A MULTIPLE-CURVE LÉVY FORWARD RATE MODEL IN A TWO-PRICE ECONOMY

ERNST EBERLEIN AND CHRISTOPH GERHART

ABSTRACT. A multiple-curve Heath-Jarrow-Morton (HJM) forward rate model driven by time-inhomogeneous Lévy processes is presented. In this approach to model spreads between curves for different tenors, credit as well as liquidity risk is taken into account. Deterministic conditions are derived to ensure the positivity of spreads and thus the monotonicity of the curves for the various tenors. Valuation formulas for standard interest rate derivatives such as caps, floors and digital options are established. These formulas can numerically be evaluated very fast by using Fourier based valuation methods. In order to be able to exploit bid and ask prices we develop this approach in the context of a two-price setting. Explicit formulas for bid as well as ask prices of the derivatives are stated. A specific model framework based on Normal Inverse Gaussian and Gamma processes is proposed which allows for calibration to market data. Calibration results are presented based on cap market quotes from September 2009.

KEYWORDS. Multiple curve model, HJM, time-inhomogeneous Lévy processes, monotonicity of curves, two-price theory, interest rate derivatives, calibration

1. INTRODUCTION

The global financial crisis which started in early August 2007 had a lasting effect on financial markets. This concerns in particular the fixed income markets. As a consequence of a new perception of risk a number of interest rates, which until then had been roughly equivalent, drifted apart. In particular the basic rates, which are relevant for the interbank market, became tenor-dependent after market participants became aware of credit, liquidity and funding risks. These risks had been assumed to be negligible in this market segment before. In the new reality classical modelling approaches which are based on arbitrage considerations assuming tenor-independence cannot reflect the market behaviour any more. More sophisticated approaches, so-called multiple curve models, are needed to take the increased diversity of risks into account. In this paper we focus on the tenor-dependence of the Euribor rates. Before the crisis Euribor as well as Libor rates typically differed by a few basis points only. Starting in early August 2007 the spreads widened and reached levels up to 200 basis points (see Figure 1).

There are essentially three types of interest rate model approaches, namely short rate models, forward rate or HJM-type models and market models. Hereafter we will discuss the multiple curve approach in a framework where (instantaneous) forward rates and forward spreads are taken as basic modeling...
quantities. The possibility that interest rates become negative in a classical single curve HJM-model has often been considered to be an undesirable property. However in recent years rates for several of the major currencies, as for example for Euro and Yen, entered deep into negative territory. By mid-June 2016 the whole AAA Euro yield curve up to ten years was below zero. Therefore, the potential that the basic (risk-free) rates become negative or even start from negative values, is an important characteristic which a realistic model has to take into account.

Given its importance for the financial industry, there is in the meanwhile an impressive literature on multiple curve approaches and related topics. We mention here only a few of them. Ametrano and Bianchetti (2009, 2013) explain which market instruments can be selected to obtain the current rates and how the bootstrapping works. A short rate approach is developed in Kijima, Tanaka, and Wong (2009). In an article which appeared just before the start of the crisis Henrard (2007) pointed to the fact that a single discount curve for all market participants is not appropriate. Henrard (2010) proposes a multiple curve approach. In a Gaussian framework, deterministic spreads between the curves are assumed. A diffusion driven double-curve model with regard to a foreign exchange analogy is introduced in Bianchetti (2010). The classical Libor market model is extended to a diffusion driven double-curve model by Mercurio (2010). Moreni and Pallavicini (2014) develop a parsimonious approach in a HJM-framework where spreads are assumed to be deterministic functions. In a very recent paper, Cuchiero, Fontana, and Gnoatto (2016) study a general multiple-curve approach based on multiplicative spreads with semimartingales as driving processes.

Our approach is closely related in spirit to the HJM-type approach in Crépey, Grbac, and Nguyen (2012), where a single risky rate is considered in addition
to the risk-free rate. Another conceptual difference is that we apply multiple curves in the context of a two price market. Conic finance (see Cherny and Madan (2010)), as the latter is also called, is able to exploit explicitly bid and ask price data. This is highly appropriate in this context, since multiple interest rate curves emerged from taking tenor-dependent credit and liquidity risks into account and the bid-ask spread is an important indicator for the liquidity of the corresponding market.

In section 1 we present first the class of processes which we use as drivers, namely time-inhomogeneous Lévy process. They are also called processes with independent increments and absolutely continuous characteristics (PIIAC) by Jacod and Shiryaev (2003). This class of processes has proved to be particularly appropriate for fixed income models (see e.g. Eberlein and Kluge (2006)). In section 2 the multiple-curve forward rate model is introduced. We start with the discount curve which corresponds to the risk-free term structure. OIS-zero coupon bond prices can be considered as a good approximation for the initial discount term structure. We proceed by introducing individual curves for various tenors. They are constructed by considering non-traded, fictitious bonds, which can be interpreted as risky bonds issued by a typical Euribor panel bank. The drift function in the dynamics of the underlying instantaneous forward rates is chosen such that credit and liquidity risk are explicitly represented. As it is evident from Figure 1 the spreads are monotone in the tenor length. A sufficient condition to achieve this goal is discussed in section 3, where we also propose a suitable model framework which guarantees the monotonicity of the curves.

In order to be able to calibrate this model, explicit valuation formulas for derivatives are needed. We consider caps, floors as well as digital options. Fourier-based methods are used to obtain numerically efficient versions of the pricing formulas. Since the price data consists of bid and ask prices with non-negligible spreads the analysis is based on valuation formulas in a two-price economy. Using distortions we derive numerically efficient formulas for bid and ask prices of caplets, floorlets as well as digital options. For this purpose the cumulative distribution function of the stochastic component in the modeling of the underlying quantity is needed. In the last section some calibration results based on specific driving processes are presented.

2. The driving process

Let $T^* > 0$ be a finite time horizon and $\mathcal{B} := (\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T^*]}, P)$ a stochastic basis that satisfies the usual conditions in the sense of Jacod and Shiryaev (2003, Definition I.1.2 and Definition I.1.3). We will consider as driving process a $d$-dimensional time-inhomogeneous Lévy process $L = (L^1, \ldots, L^d)^T$ on $\mathcal{B}$ with $L^i = (L^i_t)_{t \in [0,T^*]}$ for every $i \in \{1, \ldots, d\}$. This means that $L$ is a $\mathbb{F}$-adapted process with independent increments and absolutely continuous characteristics (abbreviated by PIIAC) defined on $\mathcal{B}$ (see Eberlein, Jacod, and Raible (2005) and Jacod and Shiryaev (2003)). We emphasise that $L$ is a $d$-dimensional semimartingale (see Jacod and Shiryaev (2003, §5)).

We can assume that the paths of each component of $L$ are càdlàg. This means that these paths are right-continuous and admit left-hand limits (almost surely). We also postulate that each component $L^i$ starts at zero. The
semimartingale characteristics of \( L \) are given by the triplet \((B, C, \nu)\) with
\[
B = \int_0^t b_s \, ds, \quad C = \int_0^t c_s \, ds, \quad \nu(ds, dx) = F_s(dx) \, ds,
\]
where \( b_s = (b_1^s, \ldots, b_d^s)^T : [0, T^*) \to \mathbb{R}^d \), \( c_s = (c_{ij}^s)_{i,j \leq d} : [0, T^*) \to \mathbb{R}^{d \times d} \)
whose values are in the set of symmetric nonnegative-definite \( d \times d \)-matrices and \( F_s \) is
a Lévy measure for every \( s \in [0, T^*) \), i.e. a nonnegative measure on \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\)
that integrates \((|x|^2 \wedge 1)\) and satisfies \( F_s(\{0\}) = 0 \). The Euclidean scalar product
on \( \mathbb{R}^d \) is denoted by \( \langle \cdot, \cdot \rangle \) and \(|\cdot|\) is the corresponding norm. The scalar product
on \( \mathbb{R}^d \) is extended to complex numbers by setting \( \langle w, z \rangle := \sum_{j=1}^d w_j z_j \) for every \( w, z \in \mathbb{C}^d \). Thus, \( \langle \cdot, \cdot \rangle \) is not the Hermitian scalar product here. We further
assume that
\[
\int_0^{T^*} \left[ |b_s| + \|c_s\| + \int_{\mathbb{R}^d} \left( |x|^2 \wedge 1 \right) F_s(dx) \right] ds < \infty,
\]
where \( \|\cdot\| \) denotes any norm on the set of \( d \times d \)-matrices.

Since we shall consider exponentials of stochastic integrals with respect to the
process \( L \), we will need a priori an appropriate exponential moment condition

**Assumption 2.1.** \((\text{EM})\) There are constants \( M, \epsilon > 0 \), such that, for every
\( u \in \left[ -(1 + \epsilon)M, (1 + \epsilon)M \right]^d \),
we have
\[
\int_0^{T^*} \int_{\{ |x| > 1 \}} \exp(\langle u, x \rangle) F_s(dx) ds < \infty.
\]

Assumption \((\text{EM})\) is equivalent to \( \mathbb{E}[\exp(\langle u, L_t \rangle)] < \infty \) for all \( t \in [0, T^*) \) and
\( u \in \left[ -(1 + \epsilon)M, (1 + \epsilon)M \right]^d \). A direct consequence of \((\text{EM})\) is that in particular
the expectation of \( L_t \) is finite. In this situation one does not need a truncation
function in the Lévy–Khintchine representation of its characteristic function.
Consequently, the law of \( L_t \) is determined by the characteristic function in the form
\[
\mathbb{E}[e^{i\langle u, L_t \rangle}] = \exp \left( \int_0^t \left[ i\langle u, b_s \rangle - \frac{1}{2}\langle u, c_s u \rangle + \int_{\mathbb{R}^d} \left( e^{i\langle u, x \rangle} - 1 - i\langle u, x \rangle \right) F_s(dx) \right] ds \right) \quad (u \in \mathbb{R}^d).
\]

Likewise the canonical representation of the process \( L \) can be written in the simple form
\[
L_t = \int_0^t b_s \, ds + \int_0^t \sqrt{c_s} \, dW_s + \int_0^t \int_{\mathbb{R}^d} \left( \mu^L - \nu \right)(ds, dx),
\]
i.e. without considering an additional term for the big jumps. Here \( W_t = (W_t)_{t \in [0, T^*)} \) is a standard \( d \)-dimensional Brownian motion (Wiener process),
\( \sqrt{c_s} \) is a measurable version of the square root of \( c_s \), and \( \mu_L \) is the random measure of jumps of \( L \) with compensator \( \nu(ds, dx) = F_s(dx)ds \) (cf. Jacod and Shiryaev (2003, Corollary II.2.38)). The (extended) cumulant process associated with the process \( L \) is denoted by \( \theta_s \) and given by

\[
\theta_s(z) = \langle z, b_s \rangle + \frac{1}{2} \langle z, c_s z \rangle + \int_{\mathbb{R}^d} \left( e^{\langle z, x \rangle} - 1 - \langle z, x \rangle \right) F_s(dx)
\]

for every \( z \in \mathbb{C}^d \) where this function is defined. Kallsen and Shiryaev (2002) provide a detailed analysis of the cumulant process for semimartingales. Note that if \( L \) is a (homogeneous) Lévy process, i.e. if the increments of \( L \) are stationary, the triplet \( (b_s, c_s, F_s) \) and thus \( \theta_s \) do not depend on \( s \). In this case, we write \( \theta \) for short. It then equals the cumulant (also called log moment generating function) of \( L_1 \). Observe that the cumulant process is related to a specific measure. Which measure is meant in the following can unambiguously be seen from the notation.

3. The Multiple-Curve Lévy Forward Rate Model

Let us consider a complete stochastic basis \( \hat{\mathcal{B}} := (\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t)_{t \in [0, T^*]}, \hat{\mathcal{P}}) \) and a \( d \)-dimensional time-inhomogeneous Lévy process \( L = (L^1, \ldots, L^d)^T \) defined on \( \hat{\mathcal{B}} \). In the multiple-curve setting we will consider a discount curve which we will denote with the index \( d \) (later sometimes written as 0) and \( m \) different term structures of interest rates for \( m \in \mathbb{N} = \{1, 2, 3, \ldots\} \). Note that this framework contains the single-curve approach by setting \( m = 0 \).

3.1. The Discount Curve. Let us begin with the specification of the discount curve. To this end, we consider an arbitrary equidistant discrete tenor structure \( \mathcal{T}_d := \{T_0, \ldots, T_n\} \), where \( T_0 < \cdots < T_n \) and \( T_n = T^* \). We denote by \( \delta := \delta(T_{k-1}, T_k) \) the year fraction between the dates \( T_{k-1} \) and \( T_k \). \( \delta \) is assumed to be independent of \( k \) and is usually called the tenor of \( \mathcal{T}_d \). The time-\( t \) price of a risk-free zero-coupon bond maturing at date \( T \in [0, T^*] \) is denoted by \( B^d_t(T) \) (discount bond).

We denote by \( \mathcal{P} \) the predictable \( \sigma \)-field, which is the \( \sigma \)-field on \( \hat{\Omega} \times [0, T^*] \) generated by all adapted processes with left-continuous paths (cf. Jacod and Shiryaev (2003, Definition I.2.1)). The optional \( \sigma \)-field \( \mathcal{O} \) is defined as the \( \sigma \)-field on \( \hat{\Omega} \times [0, T^*] \) which is generated by all adapted processes with càdlàg paths (cf. Jacod and Shiryaev (2003, Definition I.1.20)).

The following two ingredients are needed to develop the model for the discount curve.

(D.1) The initial discount curve \( B^d_0 \) defined by

\[
B^d_0 : [0, T^*] \to (0, \infty) \quad T \mapsto B^d_0(T)
\]

is given.

The initial discount curve can be derived from market data by using an appropriate bootstrapping technique. A general explanation of such a technique is given in Hull (2012) and Ametrano and Bianchetti (2013). One typically takes the OIS-zero-coupon bond price as an approximation of \( B^d_0 \).
We consider a drift function \( \alpha^d \) and a volatility structure \( \sigma^d \) defined by
\[
\begin{aligned}
\alpha^d : \hat{\Omega} \times [0, T^*] \times [0, T^*] &\to \mathbb{R} \\
(\hat{\omega}, s, T) &\mapsto \alpha^d(\hat{\omega}, s, T)
\end{aligned}
\]
and
\[
\begin{aligned}
\sigma^d : \hat{\Omega} \times [0, T^*] \times [0, T^*] &\to \mathbb{R}^d \\
(\hat{\omega}, s, T) &\mapsto \sigma^d(\hat{\omega}, s, T) = (\sigma^d_1(\hat{\omega}, s, T), \ldots, \sigma^d_d(\hat{\omega}, s, T))
\end{aligned}
\]
which satisfy the usual measurability and boundedness conditions (cf. Eberlein, Jacod, and Raible (2005))

(a) \( \alpha^d \) and \( \sigma^d = (\sigma^d_1, \ldots, \sigma^d_d) \) are measurable with respect to \( \mathcal{P} \otimes \mathcal{B}([0, T^*]) \).

(b) The random functions are bounded for each \( \hat{\omega} \in \hat{\Omega} \) in the sense of \( \sup_{0 \leq s, T \leq T^*} (|\alpha^d(\hat{\omega}, s, T)| + |\sigma^d(\hat{\omega}, s, T)|) < \infty \).

(c) For every \( (\hat{\omega}, s, T) \in \hat{\Omega} \times [0, T^*] \times [0, T^*] \) with \( T < s \), we have \( \alpha^d(\hat{\omega}, s, T) = 0 \) and \( \sigma^d(\hat{\omega}, s, T) = (0, \ldots, 0) \).

Now let us postulate that, for every fixed maturity \( T \in [0, T^*] \), the dynamics of the discount instantaneous forward rate \( f^d(T) = f^d_t(T) \) is given by
\[
f^d_t(T) = f^d_0(T) + \int_0^t \alpha^d(s, T)ds - \int_0^t \sigma^d(s, T)dL_s. \tag{3.1}
\]
The initial values \( f^d_0(T) \) are assumed to be deterministic and bounded. We also require that they form a measurable mapping
\[
[0, T^*] \ni T \mapsto f^d_0(T) \in \mathbb{R}.
\]

It is shown by Eberlein, Jacod, and Raible (2005) that under these assumptions we can find a joint-version of all \( f^d_t(T) \) such that the map \((\hat{\omega}, t, T) \mapsto f^d_t(\hat{\omega}, T)1_{t \leq T} \) is \( \mathcal{P} \otimes \mathcal{B}([0, T^*]) \)-measurable. By applying Fubini’s Theorem (cf. Protter (2004, Theorem 64)) it follows from the forward rate dynamics (3.1) that the bond price \( B^d_t(T) \) at time \( t \) given by
\[
B^d_t(T) = \exp \left( - \int_0^T f^d_t(u)du \right)
\]
can be expressed as
\[
B^d_t(T) = B^d_0(T) \exp \left( \int_0^t (r^d_s - A^d(s, T))ds + \int_0^t \Sigma^d(s, T)dL_s \right), \tag{3.2}
\]
where the short rate \( r^d_t \) is specified by \( r^d_t = f^d_t(t) \) and where we have set
\[
A^d(s, T) := \int_{s \wedge T}^T \alpha^d(s, u)du \quad \text{and} \quad \Sigma^d(s, T) := \int_{s \wedge T}^T \sigma^d(s, u)du \tag{3.3}
\]
(cf. Eberlein and Kluge (2006)). Note that the integral \( \int_{s \wedge T}^T \sigma^d(s, u)du \) is understood componentwise and the initial values \( f^d_0(T) \) can be obtained from the
relation
\[ f_0^d(T) = -\frac{\partial \ln B_0^d(T)}{\partial T} \]
if this derivative exists.

The discount factor process \( \beta^d = (\beta^d_t)_{t \in [0,T^*]} \) given by
\[ \beta^d_t = \exp \left( -\int_0^t r^d_s ds \right) \]
is an adapted process with continuous paths. It can obviously be written as
\[ \beta^d_t = B_0^d(t) \exp \left( -\int_0^t A^d(s,t) ds + \int_0^t \Sigma^d(s,t) dL_s \right). \]  
(3.4)

From (3.2) together with (3.4) we can easily get another representation which turns out to be useful
\[ B_0^d(T) = B_0^d(t) \exp \left( \int_0^t (A^d(s,t) - A^d(s,T)) ds + \int_0^t \Sigma^d(s,t) dL_s \right). \]  
(3.5)

Here we have set \( \Sigma^d(s,t,T) := \Sigma^d(s,T) - \Sigma^d(s,t) \). To get a tractable model and guarantee the existence of all related functions, we require in the following the model to be based on a deterministic, bounded and continuous volatility structure.

**Assumption 3.1.** (DET) The volatility structure \( \sigma^d \) is a deterministic and bounded function such that, for every \( s \) and \( T \) with \( 0 \leq s, T \leq T^* \)
\[ 0 \leq \Sigma_j^d(s,T) \leq \hat{M} < M, \quad \text{for every } j \in \{1, \ldots, d\}, \]
where \( \Sigma_j^d \) arises from definition (3.3) \( (j^{\text{th}} \text{ component of } \Sigma^d) \) and the constant \( M \) is from assumption (EM). Moreover, the mapping
\[ [0, T^*] \ni s \mapsto \sigma^d(s,T) \in \mathbb{R}^d \]
is continuous for each given \( T \in [0, T^*] \).

The market will be free of arbitrage if the drift function \( A^d \) is defined in such a way that the discounted bond price processes \( Z^d(T) = (Z^d_t(T))_{0 \leq t \leq T} \) given by \( Z^d_t(T) := \beta^d_t B^d_t(T) \) are martingales. We will repeatedly use the fact (see Eberlein and Raible (1999, Lemma 3.1)) that expectations of exponentials of stochastic integrals can be explicitly computed, namely
\[ \mathbb{E}_{\mathcal{F}_t^d} \left[ \exp \left( \int_0^t \Sigma^d(s,T) dL_s \right) \right] = \exp \left( \int_0^t \theta_s(\Sigma^d(s,T)) ds \right). \]  
(3.6)

Note that the integral process in the exponent on the left side has independent increments. Using this fact, it is easy to show that the process \( M = (M_t)_{t \in [0,T]} \)
with

\[ M_t = \frac{\exp\left( \int_0^t \Sigma^d(s, T) dL_s \right)}{E_{\mathcal{P}} \left[ \exp\left( \int_0^t \Sigma^d(s, T) dL_s \right) \right]} \]

is a martingale. Combining this observation with (3.6), one sees that the drift assumption

\[ A^d(s, T) = \theta_s(\Sigma^d(s, T)) \]

guarantees that the discounted bond price processes \( Z^d(T) \) are martingales for all \( T \in [0, T^*] \). Notice that this approach for the discount curve (following the Lévy forward rate model of Eberlein and Kluge (2006)) works directly under the risk-neutral measure. This choice of the drift term is closely related to the notion of the exponential compensator (see Kallsen and Shiryaev (2002) and Jacod and Shiryaev (2003, Section II.8)). We repeat that the drift function \( A^d \) and therefore also \( \alpha^d \) as well as the volatility structure \( \Sigma^d \) (and \( \sigma^d \)) are deterministic in this setting.

It follows that the bond price process (3.2) is given in the more specific form

\[ B^d_t(T) = \frac{B^d_0(T)}{B^d_0(t)} \exp\left( - \int_0^t \theta_s(\Sigma^d(s, T)) ds + \int_0^t \Sigma^d(s, T) dL_s \right) \]

from which we conclude that

\[ \frac{Z^d_t(T)}{Z^d_0(T)} = \exp\left( - \int_0^t \theta_s(\Sigma^d(s, T)) ds + \int_0^t \Sigma^d(s, T) dL_s \right). \quad (3.7) \]

By applying Jacod and Shiryaev (2003, Theorem II.8.10) we can express the latter process also as a stochastic exponential

\[ \frac{Z^d_t(T)}{Z^d_0(T)} = \delta_t(\hat{Y}^d), \]

where the process \( \hat{Y}^d \) is specified by the stochastic logarithm \( \hat{Y}^d = \mathcal{L}(\exp(Y^d)) \) with

\[ Y^d_t = - \int_0^t \theta_s(\Sigma^d(s, T)) ds + \int_0^t \Sigma^d(s, T) dL_s. \]

In other words, the discounted bond price process \( Z^d(T) \) is the solution of the stochastic differential equation

\[ dZ^d(T) = Z^d(T) d\hat{Y}^d. \]

The explicit form of the process \( \hat{Y}^d \) is

\[ \hat{Y}^d_t = \int_0^t \Sigma^d(s, T) \sqrt{c_s} dW_s + \int_0^t \int_{\mathbb{R}^d} (e^{\Sigma^d(s, T, x)} - 1)(\mu^L_s - \nu)(ds, dx). \]

This representation confirms once again that \( Z^d(T) \) is a \( \mathcal{P}_d \)-(local) martingale (cf. Kallsen and Shiryaev (2002)).
To end this subsection, we introduce some useful definitions. We observe that the representation (3.5) results in
\[
B_t^d(T) = B_0^d(T) \exp \left( \int_0^t \left[ \theta_s(\Sigma^d(s, t)) - \theta_s(\Sigma^d(s, T)) \right] ds \\
+ \int_0^t \Sigma^d(s, t, T) dL_s \right).
\]
(3.8)

For fixed \(t, T \in [0, T^*]\) with \(t \leq T\), the last expression is decomposed into its deterministic part
\[
D^d(t, T) := B_0^d(T) \exp \left( \int_0^t \left[ \theta_s(\Sigma^d(s, t)) - \theta_s(\Sigma^d(s, T)) \right] ds \right)
\]
and its stochastic part given as the exponential of the \(\mathcal{F}_t\)-measurable random variable
\[
X^d(t, T) := \int_0^t \Sigma^d(s, t, T) dL_s.
\]

Hence, we obtain the compact form
\[
B_t^d(T) = D^d(t, T) \exp(X^d(t, T)).
\]

3.2. Multiple Curves. Now we address the risky curves. We will consider \(m\) different curves. Since each term structure corresponds to a discrete tenor structure, we introduce the equidistant tenor structure \(T_k := \{T_{k0}, ..., T_{knk}\}\) for every \(k \in \{1, ..., m\}\) and \(n_k \in \mathbb{N}\). We assume that \(T^* = T_0^* = T^*_n = T^*_n\) for all \(k \in \{1, ..., m\}\). The year fraction between the dates \(T_{j-1}^k\) and \(T_j^k\) is denoted by \(\delta^k := \delta^k(T_{j-1}^k, T_j^k)\) for \(j \in \{1, ..., n_k\}\). We call \(\delta^k\) the tenor of \(\mathcal{T}_k\).

Moreover, for all \(k, l \in \{1, ..., m\}\) with \(k \leq l\), we postulate
\[
\mathcal{T}_l \subset \mathcal{T}_k \subset \mathcal{T}^d \subset [0, T^*].
\]
(3.9)

This assumption means that the curves will reflect liquidity and credit risk in decreasing order of magnitude (\(\delta^k < \delta^l\)). For every \(k \in \{1, ..., m\}\) we interpret \(B_t^k(T)\) as time-\(t\) price of a fictitious risky zero-coupon bond with maturity \(T\) that corresponds to curve \(k\).

We need two ingredients to model the multiple curves.

\((MC.1)\) The initial multiple term structure curves \(B_0^1, ..., B_0^m\):

\[
B_0^k : [0, T^*] \to (0, \infty) \\
T \mapsto B_0^k(T)
\]

are given for every \(k \in \{1, ..., m\}\).

Ametrano and Bianchetti (2009, 2013) developed a bootstrapping method dealing with this multiple-curve setting. By using this bootstrapping procedure, the initial values typically satisfy
\[
B_0^l(T) \leq B_0^k(T) \leq B_0^d(T)
\]
for every \( k, l \in \{1, \ldots, m\} \) with \( k \leq l \) and \( T \in [0, T^*] \).

\((\text{MC}.2)\) For every \( k \in \{1, \ldots, m\} \), we consider the drift function \( \alpha^k \) and the volatility structure \( \sigma^k \) defined by

\[
\alpha^k : \hat{\Omega} \times [0, T^*] \times [0, T^*] \to \mathbb{R} \\
(\hat{\omega}, s, T) \mapsto \alpha^k(\hat{\omega}, s, T)
\]

and

\[
\sigma^k : \hat{\Omega} \times [0, T^*] \times [0, T^*] \to \mathbb{R}^d \\
(\hat{\omega}, s, T) \mapsto \sigma^k(\hat{\omega}, s, T) = (\sigma^k_1(\hat{\omega}, s, T), \ldots, \sigma^k_d(\hat{\omega}, s, T))
\]

which satisfy the same (measurability and boundedness) conditions as \( \alpha^d \) and \( \sigma^d \) in \((\text{D}.2)\).

For every \( k \in \{1, \ldots, m\} \) and \( T \in [0, T^*] \), the dynamics of the instantaneous forward rates \( f^k(T) = (f^k_t(T))_{t \in [0, T]} \) are postulated to be

\[
f^k_t(T) = f^k_0(T) + \int_0^t \alpha^k(s, T)ds - \int_0^t \sigma^k(s, T)dL_s,
\]

where the initial values \( f^k_0(T) \) are assumed to be deterministic, bounded and measurable in \( T \). Those values can be determined by the formula

\[
f^k_0(T) = -\frac{\partial \ln B^k_0(T)}{\partial T}
\]

if the derivative exists. In the same way as one gets representation \((3.2)\), we obtain the form

\[
B^k_t(T) = B^k_0(T) \exp \left( \int_0^t (r^k_s - A^k(s, T))ds + \int_0^t \Sigma^k(s, T)dL_s \right)
\]

from the relation

\[
B^k_t(T) = \exp \left( -\int_0^T f^k_\tau(\tau)d\tau \right)
\]

for each \( k \in \{1, \ldots, m\} \). The rate \( r^k_t \) at \( t \) is given by \( r^k_t = f^k_t(t) \) and we similarly define

\[
A^k(s, T) := \int_0^{T \wedge T} \alpha^k(s, u)du \quad \text{and} \quad \Sigma^k(s, T) := \int_0^{T \wedge T} \sigma^k(s, u)du.
\]

To ensure the existence of the cumulant process, we need the following

**Assumption 3.2.** \((\text{MC}.\text{DET})\) For any \( k \in \{1, \ldots, m\} \) and all \( s, T \in [0, T^*] \) the volatility structure \( \sigma^k \) is deterministic and bounded in the sense of

\[
0 \leq \Sigma^k_j(s, T) \leq \hat{M} < M, \text{ for every } j \in \{1, \ldots, d\}.
\]

The mapping \([0, T^*] \ni s \mapsto \sigma^k(s, T) \in \mathbb{R}^d\) is continuous. As usual, \( M \) is the constant from assumption \((\text{EM})\).
Note that the constant $\hat{M}$ in this assumption does not have to coincide with the constant $\hat{M}$ from assumption (DET). The discounted bond price process $Z^k_t(T) = (Z^k_t(T))_{0 \leq t \leq T}$ corresponding to curve $k$ is defined by $Z^k_t(T) := \beta^d t B^k_t(T)$ for each date $T \in [0, T^*]$. One easily verifies that
\[
\frac{Z^k_t(T)}{Z^0_t(T)} = \exp \left( \int_0^t [r^k_s - r^d_s - A^k(s, T)] ds + \int_0^t \Sigma^k(s, T) dL_s \right).
\] (3.11)

By a further application of Jacod and Shiryaev (2003, Theorem II.8.10) we can represent this as a stochastic exponential $Z^k_t(T) = \mathcal{E}_t(\bar{Y}^k)$, where the process $\bar{Y}^k$ is the stochastic logarithm $\bar{Y}^k = \mathcal{L}(\exp(Y^k))$ with
\[
Y^k_t = \int_0^t (r^k_s - r^d_s - A^k(s, T)) ds + \int_0^t \Sigma^k(s, T) dL_s.
\]
This means that the discounted bond price process $Z^k(t)$ is the solution of
\[
dZ^k(t) = Z^k(t) d\bar{Y}^k,
\]
where the process $\bar{Y}^k = (Y^k_t)_{0 \leq t \leq T}$ is explicitly given by
\[
\bar{Y}^k_t = \int_0^t [r^k_s - r^d_s - A^k(s, T) + \theta_s(\Sigma^k(s, T))] ds + \int_0^t \Sigma^k(s, T) \sqrt{\sigma_x} dW_s
\]
\[+ \int_0^t \int_{\mathbb{R}^x} \left( e^{\Sigma^k(s, T), x} - 1 \right) (\mu^L - \nu)(ds, dx).
\]

The last expression shows that $Z^k(T)$ is not a $\hat{P}^d$-(local) martingale in general. Similarly as in (3.5), we rewrite the expression (3.10) in the form
\[
B^k_t(T) = \frac{B^k_0(T)}{B^0_0(t)} \exp \left( \int_0^t [A^k(s, t) - A^k(s, T)] ds + \int_0^t \Sigma^k(s, t, T) dL_s \right),
\] (3.12)
where we have set
\[
\Sigma^k(s, t, T) := \Sigma^k(s, T) - \Sigma^k(s, t).
\]
To simplify the notation, for fixed $t, T \in [0, T^*]$ with $t \leq T$, we define the factor
\[
D^k(t, T) := \frac{B^k_0(T)}{B^0_0(t)} \exp \left( \int_0^t [A^k(s, t) - A^k(s, T)] ds \right)
\]
and the $\hat{F}_t$-measurable random variable
\[
X^k(t, T) := \int_0^t \Sigma^k(s, t, T) dL_s.
\]
Then, we obtain
\[ B^k_t(T) = D^k(t, T) \exp(X^k(t, T)), \]
where at this stage \( D^k(t, T) \) could be random.

Now, we will specify the drift function \( A^k \) in such a manner that credit and liquidity risk are taken into account. We follow the line of thought of Crépey, Grbac, and Nguyen (2012, Section 2.3.2). Their idea is based on no-arbitrage requirements in defaultable HJM-models with an additional liquidity component which lead to the required drift condition.

Let us temporarily assume that defaultable bonds with respect to each curve can be traded in the market. The time-\( t \) price of such a bond maturing at \( T \) is denoted by \( \bar{B}^k_t(T) \). Keep in mind that such bonds are actually not traded in the market but can be considered as bonds that are issued by an average Libor or Euribor panel member. In fact, they are rather mathematical concepts which represent the credit risk of the panel bank members but are not defaultable in the classical sense.

Hereinafter, we specify the intensity-based credit risk model and construct default times \( \tau^1, \ldots, \tau^m \). To this end, we need to enlarge the initial stochastic basis \( \tilde{\mathcal{B}} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) \) as follows. Let \( \Gamma^1, \ldots, \Gamma^m \) be real-valued, \( \mathbb{F} \)-adapted, continuous and increasing stochastic processes defined on \( \tilde{\mathcal{B}} \). It is assumed that \( \Gamma^k_0 = 0 \) and \( \Gamma^k_\infty := \lim_{t \rightarrow \infty} \Gamma^k_t = \infty \) for every \( k \in \{1, \ldots, m\} \).

We additionally consider an auxiliary probability space \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) \) endowed with a family of independent random variables \( \xi_1, \ldots, \xi_m \) that are uniformly distributed on the interval \([0, 1]\). We state the product space
\[ (\Omega, \mathcal{G}, P^d) := (\tilde{\Omega} \times \tilde{\Omega}, \tilde{\mathcal{F}} \otimes \tilde{\mathcal{F}}, \tilde{\mathbb{P}} \otimes \tilde{\mathbb{P}}) \]
and denote by \( \mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]} \) the trivial extension of \( \tilde{\mathbb{P}} \) to the enlarged probability space \( (\Omega, \mathcal{G}, P^d) \). This means that each \( A \in \mathcal{G}_t \) is of the form \( A \times \tilde{\Omega} \) for some \( A \in \tilde{\mathcal{F}}_t \). Observe that \( \mathbb{F} \) is right-continuous and denotes the reference filtration here. All the random variables (functions) and stochastic processes defined on \( \tilde{\mathcal{B}} \) or \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) \) are extended to the enlarged filtered probability space \( (\Omega, \mathcal{G}, \mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P^d) \) in the usual canonical way. We retain their names when we consider them on this complete stochastic basis to avoid unnecessary and confusing notation (cf. Eberlein and Özkan (2003) and Kluge (2005, section 4.2)). Observe that each \( (\tilde{\mathbb{F}}, P^d) \)-(local) martingale is also a \( (\mathbb{F}, P^d) \)-(local) martingale.

For every \( k \in \{1, \ldots, m\} \), let us define a random time \( \tau^k : \Omega \rightarrow \mathbb{R}_+ \) on \( (\Omega, \mathcal{G}, P^d) \) by setting
\[ \tau^k := \inf\{t \in \mathbb{R}_+ \mid e^{-\Gamma^k_t} \leq \xi_k\} = \inf\{t \in \mathbb{R}_+ \mid \Gamma^k_t \geq \eta_k\}, \]
where the random variable \( \eta_k := -\ln \xi_k \) is exponentially distributed with mean one under \( P^d \). Obviously, \( \eta_1, \ldots, \eta_m \) is a family of independent random variables. For every \( k \in \{1, \ldots, m\} \), we denote by \( \mathcal{H}^k = (\mathcal{H}^k_t)_{t \in \mathbb{R}_+} \) the right-continuous filtration generated by the default process \( H^k = (H^k_t)_{t \in \mathbb{R}_+} \), that is defined by \( H^k_t = \mathbb{1}_{\{\tau^k \leq t\}} \). More precisely we have \( \mathcal{H}^k_t = \sigma(H^k_u : 0 \leq u \leq t) = \sigma(\{\tau^k \leq u\} : 0 \leq u \leq t) \). We postulate that each random time \( \tau^k \) possesses
a $\mathbb{F}$-intensity $\gamma^k$, i.e., $\Gamma^k_t = \int_0^t \gamma^k_s ds$ for any $t \geq 0$ and some non-negative, $\mathbb{F}$-progressively measurable stochastic process $\gamma^k$ with integrable sample paths (see Bielecki and Rutkowski (2002, chapter 5 and 8)).

Further, we assume that the defaultable bonds pay a certain recovery upon default. This recovery payment is specified by the terminal recovery process $R^k = (R^k_t)_{t \in [0, T^*]}$ for every curve $k \in \{1, \ldots, m\}$. The process $R^k$ is $\mathbb{F}$-adapted and (locally) bounded on $(\hat{\Omega}, \mathcal{G}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$ (cf. Bielecki and Rutkowski (2002, Section 13.1.9)). In financial interpretation, the amount $R^k_t \in \mathbb{R}$ is the recovery payment made at maturity $T$ if the default of the bond issuer occurs at time $\tau^k \leq T$. More specifically, the value of the defaultable bond price $\bar{B}^k_t(T)$ at time $T$ is given by

\[
\bar{B}^k_T(T) = 1_{\{\tau^k > T\}} B^k_T(T) + B^k_T(T) R^k_{\tau^k} 1_{\{\tau^k \leq T\}}.
\]

Then, the time-$t$ price results in

\[
\bar{B}^k_t(T) = 1_{\{\tau^k > t\}} B^k_t(T) + B^k_t(T) R^k_{\tau^k} 1_{\{\tau^k \leq t\}}
\]

and we obtain its discounted value $\bar{Z}^k_t(T) := \beta^k T^k(T)$ as

\[
\bar{Z}^k_t(T) = 1_{\{\tau^k > t\}} Z^k_t(T) + R^k_{\tau^k} 1_{\{\tau^k \leq t\}} Z^k_t(T).
\]

Consequently, the time-$t$ bond price $B^k_t(T)$ is interpreted as the pre-default price of the associated defaultable zero-coupon bond.

As mentioned above, we require that the process $\bar{Z}^k(T) = (\bar{Z}^k_t(T))_{t \in [0, T]}$ to be a local martingale. Note that, in general, the default times $\tau^1, \ldots, \tau^m$ are not stopping times with respect to the reference filtration $\mathbb{F}$. Therefore, we need to enlarge $\mathbb{F}$. Unfortunately, the filtration $\mathbb{G} := (\mathcal{G}_t)_{t \in [0, T^*]}$ induced by $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}^1_t \vee \cdots \vee \mathcal{H}^m_t := \sigma(\mathcal{F}_t, \mathcal{H}^1_t, \ldots, \mathcal{H}^m_t)$ does not have to be right-continuous (cf. Song (2013)). Therefore, we endow the probability space $(\Omega, \mathcal{G}, \mathbb{P}^d)$ with the filtration $\mathbb{G} = (\mathcal{G}_t)_{t \in [0, T^*]}$ given by

\[
\mathcal{G}_t := \bigcap_{s > t} \tilde{\mathcal{G}}_s, \quad \text{for any } t \in [0, T^*].
\]

This filtration trivially satisfies the right-continuity. Since the stochastic basis $(\Omega, \mathcal{G}, \mathbb{P}, \mathbb{P}^d)$ is complete (see also Jacod and Shiryaev (2003, §1a. 1.4)) it easily follows that the enlarged stochastic basis $(\Omega, \mathcal{G}, \mathbb{G} = (\mathcal{G}_t)_{t \in [0, T^*]}, \mathbb{P}^d)$ is also complete. We conclude that $\mathbb{G}$ is specified as the smallest enlargement of $\mathbb{F}$ containing $\tilde{\mathcal{G}}$. It can be shown that the martingale invariance property (see Brémaud and Yor (1978), Dellacherie and Meyer (1978) and Bielecki and Rutkowski (2002)) is satisfied in this framework.

The following theorem states the conditions which ensure the absence of arbitrage. To be precise, we derive conditions such that for every $k \in \{1, \ldots, m\}$, the discounted defaultable bond price processes $\bar{Z}^k(T)$ are $(\mathbb{G}, \mathbb{P}^d)$-local martingales for each $T \in [0, T^*]$.

**Theorem 3.3.** Assume that, for each $k \in \{1, \ldots, m\}$ and $T \in [0, T^*]$, the condition

\[
\bar{Z}^k_{t-}(T) \left[ \lambda^k_t - A^k(t, T) + \theta_t(\Sigma^k(t, T)) \right] = (\bar{Z}^k_{t-}(T) - R^k_t \bar{Z}^d_t(T)) \gamma^k_t
\]

(3.13)
is satisfied for all $t \in [0, T]$, where we set $\lambda_t^{k,d} := r_t^k - r_t^d$. Then, for each $k \in \{1, \ldots, m\}$ and $T \in [0, T^*]$, the process $\tilde{Z}^k(T)$ is a $(\mathcal{G}, P^d)$-local martingale.

**Proof.** For every $k \in \{1, \ldots, m\}$ and $T \in [0, T^*]$, we obviously have that $H^k = \mathbf{1}_{\{r^k \leq \cdot\}}$ as well as $1 - H^k = \mathbf{1}_{\{r^k > \cdot\}}$ are finite variation processes and $Z^k(T)$ is a semimartingale. Let us define by $\hat{H}_t^k := R_{t,r^k}^k H_t^k$ a $\mathbb{G}$-adapted finite variation process $H^k := (\hat{H}_t^k)_{t \in [0, T^*]}$. Then, by Jacod and Shiryaev (2003, Proposition I.4.49), we obtain that

$$
\tilde{Z}_t^k(T) = \int_0^t (1 - H_{s-}^k) dZ_{s-}^k(T) + \int_0^t Z_s^k(T) d(1 - H_s^k) + \tilde{Z}_0^k(T)
$$

$$
= \int_0^t (1 - H_{s-}^k) dZ_{s-}^k(T) - \int_0^t Z_s^k(T) dH_s^k + \tilde{Z}_0^k(T)
$$

$$
+ \int_0^t \hat{H}_{s-}^k dZ_s^d(T) + \int_0^t R_s^k Z_s^d(T) dH_s^k.
$$

One gets

$$
\tilde{Z}_t^k(T) = \int_0^t (1 - H_{s-}^k) Z_{s-}^k(T) \left[ \lambda_s^{kd} - A^k(s, T) + \theta_s(\Sigma^k(s, T)) \right] ds + \tilde{Z}_0^k(T)
$$

$$
+ \int_0^t (1 - H_{s-}^k) Z_s^k(T) \Sigma^k(s, T) \sqrt{c_s} dW_s
$$

$$
+ \int_0^t \int_{\mathbb{R}^d} (1 - H_{s-}^k) Z_{s-}^k(T) \left( e^{(\Sigma^k(s, T), x)} - 1 \right) (\mu^L - \nu)(ds, dx)
$$

$$
+ \int_0^t \hat{H}_{s-}^k dZ_s^d(T) + \int_0^t (R_s^k Z_s^d(T) - Z_s^k(T)) dH_s^k.
$$

Since condition (3.13) is assumed to be satisfied, we have

$$
\int_0^t (1 - H_s^k) Z_{s-}^k(T) \left[ \lambda_s^{kd} - A^k(s, T) + \theta_s(\Sigma^k(s, T)) \right] ds
$$

$$
= \int_0^t (1 - H_s^k) (Z_s^k(T) - R_s^k Z_s^d(T)) \gamma_s^k ds
$$

$$
= \int_0^t (1 - H_s^k) (Z_s^k(T) - R_s^k Z_s^d(T)) \gamma_s^k ds
$$
A MULTIPLE-CURVE LÉVY FORWARD RATE MODEL IN A TWO-PRICE ECONOMY

and it follows that

$$\tilde{Z}_k^k(T) = \tilde{Z}_0^k(T) + \int_0^t (1 - H_{s-}^k) Z^k_{s-}(T) \Sigma^k(s, T) \sqrt{e_s} dW_s$$

$$+ \int_0^t \int_{\mathbb{R}^d} (1 - H_{s-}^k) Z^k_{s-}(T) \left( e^{(\Sigma^k(s, T) \cdot x)} - 1 \right) (\mu^L - \nu) (ds, dx)$$

$$+ \int_0^t \hat{H}_{s-}^k dZ_s^d(T) + \int_0^t (R_s^k Z_s^d(T) - Z_s^k(T)) dM_s^k, \quad (3.14)$$

where the $(G, P^d)$-martingale $M^k = (M_t^k)_{t \in [0, T^*]}$ is defined by

$$M_t^k = H_t^k - \int_0^t (1 - H_s^k) \gamma_s^k ds.$$ 

By taking into account the valid martingale invariance property, we observe that all the considered stochastic integrals in equation (3.14) have $(G, P^d)$-local martingales as integrators. Hence, we conclude that $\tilde{Z}^k(T)$ is a $(G, P^d)$-local martingale. \(\square\)

Let us assume that the terminal recovery process is of the form

$$R_t^k = R^k B_{t-}^k(T) B^d_t(T)^{-1},$$

where $R^k \in [0, 1)$. Note that this choice corresponds to the fractional recovery of market value (see Bielecki and Rutkowski (2002, section 1.1.1)). By easy computations, we obtain the following convenient form of condition (3.13). For each $k \in \{1, \ldots, m\}$ and $T \in [0, T^*]$,

$$\lambda_t^{k,d} - A^k(t, T) + \theta_t(\Sigma^k(t, T)) = (1 - R^k) \gamma_t^k \quad (3.15)$$

for every $t \in [0, T]$. One verifies that condition (3.15) can equivalently be formulated as

$$\lambda_t^{k,d} = (1 - R^k) \gamma_t^k$$

$$A^k(t, T) = \theta_t(\Sigma^k(t, T)). \quad (3.16)$$

Hence, the credit risk component of the model is given by equation (3.16).

Since the crisis was caused by a mixture of credit and liquidity risk (cf. Filipović and Trolle (2013) and Eberlein (2015)), we add an additional liquidity component to the pure credit risk factor $\theta_t(\Sigma^k(t, T))$ in (3.16). For this reason, we need another ingredient in the model.

(MC.3) For every $k \in \{1, \ldots, m\}$, we consider the liquidity component $l^k$ defined by

$$l^k : \begin{cases} [0, T^*] \times [0, T^*] \to \mathbb{R} \\ (t, T) \mapsto l^k(t, T) \end{cases}$$

which is assumed to be a bounded function.
Finally, we consider the drift function
\[ A_k(t, T) = \theta(t) \Sigma_k(t, T) + l_k(t, T) \tag{3.17} \]
for every \( k \in \{1, \ldots, m\} \). We stress that the drift function \( A_k \) (and \( \alpha_k \)) and the volatility function \( \Sigma_k \) (and \( \sigma_k \)) are deterministic functions. Therefore, the factor \( D_k(t, T) \) is deterministic.

4. Monotonicity of the Curves

Assumption (3.9) implies some kind of monotonicity of the curves. More specifically, bonds that are related to a riskier curve should have a lower price compared to bonds that correspond to a curve associated with less credit and liquidity risk. Figure 1 implicitly confirms this fact by means of market data.

4.1. The Monotonicity Condition. Recall that we generally have for every \( k, l \in \{1, \ldots, m\} \) with \( k \leq l \) and all \( T \in [0, T^*] \) the monotonicity of the initial curves

\[ B^k_0(T) \leq B^l_0(T) \leq B^d_0(T). \]

We have to design the model such that the monotonicity holds at any time. This means that we want to achieve

\[ B^k_t(T) \leq B^l_t(T) \leq B^d_t(T) \tag{4.1} \]

for every \( t, T \in [0, T^*] \) satisfying \( t \leq T \). This reflects the fact that the higher the risk is, the lower the price of the bond should be. The monotonicity will be guaranteed by additional restrictions on the model parameters. The inequalities (4.1) can obviously be achieved if

\[ f^k_t(T) \leq f^l_t(T) \leq f^d_t(T). \tag{4.2} \]

For every \( k, j \in \{d, 1, \ldots, m\} \) and \( T \in [0, T^*] \), we define the additive (forward) spread between the curves \( k \) and \( j \) by

\[ s^{k,j}_t(T) := f^k_t(T) - f^j_t(T). \tag{4.3} \]

One sees that the dynamics \( s^{k,j}(T) = (s^{k,j}_t(T))_{t \in [0, T]} \) are given by

\[ s^{k,j}_t(T) = s^{k,j}_0(T) + \int_0^t \alpha^{k,j}(s, T) ds - \int_0^t \sigma^{k,j}(s, T) dL_s, \tag{4.4} \]

where we set \( \alpha^{k,j}(s, T) := \alpha^k(s, T) - \alpha^j(s, T) \) and \( \sigma^{k,j}(s, T) := \sigma^k(s, T) - \sigma^j(s, T) \). Note that \( s^{k,k}_t(T) = 0 \) for every \( k \in \{d, 1, \ldots, m\} \) and we deduce from (4.2) that

\[ 0 \leq s^{k,d}_0(T) \leq s^{l,d}_0(T) \quad \text{and} \quad 0 \leq s^{l,k}_0(T) \]

for all \( k, l \in \{1, \ldots, m\} \) with \( k \leq l \) and \( T \in [0, T^*] \). For every \( k, j \in \{d, 1, \ldots, m\} \), the short term spread between \( k \) and \( j \) is defined by \( \lambda^{k,j}_t := r^k_t - r^j_t \) and we set

\[ A^{k,j}(s, T) := \int_{s \wedge T}^{T} \alpha^{k,j}(s, u) du \quad \text{and} \quad \Sigma^{k,j}(s, T) := \int_{s \wedge T}^{T} \sigma^{k,j}(s, u) du. \]
Then, we clearly have the relations
\[ A^{k,j}(s,T) = A^{k}(s,T) - A^{j}(s,T) \quad \text{and} \quad \Sigma^{k,j}(s,T) = \Sigma^{k}(s,T) - \Sigma^{j}(s,T). \]
For any \( k \in \{1, \ldots, m\} \), we obviously have
\[ s^{k,d}_t(T) = \sum_{j=1}^{k} s^{j-1}(t), \tag{4.5} \]
where we define \( s^{1,0}_t(T) := s^{1,d}_t(T) \). Therefore, we obtain
\[ f^{k}_t(T) = f^{d}_t(T) + \sum_{j=1}^{k} s^{j-1}_t(T). \]
Note that, for every \( k \in \{1, \ldots, m\} \), the short term spread \( \lambda^{k,d} \) is given by
\[ \lambda^{k,d}_t = \sum_{j=1}^{k} \lambda^{j-1}(t), \tag{4.6} \]
where for notational convenience we will write sometimes 0 instead of \( d \), i.e. \( \lambda^{1,0}_t = \lambda^{1,d}_t \). Moreover, one easily sees that the drift function and the volatility structure related to curve \( k \) can be represented by
\[ \alpha^{k}(s,T) = \alpha^{d}(s,T) + \sum_{j=1}^{k} \alpha^{j-1}(s,T) \]
and
\[ \sigma^{k}(s,T) = \sigma^{d}(s,T) + \sum_{j=1}^{k} \sigma^{j-1}(s,T), \]
where we denote as mentioned above \( \alpha^{1,0}(s,T) := \alpha^{1,d}(s,T) \) and \( \sigma^{1,0}(s,T) := \sigma^{1,d}(s,T) \). It follows that we have
\[ A^{k}(s,T) = A^{d}(s,T) + \sum_{j=1}^{k} A^{j-1}(s,T) \]
and
\[ \Sigma^{k}(s,T) = \Sigma^{d}(s,T) + \sum_{j=1}^{k} \Sigma^{j-1}(s,T), \]
where we define \( A^{1,0}(s,T) := A^{1,d}(s,T) \) and \( \Sigma^{1,0}(s,T) := \Sigma^{1,d}(s,T) \). Consequently, due to relation (4.3), we can specify the dynamics of the quantity \( f^{k}(T) \) by modelling the forward spreads \( s^{k,j}(T) \) and the forward rates \( f^{j}(T) \).
It is evident that the relation (4.2) is equivalent to the condition
\[ 0 \leq s^{k,d}_t(T) \leq s^{l,d}_t(T) \tag{4.7} \]
for every \( k, l \in \{1, \ldots, m\} \) with \( k \leq l \) and \( t, T \in [0, T^*] \) satisfying \( t \leq T \). Then, we conclude from representation (4.5) that condition (4.7) is valid if, for all \( j \in \{1, \ldots, m\} \) and \( t, T \in [0, T^*] \) with \( t \leq T \), we have
\[ 0 \leq s^{j-1}_t(T). \tag{4.8} \]
To sum up, this approach results in the non-negative specification of the forward spreads between two subsequent curves (4.8). Then, condition (4.7) is automatically satisfied. This fact implicitly guarantees the relations (4.2) and we therefore ensure the required monotonicity (4.1).

4.2. A Suitable Model Framework. Below, we present a tractable model which ensures the non-negativity of the consecutive forward spreads. Through the analysis made in the previous subsection, it follows that the monotonicity which ensures the non-negativity of the consecutive forward spreads. Through this framework is motivated by Crépey, Grbac, and Nguyen (2012). We emphasise that we specify the model quantities corresponding to each curve \( k \in \{1, \ldots, m\} \) by modelling the relevant spreads and the discount curve.

Let \( d, m, l \in \mathbb{N}^* = \{1, 2, 3, \ldots\} \) with \( l + m \leq d \). The \( d \)-dimensional driving process \( L = (L^1, \ldots, L^d)^T \) is given on the enlarged stochastic basis \((\Omega, \mathcal{G}, G, P_d^d)\). Its components are divided into \( l \) real-valued Lévy processes and \( d - l \) Lévy processes with negative values. More precisely, we specify the \( d \)-dimensional (time-homogeneous) Lévy process \( L \) as follows

1. \( Y^1 := (L^1, \ldots, L^l)^T \) is an \( \mathbb{R}^l \)-valued Lévy process.
2. \( Y^2 := (L^{l+1}, \ldots, L^{d})^T = (-Z^{l+1}, \ldots, -Z^d)^T \), where \( Z := (Z^{l+1}, \ldots, Z^d)^T = -Y^2 \) is an \( \mathbb{R}^{d-l}_+ \)-valued Lévy process whose components are subordinators (see Sato (1999, Definition 21.4.)). The cumulant process of \( Z \) is of the form
   \[ \theta^Z(z) = \langle z, b \rangle + \int_{\mathbb{R}^{d-l}_+} (e^{\langle z, x \rangle} - 1) F(dx), \]
   where \( z \in \mathbb{C}^{d-l} \) such that \( \Re(z) \in [-(1 + \epsilon)M, (1 + \epsilon)M]^{d-l} \). The drift term \( b \) satisfies \( b^j \geq 0 \) for any \( j \in \{1, \ldots, d - l\} \) and the Lévy measure \( F \) has its support on \( \mathbb{R}^{d-l}_+ \).

We make the following

**Assumption 4.1.** \((\text{VL})\) For every \( k \in \{1, \ldots, m\} \), the non-negative volatility functions \( \Sigma^d \) and \( \Sigma^{k,k-1} \) as well as the liquidity function \( l^k \) are deterministic, differentiable and stationary in the following sense: For every \( k \in \{1, \ldots, m\} \), \( j \in \{1, \ldots, d\} \) and \( s, T \) with \( 0 \leq s \leq T \leq T^* \), the functions are of the form

\[
\begin{align*}
\Sigma^d_j(s, T) &= G_j(T - s) \\
\Sigma^{k,k-1}_j(s, T) &= G^k_j(T - s) \\
l^k_j(s, T) &= G^l_j(T - s),
\end{align*}
\]

where \( G_j : [0, T^*] \to \mathbb{R}_+ \) and \( G^k_j : [0, T^*] \to \mathbb{R}_+ \) are differentiable functions satisfying \( G_j(0) = G^k_j(0) = 0 \) that are bounded in the sense of

\[
G_j(s) + \sum_{k=1}^m G^k_j(s) \leq M < M
\]

for all \( s \in [0, T^*] \). The function \( G^l_j : [0, T^*] \to \mathbb{R} \) is differentiable and bounded satisfying \( G^l_j(0) = 0 \).

It follows that the conditions (\text{DET}) and (\text{MC.DET}) are fulfilled under assumption (\text{VL}).
Proposition 4.2. The forward spread $s^1_t(T)$ can be written as
\[
s^1_t(T) = s^1_0(T) - \theta(\Sigma^d(t, T) + \Sigma^1_d(t, T)) + \theta(\Sigma^d(0, T) + \Sigma^1_d(0, T)) - \int_0^t \sigma^1_s(T) dL_s
\] (4.9)

and, in the case where $m \geq 2$, the forward spread $s^{j,j-1}_t(T)$ is given by
\[
s^{j,j-1}_t(T) = s^{j,j-1}_0(T) - \theta(\Sigma^d(t, T) + \sum_{i=1}^j \Sigma^{i,i-1}(t, T)) + \theta(\Sigma^d(0, T) + \sum_{i=1}^j \Sigma^{i,i-1}(0, T)) - \int_0^t \sigma^{j,j-1}_s(T) dL_s
\] (4.10)

for any $j \in \{2, \ldots, m\}$. The corresponding short term spreads result in
\[
\lambda^{1}_t = s^{1}_0(t) + \theta(\Sigma^d(0, t) + \Sigma^1_d(0, t)) + l^1(0, t) - \theta(\Sigma^d(0, t)) - \int_0^t \sigma^1_s(t) dL_s
\]

and
\[
\lambda^{j,j-1}_t = s^{j,j-1}_0(t) + \theta(\Sigma^d(0, t) + \sum_{i=1}^j \Sigma^{i,i-1}(0, t)) + l^{j,j-1}(0, t) - \theta(\Sigma^d(0, t) + \sum_{i=1}^{j-1} \Sigma^{i,i-1}(0, t)) + \int_0^t \sigma^{j,j-1}_s(t) dL_s.
\]

Proof. We observe that
\[
\frac{\partial}{\partial T} G_j(T - s) = -\frac{\partial}{\partial s} G_j(T - s)
\]
\[
\frac{\partial}{\partial T} G^{k}_j(T - s) = -\frac{\partial}{\partial s} G^{k}_j(T - s)
\]
\[
\frac{\partial}{\partial T} G^k(T - s) = -\frac{\partial}{\partial s} G^k(T - s).
\]

This implies
\[
\frac{\partial}{\partial T} \theta(\Sigma^d(s, T) + \Sigma^1_d(s, T)) + \frac{\partial}{\partial T} l^1(s, T) - \frac{\partial}{\partial T} \theta(\Sigma^d(s, T))
\]
from which we get

\[
\int_0^t \left[ \frac{\partial}{\partial T} \theta(\Sigma^d(s, T) + \Sigma^{1,d}(s, T)) + \frac{\partial}{\partial T} l^1(s, T) - \frac{\partial}{\partial T} \theta(\Sigma^d(s, T)) \right] ds
\]

\[
= - \int_0^t \left[ \frac{\partial}{\partial s} \theta(\Sigma^d(s, T) + \Sigma^{1,d}(s, T)) + \frac{\partial}{\partial s} l^1(s, T) - \frac{\partial}{\partial s} \theta(\Sigma^d(s, T)) \right] ds
\]

\[
= -\theta(\Sigma^d(t, T) + \Sigma^{1,d}(t, T)) + \theta(\Sigma^d(0, T) + \Sigma^{1,d}(0, T)) - l^1(t, T) + l^1(0, T)
\]

\[
+ \theta(\Sigma^d(t, T)) - \theta(\Sigma^d(0, T)).
\]

Equation (4.9) follows now if we combine the representation of \(s_1^{1,d}(T)\) in (4.4) with the choice of the drift coefficient according to (3.17). Analogously, one shows that

\[
\frac{\partial}{\partial T} \theta(\Sigma^d(s, T) + \sum_{i=1}^j \Sigma^{i,i-1}(s, T)) + \frac{\partial}{\partial T} l^{j-1}(s, T)
\]

\[
- \frac{\partial}{\partial T} \theta(\Sigma^d(s, T) + \sum_{i=1}^{j-1} \Sigma^{i,i-1}(s, T))
\]

\[
= - \frac{\partial}{\partial s} \theta(\Sigma^d(s, T) + \sum_{i=1}^j \Sigma^{i,i-1}(s, T)) - \frac{\partial}{\partial s} l^{j-1}(s, T)
\]

\[
+ \frac{\partial}{\partial s} \theta(\Sigma^d(s, T) + \sum_{i=1}^{j-1} \Sigma^{i,i-1}(s, T)).
\]

From this we obtain

\[
\int_0^t \left[ \frac{\partial}{\partial T} \theta(\Sigma^d(s, T) + \sum_{i=1}^j \Sigma^{i,i-1}(s, T)) + \frac{\partial}{\partial T} l^{j-1}(s, T)
\]

\[
- \frac{\partial}{\partial T} \theta(\Sigma^d(s, T) + \sum_{i=1}^{j-1} \Sigma^{i,i-1}(s, T)) \right] ds
\]

\[
= - \int_0^t \left[ \frac{\partial}{\partial s} \theta(\Sigma^d(s, T) + \sum_{i=1}^j \Sigma^{i,i-1}(s, T)) + \frac{\partial}{\partial s} l^{j-1}(s, T)
\]

\[
- \frac{\partial}{\partial s} \theta(\Sigma^d(s, T) + \sum_{i=1}^{j-1} \Sigma^{i,i-1}(s, T)) \right] ds
\]

\[
= - \theta(\Sigma^d(t, T) + \sum_{i=1}^j \Sigma^{i,i-1}(t, T)) + \theta(\Sigma^d(0, T) + \sum_{i=1}^j \Sigma^{i,i-1}(0, T)) - l^{j-1}(t, T)
\]

\[
+ l^{j-1}(0, T) + \theta(\Sigma^d(t, T) + \sum_{i=1}^{j-1} \Sigma^{i,i-1}(t, T)) - \theta(\Sigma^d(0, T) + \sum_{i=1}^{j-1} \Sigma^{i,i-1}(0, T)).
\]

The last expressions immediately lead to the form (4.10). The representations of the short term spreads follow from (4.6) \(\square\)
Next, we derive necessary and sufficient deterministic conditions for the non-
negativity of the (consecutive) forward spreads. Let $m \geq 2$. We shall exploit
the representations obtained in the previous Proposition by considering the
following deterministic terms

$$
\mu^1_d(t, T) := s^1_d(T) + \int_0^t \sigma^1_d(s, T) dL_s
$$

$$
= s^1_d(T) - \theta(\Sigma^d(t, T) + \Sigma^1_d(t, T)) + \theta(\Sigma^d(0, T) + \Sigma^1_d(0, T))
- l^1(t, T) + l^1(0, T) + \theta(\Sigma^d(t, T)) - \theta(\Sigma^d(0, T))
$$

$$
\mu^j_d(t) := \lambda^1_d + \int_0^t \sigma^1_d(s, t) dL_s
$$

$$
= s^1_d(t) + \theta(\Sigma^d(0, t) + \Sigma^1_d(0, t)) + l^1(0, t) - \theta(\Sigma^d(0, t))
$$

and

$$
\mu^{j-1}_d(t, T) := s^{j-1}_d(T) + \int_0^t \sigma^{j-1}_d(s, T) dL_s
$$

$$
= s^{j-1}_d(T) - \theta(\Sigma^d(t, T) + \sum_{i=1}^j \Sigma^{i,j-1}(t, T))
+ \theta(\Sigma^d(0, T) + \sum_{i=1}^j \Sigma^{i,j-1}(0, T)) - l^{j-1}(t, T) + l^{j-1}(0, T)
$$

$$
+ \theta(\Sigma^d(t, T) + \sum_{i=1}^{j-1} \Sigma^{i,j-1}(t, T)) - \theta(\Sigma^d(0, T) + \sum_{i=1}^{j-1} \Sigma^{i,j-1}(0, T))
$$

$$
\mu^{j-1}_d(t) := \lambda^{j-1}_d + \int_0^t \sigma^{j-1}_d(s, t) dL_s
$$

$$
= s^{j-1}_d(t) + \theta(\Sigma^d(0, t) + \sum_{i=1}^j \Sigma^{i,j-1}(0, t)) + l^{j-1}(0, t)
$$

$$
- \theta(\Sigma^d(0, t) + \sum_{i=1}^{j-1} \Sigma^{i,j-1}(0, t)).
$$

**Proposition 4.3.** Let $T \in [0, T^*]$. We assume that the following two conditions
are satisfied

1. For any $t \in [0, T]$ and each $k \in \{1, \ldots, l\}$, we have

$$
\sigma^1_k(t, T) = 0
$$

and, if $m \geq 2$, we have for every $j \in \{2, \ldots, m\}$

$$
\sigma^{j-1}_k(t, T) = 0
$$
(2) For all \( t \in [0, T] \), we have

\[
0 \leq \mu^{1,d}(t, T) = \mu^{1,d}(T) - \theta(\Sigma^d(t, T) + \Sigma^{1,d}(t, T)) - l^1(t, T) + \theta(\Sigma^d(t, T))
\]

and, if \( m \geq 2 \), we have for every \( j \in \{2, \ldots, m\} \)

\[
0 \leq \mu^{j,j-1}(t, T) = \mu^{j,j-1}(T) - \theta(\Sigma^d(t, T) + \sum_{i=1}^{j} \Sigma^{i,i-1}(t, T)) - l^{j-1}(t, T) + \theta(\Sigma^d(t, T)) + \sum_{i=1}^{j-1} \Sigma^{i,i-1}(t, T)).
\]

Then the forward spreads satisfy

\[
0 \leq s^{1,d}_t(T) \quad \text{and} \quad 0 \leq s^{j,j-1}_t(T)
\]

for every \( t, T \in [0, T^*] \) with \( t \leq T \) and each \( j \in \{2, \ldots, m\} \).

Proof. It follows from the definition of \( \mu^{1,d}(t, T) \) that the non-negativity of the term \(-\int_0^t \sigma^{1,d}(s, T) dL_s\), for any \( t \in [0, T] \), together with assumption (4.11) imply \( s^{1,d}_t(T) \geq 0 \) for all \( t \in [0, T] \). Similar arguments lead to the non-negativity of the spread \( s^{j,j-1}_t(T) \) for any \( t \in [0, T] \) with \( t \leq T \) and every \( j \in \{2, \ldots, m\} \) if \( m \geq 2 \).

Note that the conditions (4.11) and (4.12) result in additional restrictions on the considered distribution parameters of the driving process as well as the parameters of the volatility and liquidity functions.

5. Valuation Formulas in a Two-Price Economy

5.1. Preface. We begin with some preliminary remarks related to the valuation approach. Let \( m \in \mathbb{N} \) be the number of the risky curves and \( \mathcal{F}^k = \{T_0^k, \ldots, T_n^k\} \) be the discrete tenor structure with tenor \( \delta^k \). To simplify the notation, we omit the superscripts in the symbols for the dates and tenors related to the discount curve.

We conclude from the equations (3.8) and (3.12) that the time-\( T_{j-1}^k \) price of the bond maturing at date \( T_j^k \) can be represented in the case \( k = 0 = d \) as

\[
B_{T_{j-1}^k}^d(T_j) = \frac{B_d^0(T_j)}{B_d^0(T_{j-1})} \exp \left( \int_0^{T_{j-1}} \Sigma^d(s, T_{j-1}, T_j) dL_s 
+ \int_0^{T_{j-1}} \left[ \theta_s(\Sigma^d(s, T_{j-1})) - \theta_s(\Sigma^d(s, T_j)) \right] ds \right)
\]
for every $j \in \{1, \ldots, n\}$ and

$$B^k_{T_j^{-1}}(T_j^k) = \frac{B^k_{0}(T_j^k)}{B^k_{0}(T_j^{-1})} \exp \left( \int_{0}^{T_j^{k-1}} \sum^k(s, T_j^k, T_j^k) dL_s ight)$$

for every $j \in \{1, \ldots, n_k\}$ with $k \in \{1, \ldots, m\}$. These expressions can be written in a compact form as

$$B^k_{T_j^{-1}}(T_j^k) = D^k_j \exp(X^k_j),$$

where we set the deterministic part as

$$D^k_j := D^k_j(T_j^{-1}, T_j^k) = \frac{B^k_{0}(T_j^k)}{B^k_{0}(T_j^{-1})} \exp \left( \int_{0}^{T_j^{k-1}} [\theta(s(\sum^k(s, T_j^k))) + l^k(s, T_j^k) - \theta(s(\sum^k(s, T_j^k))) - l^k(s, T_j^k)] ds \right)$$

and the stochastic term as

$$X^k_j := X^k_j(T_j^{-1}, T_j^k) = \int_{0}^{T_j^{k-1}} \sum^k(s, T_j^k, T_j^k) dL_s$$

for every $k \in \{d, 1, \ldots, m\}$. Furthermore, the discount factor process $\beta^d$ at date $T \in [0, T^*)$ is calculated as

$$\beta^d_T = B^d_0(T) \exp \left( - \int_{0}^{T} \theta(s(\sum^d(s, T))) ds + \int_{0}^{T} \sum^d(s, T) dL_s \right). \quad (5.1)$$

The forward martingale measure for the date $T \in [0, T^*)$, denoted by $P^d_T$, is defined by the Radon-Nikodym derivative

$$dP^d_T|_{\mathcal{G}_T} = \frac{\beta^d_T}{B^d_0(T)}.$$

(5.2)
By using Eberlein and Kluge (2006), we observe that the characteristic function of \( X_j^k \) under \( P^d_t \) can be determined as

\[
\phi_{X_j^k}^T(u) = \exp \left( \int_0^{T_{j-1}^k} \theta_s^T \left( iu \Sigma^k(s, T_{j-1}^k, T_j^k) \right) ds \right),
\]

where \( \theta_s^T \) denotes the cumulant with respect to \( P^d_t \) and \( T \in \mathcal{F}_t \).

By changing to the spot martingale measure \( P^d \) with the use of (5.2) together with equation (5.1), we get the useful representation

\[
\phi_{X_j^k}^{T_k}(u) = \exp \left( \int_0^{T_{j-1}^k} \left[ \theta_s \left( \Sigma^d(s, T_j^k) + iu \Sigma^k(s, T_{j-1}^k, T_j^k) \right) - \theta_s(\Sigma^d(s, T_j^k)) \right] ds \right).
\] (5.3)

Note that this expression can be extended to complex numbers by using assumption (EM) (cf. Sato (1999, Theorem 25.17)).

The payoffs of the derivatives that we will consider are functions of the reference rates \( L_{T_{j-1}^k}^k (T_{j-1}^k, T_j^k) \). The (forward) rates are related to the different bond curves by

\[
L_i^k(T_{j-1}^k, T_j^k) = \frac{1}{\delta^k} \left( \frac{B_i^k(T_{j-1}^k)}{B_i^k(T_j^k)} - 1 \right) \quad (t \leq T_{j-1}^k).
\] (5.4)

Note that from (5.4) we get in particular

\[
1 + \delta^k L_{T_{j-1}^k}^k (T_{j-1}^k, T_j^k) = \frac{1}{B_{T_{j-1}^k}^k (T_j^k)}.
\]

Since the martingale invariance property between \( \mathbb{F} \) and \( \mathbb{G} \) under \( P^d \) is valid, we conclude that

\[
(\beta_t^d)^{-1} \mathbb{E}^P[\beta_t^d f(L_{T_{j-1}^k}^k (T_{j-1}^k, T_j^k))] G_t = (\beta_t^d)^{-1} \mathbb{E}^P[\beta_t^d f(L_{T_{j-1}^k}^k (T_{j-1}^k, T_j^k))] \mathcal{F}_t
\]

for every \( t \in [0, T_{j-1}^k] \) and each Borel-measurable function. Note that \( f(L_{T_{j-1}^k}^k (T_{j-1}^k, T_j^k)) \) is assumed to be an integrable, \( \mathcal{F}_{T_{j-1}^k} \)-measurable random variable. By using the abstract Bayes rule, we obtain

\[
(\beta_t^d)^{-1} \mathbb{E}^P_{\mathcal{F}_t} \left[ \beta_t^d f(L_{T_{j-1}^k}^k (T_{j-1}^k, T_j^k)) G_t \right] = \mathbb{E}^P_{\mathcal{F}_t} \left[ B_t^d(T_j^k) f(L_{T_{j-1}^k}^k (T_{j-1}^k, T_j^k)) G_t \right].
\]

Since the discount factor process \( \beta_t^d \) is \( \mathbb{F} \)-adapted, we conclude that the martingale invariance property between \( \mathbb{F} \) and \( \mathbb{G} \) is also satisfied for any forward measure \( P^d_T \) with \( T \in [0, T^*] \). Hence, we have

\[
\mathbb{E}^P_{\mathcal{F}_t} \left[ B_t^d(T_j^k) f(L_{T_{j-1}^k}^k (T_{j-1}^k, T_j^k)) G_t \right] = \mathbb{E}^P_{\mathcal{F}_t} \left[ B_t^d(T_j^k) f(L_{T_{j-1}^k}^k (T_{j-1}^k, T_j^k)) G_t \right]
\]

for every \( t \in [0, T_{j-1}^k] \). To summarise, we can replace \( G_t \) by \( \mathcal{F}_t \) in all risk-neutral pricing formulas in our model. Hereafter, we consider \( \beta_t \in \{d, 1, \ldots, m\} \). The related valuation formulas for the single-curve setting are well-known and can be found in the literature. To simplify the notation, we choose the notional amount to be one.
5.2. Caps and Floors. The time-\( t \) value of a cap with strike rate \( K \) and maturity \( T^* \) results in

\[
\text{Cap}_t(\mathcal{F}^k, \delta^k, K) = \sum_{j=1}^{n_k} (\beta^d)_{t_j}^{-1} E_{\mathcal{F}^d}[\beta^d_{t_j} \delta^k(T^k_{T^k_{j-1}}, T^k_j) - K]^+ | \mathcal{F}_t]
\]

\[
= \sum_{j=1}^{n_k} E_{\mathcal{F}^d}(T^k_j) (\eta^k_j \exp(-X^k_j) - \tilde{K}^k)^+ | \mathcal{F}_t,
\]

where we have defined \( \tilde{K}^k := 1 + \delta^k K \) and \( \eta^k_j := \frac{1}{T^k_j} \). Hence, the time-0 price of a caplet with strike rate \( K \) is given by

\[
\text{Cpl}_0(T^k_{j-1}, T^k_j, K) = \sum_{j=1}^{n_k} E_{\mathcal{F}^d}(T^k_j) (\eta^k_j \exp(-X^k_j) - \tilde{K}^k)^+.
\]

To simplify the notation, we set \( Y^k_j := -X^k_j \). The extended characteristic function of \( Y^k_j \) relative to the pricing measure \( \mathcal{F}^d(T^k_j) \) can be calculated as

\[
\varphi_{T^k_{j-1}, T^k_j}(z) = \exp \left( \int_0^{T^k_{j-1}} \left[ \theta_s(\Sigma^d(s, T^k_j) - \xi^k J^k(s, T^k_{j-1}, T^k_j))
\right.
\]

\[
- \theta_s(\Sigma^d(s, T^k_j)) \right] ds \right)
\]

for every \( z \in \mathbb{C} \) where this function exists.

To evaluate the price of the caplet, we use the Fourier based valuation method (see Eberlein, Glau, and Papapantoleon (2010)).

**Proposition 5.1.** The risk-neutral price of the caplet at time 0 can be written as

\[
\text{Cpl}_0(T^k_{j-1}, T^k_j, K) = B^d_0(T^k_j) e^{-R_{T^k_j}} \frac{\pi}{\xi^k J^k} \int_0^\infty \Re \left( e^{-iu \xi^k J^k(u-iR)(\tilde{K}^k)^{1-R-\xi^k J^k}} \right) \frac{1}{(-R-i\xi^k J^k(1-R-iu))} du
\]

for any \( R \in \left( 1, \frac{M-\tilde{M}}{2} \right) \), where we set \( \xi^k_j := -\ln \eta^k_j \) and \( \tilde{M} \) is assumed to be chosen such that \( \tilde{M} < \frac{M}{2} \).

**Proof.** Clearly, we have

\[
\varphi_{T^k_{j-1}, T^k_j}(u-iR) = \exp \left( \int_0^{T^k_{j-1}} \left[ \theta_s(\Sigma^d(s, T^k_j) - (iu+R)\Sigma^k(s, T^k_{j-1}, T^k_j))
\right.
\]

\[
- \theta_s(\Sigma^d(s, T^k_j)) \right] ds \right)}
and
\[
|\text{Re}(\Sigma^d_l(s, T^k_j) - (iu + R)\Sigma^k_l(s, T^k_{j-1}, T^k_j))| = |\Sigma^d_l(s, T^k_j) - R\Sigma^k_l(s, T^k_{j-1}, T^k_j)| \\
\leq \hat{M} + |R|\hat{M} \leq \frac{M - \hat{M}}{M} \hat{M} = M
\]
for every \( l \in \{1, \ldots, d\} \). Then the result follows from Eberlein, Glau, and Papapantoleon (2010, Theorem 2.2.) and an obvious symmetry property of the integrand. \( \square \)

Note that the price of a floorlet can be obtained in an analogous way. From the pricing formula for caplets (floorlets) we immediately deduce the valuation formula of the cap floor.

5.3. Interest Rate Digital Options. A standard interest rate digital call (put) with strike rate \( B \) is an option that pays an amount of one currency unit to its owner if and only if the reference rate for the period \([T^k_{j-1}, T^k_j]\) lies above (below) \( B \) at maturity \( T^k_{j-1} \) of the option. An interest rate digital option is called delayed if the payment date \( T \) differs from the maturity date \( T^k_{j-1} \), where \( T > T^k_{j-1} \). Thus the payoff at time \( T \) can be given in the form
\[
\text{DD}^k_l(T^k_{j-1}, T^k_j, T, B, w) = 1_{\{wL^k_{T^k_{j-1}}(T^k_{j-1}, T^k_j) > wB\}},
\]
where we assume that \( B > -\frac{1}{\delta^k} \) and
\[
w = \begin{cases} 
1, & \text{for a delayed digital call} \\
-1, & \text{for a delayed digital put.}
\end{cases}
\]
Then, the time-\( t \) price for \( t \leq T^k_{j-1} \) can be represented by
\[
\text{DD}^k_l(T^k_{j-1}, T^k_j, T, B, w) = (\beta^d_l)^{-1} \mathbb{E}_{\mathbb{P}^d}[\frac{\beta^d_l}{w} 1_{\{wL^k_{T^k_{j-1}}(T^k_{j-1}, T^k_j) > wB\}} | \mathcal{F}_t] \\
= B^d_l(T) \mathbb{E}_{\mathbb{P}^d}[1_{\{w(1+\delta^k B)^{-1} > wB^k_{T^k_{j-1}}(T^k_j)\}} | \mathcal{F}_t].
\]
This formula can further be written as
\[
\text{DD}^k_l(T^k_{j-1}, T^k_j, T, B, w) = B^d_l(T) \mathbb{E}_{\mathbb{P}^d}[1_{\{w(1+\delta^k B)^{-1} > wB^k_{T^k_{j-1}}(T^k_j)\}} | \mathcal{F}_t] \\
\text{with}
\]
\[
H^k(t, T^k_{j-1}) := \exp \left( -\int_{t}^{T^k_{j-1}} A^k(s, T^k_{j-1}, T^k_j) ds + X^k_{t,j} \right),
\]
where we set
\[
A^k(s, T^k_{j-1}, T^k_j) := A^k(s, T^k_j) - A^k(s, T^k_{j-1})
\]
and
\[
X^k_{t,j} := \int_{t}^{T^k_{j-1}} \Sigma^k(s, T^k_{j-1}, T^k_j) dL_s.
\]
By the independence of \( X_{t,j} \) and \( \mathcal{F}_t \) (independent increments of the process \( L \)) and the \( \mathcal{F}_t \)-measurability of \( \frac{B^k(T_j^k)}{B^k(T_{j-1}^k)} \), we obtain from Kallenberg (2002, Theorem 6.4) that

\[
DD_t^k(t, T_{j-1}^k, T_j^k, T, B, w) = B^d(T) \cdot g_w^k \left( \frac{B^k(T_j^k)}{B^k(T_{j-1}^k)} \right) = B^d(T) \cdot g_w^k \left( F^k(t, T_{j-1}^k, T_j^k)^{-1} \right),
\]

where \( F^k(t, T_{j-1}^k, T_j^k) = \frac{B^k(T_{j-1}^k)}{B^k(T_j^k)} \) and the function \( g_w^k : \mathbb{R} \to [0, 1] \) is defined by

\[
g_w^k(y) := \mathbb{E}_{P^d_T} \left[ \mathbb{1}_{\{w(1+\delta^k B)^{-1} > wy^H(t, T_{j-1}^k)\}} \right].
\]

For every \( y > 0 \), we have

\[
g_w^k(y) = P_T^d \left( w \exp \left( X_{t,j}^k \right) < w \frac{\exp \left( \int_{t}^{T_{j-1}^k} A^k(s, T_{j-1}^k, T_j^k) ds \right)}{(1+\delta^k B)y} \right)
\]

\[
= \begin{cases}
  P_T^d,_{X_{t,j}^k} \left( x < \log \left( \frac{\exp \left( \int_{t}^{T_{j-1}^k} A^k(s, T_{j-1}^k, T_j^k) ds \right)}{(1+\delta^k B)y} \right) \right) & \text{for } w = 1 \\
  1 - P_T^d,_{X_{t,j}^k} \left( x \leq \log \left( \frac{\exp \left( \int_{t}^{T_{j-1}^k} A^k(s, T_{j-1}^k, T_j^k) ds \right)}{(1+\delta^k B)y} \right) \right) & \text{for } w = -1,
\end{cases}
\]

(5.9)

where we denote by \( P_T^d,_{X_{t,j}^k} \) the distribution of \( X_{t,j}^k \). The extended characteristic function of \( X_{t,j}^k \) under \( P_T^d \) can be determined by

\[
\varphi_{X_{t,j}^k}^T(z) = \exp \left( \int_{t}^{T_{j-1}^k} \left[ \theta_s(\Sigma^d(s, T) + iz\Sigma^k(s, T_{j-1}^k, T_j^k)) - \theta_s(\Sigma^d(s, T)) \right] ds \right)
\]

for every \( z \in \mathbb{C} \) where this function is defined.

Now, we calculate the present value \((t = 0)\) of a delayed digital option by applying the Fourier-based valuation method. Let us write

\[
DD_0^k(T_{j-1}^k, T_j^k, T, B, w) = B^d_0(T)\mathbb{E}_{P^d_0} \left[ \mathbb{1}_{\{w \exp(X_j^k - \xi_j^k) < w(1+\delta^k B)^{-1}\}} \right]
\]

\[
= \begin{cases}
  B^d_0(T) P_T^d \left( X_j^k < \log((1+\delta^k B)^{-1}) + \xi_j^k \right) & \text{for } w = 1 \\
  B^d_0(T) P_T^d \left( X_j^k > \log((1+\delta^k B)^{-1}) + \xi_j^k \right) & \text{for } w = -1
\end{cases}
\]

\[=: V_{w,j}^k(\xi_j^k),\]

where we used \( \xi_j^k = -\log(D_j^k) \in \mathbb{R} \).
Let us consider the map \( \xi^k_j \mapsto V_{w}^{j,k}(\xi^k_j) \in \mathbb{R}_+ \). Clearly, \( V_{w}^{j,k} \) has locally bounded variation. We assume that the distribution of \( X^k_j \) under \( P^d_T \) is atomless. Then, by using the symbol \( F^T_T \) for the cumulative distribution function of a random variable \( X \) under \( P^d_T \), we get

\[
V_{w}^{j,k}(\xi^k_j) = \begin{cases} B_0^d(T)F^T_T(\log((1 + \delta^k B)^{-1}) + \xi^k_j) & \text{for } w = 1 \\ B_0^d(T)(1 - F^T_T(\log((1 + \delta^k B)^{-1}) + \xi^k_j)) & \text{for } w = -1. \end{cases}
\]

It follows that \( V_{w}^{j,k} \) is a continuous function. The payoff function of a digital call option with barrier \( B \in \mathbb{R}_+ \) is given by

\[
f_w(x) = \mathbbm{1}_{\{w < x \tilde{B}_k^{-1}\}},
\]

where we set \( \tilde{B}_k = 1 + \delta^k B \). Easy calculations lead to the following form of the Fourier transform of \( f_w \) for every \( z \in \mathbb{C} \) where it is defined by

\[
f_w(z) = \begin{cases} \frac{\tilde{B}_k^{-iz}}{iz}, & \text{for } w = 1 \text{ and } \text{Im}(z) \in (-\infty, 0) \\ -\frac{\tilde{B}_k^{-iz}}{iz}, & \text{for } w = -1 \text{ and } \text{Im}(z) \in (0, \infty). \end{cases}
\]

Let us consider the dampened payoff function of the digital option given by

\[
d_w(x) = e^{-Rx}f_w(x).
\]

We easily verify that \( d_w \in L^1(\mathbb{R}) \) for every

\[
\left\{ \begin{array}{ll} R \in (-\infty, 0), & \text{for } w = 1, \\
R \in (0, \infty), & \text{for } w = -1. \end{array} \right.
\]

Observe that we can find an \( R \) that satisfies the prerequisites of Eberlein, Glau, and Papapantoleon (2010, Theorem 2.7.). Then, we conclude from this Theorem that the value of a delayed digital option at point \( \xi^k_j \) can be expressed as

\[
\text{DD}^k_0(T_{j-1}^k, T^k_j, T, B, w) = \begin{cases} B_0^d(T) \cdot \lim_{A \to -\infty} \frac{e^{-R^k_j}}{\pi} \int_0^A \Re \left( \frac{e^{-iu\xi^k_j} \varphi^{T}_{X^k_j}(u-iR)\tilde{B}_k^{R+iu}}{-R-iu} \right) du, & \text{for } w = 1 \text{ and } R < 0 \\
B_0^d(T) \cdot \lim_{A \to -\infty} \frac{e^{-R^k_j}}{\pi} \int_0^A \Re \left( \frac{e^{-iu\xi^k_j} \varphi^{T}_{X^k_j}(u-iR)\tilde{B}_k^{R+iu}}{R+iu} \right) du, & \text{for } w = -1 \text{ and } R > 0. \end{cases}
\]

Note that the price of a delayed range digital option with barriers \( B, \overline{B} \) satisfying \( 0 < B < \overline{B} \) can be determined by the formula

\[
\text{DRD}^k(T, T_1, T_2, B, \overline{B}) := \text{DD}^k_1(T_1, T_2, T, B, -1) + \text{DD}^k_1(T_1, T_2, T, \overline{B}, -1) - B_0^d(T).
\]

5.4. **Two-Price Theory.** In this section we extend the multiple-curve model according to the two-price approach. We provide bid and ask valuation formulas for some interest rate derivatives. We consider again \( m \) risky curves and the tenor structure \( \mathcal{T}^k = \{T_0^k, \ldots, T_n^k\} \) with tenor \( \delta^k \) and \( k \in \{0,1,\ldots,m\} \).
5.4.1. The Two-Price Theory based on concave distortions. Let us briefly recall the basic concept as developed in Cherny and Madan (2010). In order to get bid and ask prices, instead of a single pricing measure a whole set \( \mathcal{M} \) of pricing measures is considered. \( \mathcal{M} \) contains at least one risk-neutral probability measure. The bid price of a discounted claim \( X \) is defined as

\[
b(X) = \inf_{Q \in \mathcal{M}} E_Q[X]
\]

whereas the ask price is given by

\[
a(X) = \sup_{Q \in \mathcal{M}} E_Q[X].
\]

Under slight additional assumptions, namely co-monotonicity and law-invariance, these two values can be obtained by using an increasing concave function \( \Psi : [0, 1] \to [0, 1] \) (concave distortion) in the form

\[
b(X) = -\int_{-\infty}^{0} \Psi(F_X(y))dy + \int_{0}^{\infty} 1 - \Psi(F_X(y))dy
\]

and

\[
a(X) = \int_{0}^{\infty} \Psi(1 - F_X(y))dy - \int_{-\infty}^{0} 1 - \Psi(1 - F_X(y))dy.
\]

Observe that these bid and ask prices depend on an underlying probability measure. In our situation, this will be a basic pricing measure which is distorted by \( \Psi \) (it is assumed to be contained in the set \( \mathcal{M} \)).

Hereafter, we consider a parameterised family \( (\Psi_{\gamma})_{\gamma \geq 0} \) of distortion functions. For technical reasons we assume that the map \( \gamma \mapsto \Psi_{\gamma}(y) \) is increasing on \([0, \infty)\) and continuous on \((0, \infty)\) for every \( y \in (0, 1) \). Further, we only consider values for \( \gamma \geq 0 \) that satisfy

\[
\int_{-\infty}^{0} \Psi_{\gamma}(F_X(y))dy < \infty \quad \text{and} \quad \int_{0}^{\infty} \Psi_{\gamma}(1 - F_X(y))dy < \infty. \quad (5.10)
\]

The parameter \( \gamma \) represents the current level of market (il)liquidity in the market. We end this subsection with some crucial examples of families of distortion functions which are considered by Cherny and Madan (2009, 2010).

1. The family \( (\Psi_{\gamma}^{\text{mv}})_{\gamma \geq 0} \) of MINVAR distortion functions is defined by

\[
\Psi_{\gamma}^{\text{mv}} : \begin{cases} [0, 1] \to [0, 1] \\ y \mapsto 1 - (1 - y)^{1+\gamma} \end{cases}.
\]

2. The family \( (\Psi_{\gamma}^{\text{max}})_{\gamma \geq 0} \) of MAXVAR distortion functions is defined by

\[
\Psi_{\gamma}^{\text{max}} : \begin{cases} [0, 1] \to [0, 1] \\ y \mapsto y^{1+\gamma}. \end{cases}
\]
(3) The family \((\Psi^\text{mmv}_\gamma)_{\gamma \geq 0}\) of MINMAXVAR distortion functions is determined by
\[
\Psi^\text{mmv}_\gamma : \begin{cases} [0, 1] \rightarrow [0, 1] \\ y \mapsto 1 - (1 - y)^{1+\gamma} \end{cases}
\]
(4) The family \((\Psi^\text{mamv}_\gamma)_{\gamma \geq 0}\) of MAXMINVAR distortion functions is defined by
\[
\Psi^\text{mamv}_\gamma : \begin{cases} [0, 1] \rightarrow [0, 1] \\ y \mapsto (1 - (1 - y)^{1+\gamma})^\frac{1}{1+\gamma} \end{cases}
\]

5.4.2. Bid and Ask Price of Caplets and Floorlets. Our first aim is to derive explicit valuation formulas of bid and ask prices of caplets and floorlets. In order to determine these formulas we have to distort the cumulative distribution function of the discounted payoff
\[
\beta_{T_j}^k (\eta_j^k \exp(-X_j^k) - \tilde{K}^k)^+ \tag{5.11}
\]
with respect to measure \(P^d\) (cf. relation (5.5)). Although this can be done analytically, this calculation is, in general, a challenging task. In most cases, it has to be determined numerically and its numerical evaluation is extremely time-consuming. The main reason for this lies in the joint appearance of the random variables \(\beta_{T_j}^k\) and \(X_j^k\) in (5.11). To handle this issue, we switch to the forward martingale measure and consider the more tractable discounted payoff
\[
C_{j,K}^k := B_0^d (T_j^k) (\eta_j^k \exp(-X_j^k) - \tilde{K}^k)^+ \tag{5.11}
\]
(cf. pricing formula (5.6)). Analogously, we deal with the corresponding floorlets quantity
\[
F_{j,K}^k := B_0^d (T_j^k) (\tilde{K}^k - \eta_j^k \exp(-X_j^k))^+.
\]
Recall that \(F_T^Y\) denotes the cumulative distribution function of a random variable \(Y\) under \(P^d_T\) with \(T \in [0, T^*]\).

Lemma 5.2. (1) Let us assume that \(X_j^k\) has exponential moments of order \(M_j^k > 1\) under \(P^d_{T_j^k}\). Then, we get
(i) for any \(\gamma \geq 0\)
\[
a_{\gamma}^{T_j^k} (C_{j,K}^k) = \int_{-\infty}^{0} \Psi^\text{mv}_\gamma (F_{C_{j,K}^k}^T (x)) \, dx
\]
and
\[
b_{\gamma}^{T_j^k} (C_{j,K}^k) = \int_{0}^{\infty} \left(1 - \Psi^\text{mv}_\gamma (F_{C_{j,K}^k}^T (x)) \right) \, dx.
\]
(ii) for the the family \(\Psi = (\Psi_\gamma)_{\gamma \geq 0} \in \{\Psi^\text{mav}, \Psi^\text{mmv}, \Psi^\text{mamv}\}\) of distortion functions and every \(\gamma \in [0, u_1 - 1]\), where \(u_1\) has to satisfy
1 < u_1 \leq M^k_j, we obtain

\[ a^{T^k}_{\gamma'}(Cpl_{j,K}^k) = \int_{-\infty}^{0} \Psi_{\gamma}(F^T_{Cpl_{j,K}^k}(x))dx \]

and

\[ b^{T^k}_{\gamma'}(Cpl_{j,K}^k) = \int_{0}^{\infty} \left(1 - \Psi_{\gamma}(F^T_{Cpl_{j,K}^k}(x))\right)dx. \]

(2) For every \( \Psi = (\Psi_{\gamma})_{\gamma \geq 0} \in \{\Psi_{mv}, \Psi_{mav}, \Psi_{mmv}, \Psi_{mamv}\} \) and \( \gamma \geq 0 \), we obtain

\[ a^{T^k}_{\gamma'}(Flt_{j,K}^k) = \int_{-\infty}^{0} \Psi_{\gamma}(F^T_{Flt_{j,K}^k}(x))dx \]

and

\[ b^{T^k}_{\gamma'}(Flt_{j,K}^k) = \int_{0}^{\infty} \left(1 - \Psi_{\gamma}(F^T_{Flt_{j,K}^k}(x))\right)dx. \]

Proof. We need to verify for which \( \gamma \geq 0 \) condition (5.10) is satisfied. We frequently use the change-of-variable formula here. Since \( Cpl_{j,K}^k \geq 0 \), we only have to consider the second integral in (5.10).

(1) An application of Bernoulli’s inequality leads to \( \Psi_{\gamma}^{mv}(y) \leq (1 + \gamma)y \). For any \( \gamma \geq 0 \), we obtain

\[ \int_{0}^{\infty} \Psi_{\gamma}^{mv}(1 - F^T_{Cpl_{j,K}^k}(y))dy \leq (1 + \gamma) \int_{0}^{\infty} 1 - F^T_{Cpl_{j,K}^k}(y)dy \]

\[ = (1 + \gamma)B^k_0(T^k_j) \int_{0}^{\infty} 1 - F^T_{X^k_j}(\eta_{\gamma}^k \exp(-X^k_j) - \tilde{K}_j^k + )dy \]

\[ = (1 + \gamma)B^k_0(T^k_j) \int_{\tilde{K}_j^k}^{\infty} F^T_{X^k_j}(-\log(yD^k_j))dy \]

\[ \leq (1 + \gamma)B^k_0(T^k_j)C_1 \int_{\tilde{K}_j^k}^{\infty} e^{-u_1 \log(yD^k_j)}dy < \infty, \]

where \( C_1 > 0 \) and \( 1 < u_1 \leq M^k_j \).

(2) Since

\[ \Psi_{\gamma}^{mamv}(y) = \Psi_{\gamma}^{mav}(\Psi_{\gamma}^{mv}(y)) \leq \Psi_{\gamma}^{mav}((1 + \gamma)y) = (1 + \gamma)^{1-\gamma} \Psi_{\gamma}^{mav}(y) \]

\[ \Psi_{\gamma}^{mmv}(y) = \Psi_{\gamma}^{mav}(\Psi_{\gamma}^{mv}(y)) \leq (1 + \gamma)\Psi_{\gamma}^{mav}(y), \]

A MULTIPLE-CURVE LÉVY FORWARD RATE MODEL IN A TWO-PRICE ECONOMY
we only need to check the condition for the distortion function $\Psi^{\text{mav}}$. In a similar way as above, we get
\[
\int_0^\infty \Psi^{\text{mav}}(\frac{1}{2} - F^{T_j^k}_{X_j^k}(y)) dy \leq B_0^d(T_j^k) C_1^{\frac{1}{\gamma + 1}} \int_{K^k}^{\infty} y^{-\frac{u_1}{\gamma + 1}} dy < \infty
\]
for every $\gamma$ satisfying $0 \leq \gamma < u_1 - 1$.

(3) Clearly, we have
\[
M^{T_j^k}_{j,K}(u) = \mathbb{E}_{P^d_{T_j^k}} \left[ \exp \left( u F^{T_j^k}_{X_j^k}(y) \right) \right] < \exp \left( u B_0^d(T_j^k) \tilde{K}^k \right) < \infty
\]
for every $u \in \mathbb{R}$.

The following Proposition states useful integral representations for bid and ask prices of caplets and floorlets with reset date $T_j^k$, settlement date $T_j^k = T_{j-1}^k + \delta^k$ and strike rate $K$ at a permitted level $\gamma$.

**Proposition 5.3.** Let $(\Psi_{\gamma})_{\gamma \geq 0}$ be a family of distortion functions and $\gamma \geq 0$ be chosen such that condition (5.10) is satisfied for the caplet and floorlet. Then, the ask price of the caplet is given by
\[
a_{\gamma}^{T_j^k}(C_{j,K}) = B_0^d(T_j^k) \int_{K^k}^{\infty} \Psi_{\gamma}(F^{T_j^k}_{X_j^k}(\log(xD_j^k))) dx
\]
and the bid price of the caplet has the form
\[
b_{\gamma}^{T_j^k}(C_{j,K}) = B_0^d(T_j^k) \int_{K^k}^{\infty} \left[ 1 - \Psi_{\gamma}(1 - F^{T_j^k}_{X_j^k}(\log(xD_j^k))) \right] dx.
\]

The ask price of the floorlet is determined by
\[
a_{\gamma}^{T_j^k}(F_{j,K}) = B_0^d(T_j^k) \int_{0}^{K^k} \Psi_{\gamma}(1 - F^{T_j^k}_{X_j^k}(\log(xD_j^k))) dx
\]
and the bid price of the floorlet is represented by
\[
b_{\gamma}^{T_j^k}(F_{j,K}) = B_0^d(T_j^k) \int_{0}^{K^k} \left[ 1 - \Psi_{\gamma}(F^{T_j^k}_{X_j^k}(\log(xD_j^k))) \right] dx.
\]
Proof. By applying the change-of-variable formula, we get the ask price as

\[
T^k \gamma (Cpl^k_{j,k}) = \int_{-\infty}^{0} \Psi_\gamma (F^{T^k}_j (x)) dx
\]

\[
= \int_{-\infty}^{0} \Psi_\gamma (1 - F^{T^k}_j (\eta^k_j \exp(-X^k_j) - K^k)) + (-xB^d_0 (T^k_j))^{-1}) dx
\]

\[
= B^d_0 (T^k_j) \int_{0}^{\infty} \Psi_\gamma (1 - F^{T^k}_j (\eta^k_j \exp(-X^k_j) - K^k) + (x)) dx
\]

\[
= B^d_0 (T^k_j) \int_{K^k}^{\infty} \Psi_\gamma (F^{T^k}_j (- \log(xD^k_j))) dx.
\]

The other formulas follow by the same arguments.

On the basis of the analysis we made above, in order to determine the bid and ask prices, we have to identify the cumulative distribution function \(F^{T^k}_{X^k_j}\).

We proceed as follows

1. We determine the characteristic function of \(X^k_j\) under \(P^d_T\).
2. As an approximation of the cumulative distribution function, we consider

\[
F^{T^k}_{X^k_j} (y) = \lim_{M \to \infty} \frac{1}{M} \int_{0}^{M} \text{Re} \left( e^{-iu x} - e^{-iuy} \varphi^{T^k}_{X^k_j} (u) \right) du
\]

for a suitable \(x \in \mathbb{R}\) satisfying \(x < y\) and \(F^{d,X^k_j}_T (\{x\}) = F^{d,X^k_j}_T (\{y\}) = 0\),

where \(P^d_T, X^k_j\) denotes the distribution of \(X^k_j\) with respect to \(P^d_T\).

5.4.3. Bid and Ask Price of Digital Options. The discounted payoff of a delayed digital option is

\[
DD^k_T (w) := B^d_0 (T) \cdot 1_{\{w\exp(X^k_j) < wB^k_0 \exp(-\xi^k_j)\}}.
\]

We state its bid and ask price.

**Proposition 5.4.** The ask price of the delayed digital option at level \(\gamma \geq 0\) is given by

\[
a^T_{\gamma} (DD^k_T (w)) = B^d_0 (T) \Psi_\gamma \left( P^d_T (w\exp(X^k_j) < wB^k_0 \exp(-\xi^k_j)) \right)
\]

and the bid price at level \(\gamma \geq 0\) can be expressed as

\[
b^T_{\gamma} (DD^k_T (w)) = B^d_0 (T) \left[ 1 - \Psi_\gamma \left( P^d_T (w\exp(X^k_j) \geq wB^k_0 \exp(-\xi^k_j)) \right) \right].
\]

Proof. One verifies that the cumulative distribution functions of the random variables \(-DD^k_T (w)\) and \(DD^k_T (w)\) under \(P^d_T\) are given by

\[
F^{T}_{-DD^k_T (w)} (y) = \begin{cases} 
1, & y \geq 0 \\
\left( P^d_T (w \exp(X^k_j) < wB^k_0 \exp(-\xi^k_j)) \right), & y \in [-B^d_0 (T), 0) \\
0, & y < -B^d_0 (T)
\end{cases}
\]

\[
F^{T}_{DD^k_T (w)} (y) = \begin{cases} 
1, & y \geq 0 \\
\left( P^d_T (w \exp(X^k_j) \geq wB^k_0 \exp(-\xi^k_j)) \right), & y \in [-B^d_0 (T), 0) \\
0, & y < -B^d_0 (T)
\end{cases}
\]
and

\[ F_{DD^{k}_{T}}^{T}(w)(y) = \begin{cases} 
1, & y \geq B_{0}^{d}(T) \\
P_{T}^{d}(w \exp(X^{k}_{j}) \geq w\tilde{B}^{-1}_{k} \exp(\xi^{k}_{j})), & y \in [0, B_{0}^{d}(T)) \\
0, & y < 0.
\end{cases} \]

Then, by considering the bid and ask price formulas above, we immediately get the statement. □

The discounted payoff of a delayed range digital option with barriers \( B \) and \( \overline{B} \) is given by

\[
DRD_{T}^{k} := B_{0}^{d}(T) \cdot 1 \left\{ \exp(\xi^{k}_{j}) \overline{B}_{k}^{d} < \exp(X_{j}^{k}) < \exp(\xi^{k}_{j}) \overline{B}_{k}^{d} \right\}.
\]

**Proposition 5.5.** The ask price of the delayed range digital option at level \( \gamma \geq 0 \) is given by

\[
a_{T}^{\gamma}(DRD_{T}^{k}) = B_{0}^{d}(T) \Psi_{\gamma}\left( P_{T}^{d}\left( \frac{\exp(\xi^{k}_{j})}{\overline{B}_{k}^{d}} < \exp(X_{j}^{k}) \right) + P_{T}^{d}\left( \exp(X_{j}^{k}) < \frac{\exp(\xi^{k}_{j})}{\overline{B}_{k}^{d}} \right) \right)
\]

and the bid price at level \( \gamma \geq 0 \) can be expressed as

\[
b_{T}^{\gamma}(DRD_{T}^{k}) = B_{0}^{d}(T) \times \left[ 1 - \Phi_{\gamma}\left( P_{T}^{d}\left( \exp(X_{j}^{k}) \leq \frac{\exp(\xi^{k}_{j})}{\overline{B}_{k}^{d}} \right) + P_{T}^{d}\left( \frac{\exp(\xi^{k}_{j})}{\overline{B}_{k}^{d}} \leq \exp(X_{j}^{k}) \right) \right) \right].
\]

**Proof.** Here, the cumulative distribution functions result in

\[
\begin{align*}
F_{DRD_{T}^{k}}^{T}(y) = & \begin{cases} 
1, & y \geq B_{0}^{d}(T) \\
P_{T}^{d}\left( \frac{\exp(\xi^{k}_{j})}{\overline{B}_{k}^{d}} < \exp(X_{j}^{k}) \right) + P_{T}^{d}\left( \exp(X_{j}^{k}) < \frac{\exp(\xi^{k}_{j})}{\overline{B}_{k}^{d}} \right), & y \in [-B_{0}^{d}(T), 0) \\
0, & y < -B_{0}^{d}(T)
\end{cases} \\
& \begin{cases} 
1, & y \geq B_{0}^{d}(T) \\
P_{T}^{d}\left( \exp(X_{j}^{k}) \leq \frac{\exp(\xi^{k}_{j})}{\overline{B}_{k}^{d}} \right) + P_{T}^{d}\left( \frac{\exp(\xi^{k}_{j})}{\overline{B}_{k}^{d}} \leq \exp(X_{j}^{k}) \right), & y \in [0, B_{0}^{d}(T)) \\
0, & y < 0.
\end{cases}
\end{align*}
\]

Then, the claim follows by the pricing formulas. □

6. Model Calibration

We consider market cap volatilities quoted during the financial crisis on 15th September 2009. We emphasise that quotes up to a maturity of two years are indexed on the three-month Euribor and quotes larger than 2 years are related to the six-month Euribor.

6.1. Data Sets. The quoted cap (mid) implied volatilities based on the three-month and six-month tenor are plotted in figure 2. In figure 3, we present the bid and ask spreads between the implied (spot) volatilities of caplets for some maturities related to two tenors and different strike rates.
6.2. **Model Framework.** The model as designed in section 4.2 is used hereafter. This approach guarantees the monotonicity of the curves. We consider three term structures, namely the discount curve, the three-month curve and the six-month curve.

The volatility function, the liquidity function and the driving process are specified as follows \((d = 3, m = 2 \text{ and } l = 1)\):

(Vol) Vasichek volatility structure
- Discount curve

\[
\sigma^d(t, T) = \begin{pmatrix} \hat{\sigma}_d \exp(-\sigma_d(T - t)) & 0 \\ 0 & 0 \end{pmatrix}^T
\]
with
\[ \Sigma^d_1(t, T) = \begin{cases} \frac{\sigma_d}{a_d} (1 - \exp(-a_d(T - t))), & \text{when } t \leq T \\ 0, & \text{when } t > T \end{cases} \]

and \( \Sigma^d_2(t, T) = \Sigma^d_3(t, T) = 0 \) for any \( t, T \in [0, T^\ast] \), where \( \hat{\sigma}_d > 0 \) and \( a_d \neq 0 \).

- Spreads related to multiple term structures
  (a) Spread between curve \( d \) and 1:
  \[ \sigma^{1,d}(t, T) = \left( \begin{array}{l} 0 \\ \hat{\sigma}_{1d} \exp(-a_{1d}(T - t)) \end{array} \right)^T \]
  with
  \[ \Sigma^{1,d}_2(t, T) = \begin{cases} \frac{\sigma_{1d}}{a_{1d}} (1 - \exp(-a_{1d}(T - t))), & \text{when } t \leq T \\ 0, & \text{when } t > T \end{cases} \]
  and \( \Sigma^{1,d}_1(t, T) = \Sigma^{1,d}_3(t, T) = 0 \) for every \( t, T \in [0, T^\ast] \), with \( \hat{\sigma}_{1d} > 0 \) and \( a_{1d} \neq 0 \).
  (b) Spread between curve 1 and 2:
  \[ \sigma^{2,1}(t, T) = \left( \begin{array}{l} 0 \\ 0 \end{array} \right)^T \]
  with
  \[ \Sigma^{2,1}_2(t, T) = \begin{cases} \frac{\sigma_{21}}{a_{21}} (1 - \exp(-a_{21}(T - t))), & \text{when } t \leq T \\ 0, & \text{when } t > T \end{cases} \]
  and \( \Sigma^{2,1}_1(t, T) = \Sigma^{2,1}_3(t, T) = 0 \) for all \( t, T \in [0, T^\ast] \), where \( \hat{\sigma}_{21} > 0 \) and \( a_{21} \neq 0 \).

(L) Liquidity function (see Brigo and Mercurio (2006)):
\[ l^j(t, T) = \sigma^j \cdot (T - t) \exp(-b^j(T - t)), \]
where \( \sigma^j, b^j > 0 \) for any \( j \in \{1, 2\} \).

We can choose \( \hat{\sigma}_d = |a_d|, \hat{\sigma}_{1d} = |a_{1d}|, \) and \( \hat{\sigma}_{21} = |a_{21}| \). For \( t, T \in [0, T^\ast] \) with \( t \leq T \), we therefore obtain
\[ \Sigma^j(t, T) = \Sigma^d(t, T) + \Sigma^{1,d}(t, T) = \left( \begin{array}{l} \text{sign}(a_d) (1 - \exp(-a_d(T - t))) \\ \text{sign}(a_{1d}) (1 - \exp(-a_{1d}(T - t))) \\ 0 \end{array} \right)^T \]
and
\[ \Sigma^2(t, T) = \Sigma^d(t, T) + \Sigma^{1,d}(t, T) + \Sigma^{2,1}(t, T) = \left( \begin{array}{l} \text{sign}(a_d) (1 - \exp(-a_d(T - t))) \\ \text{sign}(a_{1d}) (1 - \exp(-a_{1d}(T - t))) \\ \text{sign}(a_{21}) (1 - \exp(-a_{21}(T - t))) \end{array} \right)^T. \]

In accordance with assumption (M\(\text{C.DET}\)), we require the parameters \( a_d, a_{1d} \) and \( a_{21} \) to be restricted to values such that the volatility function \( \Sigma^2 \) is bounded in each component by a constant \( \tilde{M} \) satisfying \( 0 < \tilde{M} < M \). More
specifically, we claim the existence of the cumulant function $\theta$ at $\Sigma^2(t, T)$ for all $t, T \in [0, T^*]$ with $t \leq T$. Then, this function also exists at $\Sigma^d(t, T)$ and $\Sigma^1(t, T)$. Clearly, the parameters of the liquidity function can be restricted to a set such that the liquidity function is bounded. Observe that the considered volatility and liquidity functions satisfy assumption (\forall L). To ensure the drift condition (3.17), we have to choose the drift terms $A^{1,d}$ and $A^{2,1}$ as

$$A^{1,d}(t, T) = \theta(\Sigma^1(t, T)) - \theta(\Sigma^d(t, T)) + l^1(t, T)$$

and

$$A^{2,1}(t, T) = \theta(\Sigma^2(t, T)) - \theta(\Sigma^1(t, T)) + l^{2,1}(t, T).$$

(DP) The driving process $L = (L^1, L^2, L^3)^T$ defined on $(\Omega, \mathcal{F}, \mathbb{G}, P^d)$ is constructed as follows:

(a) $N$ is a normal inverse Gaussian Lévy motion with parameters $\alpha, \beta, \delta, \mu$ satisfying $0 \leq |\beta| < \alpha$, $\delta > 0$ and $\mu \in \mathbb{R}$ and $Z^j$ is a Gamma process with parameters $\alpha_j, \beta_j > 0$ for any $j \in \{1, 2, 3\}$.

(b) $N, Z^1, Z^2$ and $Z^3$ are assumed to be stochastically independent.

(c) $Y^1 := L^1 = N + Z^3$.

(d) $Y^2 := (L^2, L^3)^T = -(Z^1 + Z^3, Z^2 + Z^3)^T$.

Clearly, the components $L^1, L^2$ and $L^3$ are stochastically dependent. Further, the processes $N, Z^1, Z^2, Z^3$ do not possess a continuous martingale part.

To ensure the positivity of the forward spreads, we get the following deterministic restrictions on the model parameters (see Proposition 4.3 and conditions (4.11) and (4.12)). For every $T \in [0, T^*]$, we assume that

$$\theta(\Sigma^1(t, T)) - \theta(\Sigma^d(t, T)) + l^1(t, T) \leq s^1_{0, d}(T) + \theta(\Sigma^1(0, T)) - \theta(\Sigma^d(0, T)) + l^1(0, T)$$

and

$$\theta(\Sigma^2(t, T)) - \theta(\Sigma^1(t, T)) + l^{2,1}(t, T) \leq s^{2,1}_{0, d}(T) + \theta(\Sigma^2(0, T)) - \theta(\Sigma^1(0, T)) + l^{2,1}(0, T)$$

for all $t \in [0, T]$. Then, we conclude that the desired monotonicity (4.1) is satisfied.

By using the stochastic independence of the processes $N, Z^1, Z^2$ and $Z^3$, we can express the cumulant function $\theta$ of $L$ under measure $P^d$ as

$$\theta(z) = \theta^N(z_1) + \theta^{Z^1}(-z_2) + \theta^{Z^2}(-z_3) + \theta^{Z^3}(z_1 - z_2 - z_3)$$

$$= \mu z_1 + \delta \sqrt{\alpha^2 - \beta^2} - \delta \sqrt{\alpha^2 - (\beta + z_1)^2} - \beta_1 \log \left(1 + \frac{z_2}{\alpha_1}\right)$$

$$- \beta_2 \log \left(1 + \frac{z_3}{\alpha_2}\right) - \beta_3 \log \left(1 - \frac{z_1 - z_2 - z_3}{\alpha_3}\right)$$

for every $z = (z_1, z_2, z_3) \in \mathbb{C}^3$ such that all the terms are well-defined. More specifically, the existence of the cumulant process is guaranteed for any $z = (z_1, z_2, z_3) \in \mathbb{R}^3$ satisfying:

1. $|z_1| < \min\{|-\alpha - \beta|, \alpha - \beta\}$.
2. $z_k \in (-\alpha_k, \infty)$ for each $k \in \{2, 3\}$.
3. $z_1 - z_2 - z_3 < \alpha_3$.

Then, the characteristic functions of $X^k_j$ and $Y^k_j$ under $P^d_{T_j^*}$ can be derived by using formulas (5.3) and (5.7). From the deterministic term $D^k_j$ we immediately
Figure 4. Relative errors of (mid) market and model prices

Figure 5. Relative errors of ask (top) and bid (bottom) market and model prices
get $\xi^*_j = -\ln \eta^*_j$. Finally, the ask, mid and bid model prices of the caplet are obtained by the valuation formulas (5.12), (5.8) and (5.13).

6.3. **Calibration Results.** The calibration procedure is done in two steps. First, we calibrate the model to the mid market prices by minimising the relative errors between the model and market prices of caps. Thereby, we use the representation (5.8) and extract the volatility, liquidity and distribution model parameters. In figure 4, we present the relative errors between mid market and model prices for both tenors. In the second step, we insert the calibrated parameters in the ask and bid caplet model price formulas (5.12) and (5.13) and calibrate the parameter $\gamma$ by minimising the relative errors between model and market bid and ask caplet prices for each maturity. The results are presented in figure 5. The calibrated parameters are given in table 1.
References


Ferdinando M. Ametrano and Marco Bianchetti. Everything You Always Wanted to Know About Multiple Interest Rate Curve Bootstrapping But Were Afraid to Ask. *Available at SSRN 2219548*, 2013.


**Abteilung für Mathematische Stochastik, Albert-Ludwigs-University Freiburg, Eckerstrasse 1, 79104 Freiburg, Germany**

*E-mail address: eberlein@stochastik.uni-freiburg.de*

**Abteilung für Mathematische Stochastik, Albert-Ludwigs-University Freiburg, Eckerstrasse 1, 79104 Freiburg, Germany**

*E-mail address: christoph.gerhart@finance.uni-freiburg.de*