, whereas $1-\zeta^{-1}$ would have sufficed for

Theorem 1.) Hence (3.14) holds uniformly and since $f''(0) = -c(\sigma)$ is bounded on $\partial\mathscr{R}$, to prove (4.15) it suffices to show that the expectation in (3.14) is uniformly bounded. By (4.16) the argument following (3.16) (with $\eta = K\varepsilon^{\frac{1}{2}}\zeta^2$) shows that uniformly in ξ_0 with $-\delta\varepsilon^{-\frac{1}{2}} < y_0 < -\zeta^{-3}$

$$E^{(1)}_{\xi_0,\xi_0'}[W_1^2(T)/(1-T); \{T < t_1\} \cap A] \leqq E^{(1)}_{\xi_0,\xi_0'}[\tau_\eta/(1-\tau_\eta)],$$

which equals $(|y_0| + \eta)/(|y_0| - \eta)$ by Lemma 6. This proves (ii).

As the proof of (ii) shows, to prove that (3.5) holds uniformly in $\log \varepsilon < y_0 < -\gamma$ (when $\xi_0 = \xi_1$) it suffices to show that (3.15) holds uniformly. To simplify the notation, we shall write $P = P^{(1)}_{\xi_0,\xi_0'}$ and $E = E^{(1)}_{\xi_0,\xi_0'}$. Let $B_1 = \{\max_{0<t<1} |W_1(t)| < \varepsilon_1^{-1}\}$, $B_2 = \{\max(\tau, T) < 1 - \varepsilon_2\}$, $B_3 = \{|\tau - T| < \varepsilon_3\}$, $B_4 = \{|W_1(T) - W_1(\tau)| < \varepsilon_4\}$, and $B = B_1 \cap B_2 \cap B_3 \cap B_4$, where $\varepsilon_1, \cdots, \varepsilon_4$ will be specified later.

Since $B \subset \{T < t_1\} \cap A$ (uniformly in ξ_0)

$$\begin{aligned} &|E(W_1^2(T)/(1-T); \{T < t_1\} \cap A) - E(W_1^2(\tau)/(1-\tau))| \\ &\leqq |E(W_1^2(T)/(1-T) - W_1^2(\tau)/(1-\tau); B)| \\ &\quad + E[W_1^2(T)/(1-T); A \cap \{T < t_1\} \cap B^c] + E[W_1^2(\tau)/(1-\tau); B^c]. \end{aligned}$$

The first expectation on the right-hand side is $\leqq 2\varepsilon_4/\varepsilon_1\varepsilon_2 + \varepsilon_3/(\varepsilon_1\varepsilon_2)^2$, which for any fixed ε_1, ε_2 can be made arbitrarily small by a suitable choice of ε_3 and ε_4. Scrutiny of the argument following (3.16) shows that (3.16) is uniformly integrable in ε and ξ_0, provided that ξ_0 is bounded away from $\partial\mathscr{R}$. The same is true for $W_1^2(\tau)/(1-\tau)$. Hence it suffices to show that for any $\varepsilon_5 > 0$, for suitable $\varepsilon_1, \cdots, \varepsilon_4$, for all small ε, $P(B^c) < \varepsilon_5$ uniformly in ξ_0 for which $y_0 < -\gamma$. This inequality is straightforward but somewhat tedious to prove. Note that $B^c = B_1^c \cup (B_1 \cap B_2^c) \cup (B_1 \cap B_3^c) \cup (B_1 \cap B_2 \cap B_4^c)$. We first fix ε_1 so small that $P(B_1^c) < \varepsilon_5/4$ (cf. Lemma 2). Then choose ε_2 such that for all small ε, $P(B_1 \cap B_2^c) < \varepsilon_5/4$. Next choose ε_3 and ε_4 to make $2\varepsilon_4/\varepsilon_1\varepsilon_2 + \varepsilon_3/(\varepsilon_1\varepsilon_2)^2$ small. Then choose $\eta > 0$ such that for all small ε

$$P(B_1 \cap B_3^c) \leqq P\{\tau_\eta - \tau_{-\eta} > \varepsilon_3\} < \varepsilon_5/4.$$

Finally choose η still smaller if necessary so that for all small ε

$$\begin{aligned} P(B_1 \cap B_2 \cap B_4^c) &\leqq P\left\{|W_1(\tau_{-\eta})| < \varepsilon_1^{-1}, \tau_{-\eta} < 1 - \varepsilon_2, 2 \max_{\tau_{-\eta} \leqq t \leqq \tau_\eta} |W_1(t) - W_1(\tau_{-\eta})| > \varepsilon_4\right\} \\ &< \varepsilon_5/4. \end{aligned}$$

The details are easily filled in.

Handelsman and Keller's (1966) problem involves (4.12), where $\mathscr{R} = \{\xi_0 \in \mathbb{R}^3, \|\xi_0\| \geqq a\}$. Converting (4.12) to polar coordinates yields

$$4\pi \int_0^\infty (a + |y_0|)^2 P^{(t)}_{\xi_0,\xi_0}\{T < t\}\, d\,|y_0|. \tag{4.17}$$

An expansion of $P^{(t)}_{\xi_0,\xi_0}\{T < t\}$ along the lines of Theorem 3 and substitution into

(4.17) yields

$$(2\pi)^{\frac{1}{2}}\pi a^2 t^{\frac{1}{2}} + 4\pi at/3 + (2\pi)^{\frac{1}{2}}\pi t^{\frac{3}{2}}/24 + o(t^{\frac{3}{2}}),$$

in agreement with the expansion obtained by Handelsman and Keller.

5. Numerical accuracy

In the preceding section we used approximations for $P^{(t)}_{\xi_0,\xi_1}\{T<t\}$ as an analytic ingredient in the problems of Kac and of Handelsman and Keller. It is also interesting to know how good they are as numerical approximations. In this section we briefly describe the results of a small Monte Carlo experiment addressing this question.

The obvious method to obtain a Monte Carlo estimator of $P^{(t)}_{\xi_0,\xi_1}\{T<t\}$ is to partition the time interval $[0, t]$ by the points $t_i = it/m$, generate N realizations of the $P^{(t)}_{\xi_0,\xi_1}$ discrete-time random process $W(t_i)$, $i = 0, 1, \cdots, m$ and use as an estimator the relative frequency among the N realizations that $W(t_i) \notin \mathscr{R}$ for some i. This estimator has a bias which converges to 0 as $m \to \infty$ and a variance which converges to 0 as $N \to \infty$.

A number of related examples in Siegmund (1985, especially Chapter X) and Hogan (1984) suggest that the bias is $O(m^{-\frac{1}{2}})$, and it appears quite expensive computationally to make this bias sufficiently small merely by making m large.

A simple modification to circumvent this difficulty is the following. Having generated the partial realization $W(t_0), \cdots, W(t_i)$ and decided that $T > t_i$, generate $W(t_{i+1})$. If $W(t_{i+1}) \notin \mathscr{R}$ decide $T \leqq t_{i+1} \leqq t$. If $W(t_{i+1}) \in \mathscr{R}$ decide $T \leqq t_{i+1} \leqq t$ with probability $q[W(t_i), W(t_{i+1}), t_{i+1} - t_i]$, where $q(\xi_0, \xi_1, \varepsilon)$ is some approximation for $P^{(\varepsilon)}_{\xi_0,\xi_1}\{T < \varepsilon\}$. The point is that this approximation need not be particularly good, provided m is moderately large and hence $t_{i+1} - t_i$ is small.

For example, consider

$$P^{(t)}_{\xi_0,\xi_1}\Big\{\min_{0 \leqq s \leqq t} \|W(s)\| < a\Big\}. \tag{5.1}$$

A naive approximation to (5.1) is to regard the sphere of radius a as a hyperplane, from which the points ξ_i are at distances $\|\xi_i\| - a$ $(i = 0, 1)$, so that by Lemma 2 (5.1) is approximately

$$\exp[-2(\|\xi_0\| - a)(\|\xi_1\| - a)/t]. \tag{5.2}$$

In the special case $\xi_0 = \xi_1$ Theorem 1 suggests the approximation

$$P^{(t)}_{\xi_0,\xi_0}\Big\{\max_{0 \leqq s \leqq t} \|W(s)\| < a\Big\} \cong \exp\left[-2t^{-1}y_0^2\left(1 + \frac{(d-1)t^{\frac{1}{2}}\Phi(2y_0/t^{\frac{1}{2}})}{2a\varphi(2y_0/t^{\frac{1}{2}})}\right)\right], \tag{5.3}$$

where $y_0 = a - \|\xi_0\|$, and d is the dimension of $W(\cdot)$. When $\xi_0 = \xi_1$ the large-

TABLE 1
Approximations for $P^{(1)}_{\xi_0,\xi_0}\{\min_{0\leqq s\leqq 1}\|W(s)\|\leqq a\}$

d	a	$\|\xi_0\|$	Monte Carlo	Analytic (5·3)	Analytic (5·4)
2	1	1·2	0·891	0·889	0·843
2	1	1·5	0·535	0·515	0·495
2	1	2·0	0·106	0·089	0·096
3	1	1·7	0·255	0·221	0·221
3	1	2·0	0·076	0·058	0·068
3	2	2·4	0·646	0·646	0·605
3	2	3·2	0·039	0·033	0·035

deviation approximation provided by Theorem 2 is

$$P^{(t)}_{\xi_0,\xi_0}\left\{\max_{0\leqq s\leqq t}\|W(s)\|<a\right\}\sim\exp(-2y_0^2/t)[1+|y_0|/a]^{-(d-1)/2}. \tag{5.4}$$

Table 1 compares (5.3), (5.4), and the outcome of an $N=9999$ repetition Monte Carlo experiment. For the Monte Carlo experiment the crude approximation (5.2) was used as indicated above to 'bridge' the gap between t_i and t_{i+1}. Some experimentation with values of $m=10$, 20, and 40 showed that $m=20$ points in our partition is adequate for the range of values of the other variables considered here. In all cases $t=1$.

Both approximations are reasonably good. For small tail probabilities (5.4) is somewhat better than (5.3), as it should be, but even it is only moderately accurate. For larger probabilities (5.3) is better than (5.4). The simple-minded approximation (5.2) would usually be rather poor, although it is quite useful as a tool in the Monte Carlo experiment.

It is natural to ask whether use of a better approximation than (5.2), e.g. the approximation provided by Theorem 1, would make the Monte Carlo experiment more efficient. A second Monte Carlo experiment, not reported here, indicates the answer is no. In fact the more sophisticated approximation imposes an added numerical burden at each stage, which at least in the simple case studied here actually leads to an overall decrease in efficiency.

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